# On the Use of Tight Frames for Optimal Sensor Placement in Time-difference of Arrival Localization

Cristian Rusu and John Thompson
Institute for Digital Communications, University of Edinburgh
Email: {c.rusu, john.thompson}@ed.ac.uk

***Abstract*—In this paper we analyze the use of tight frames for the problem of localizing a source from noisy time-difference of arrival measurements. Based on the Fisher information matrix, we show that positioning the sensor network according to a tight frame that also obeys some internal symmetries provides the best average localization accuracy. We connect our result to previous approaches from the literature and show experimentally that near optimal accuracy can also be provided by random tight frames. We also make the assumption that the sensors are not fixed but placed on mobile units and we study the problem of bringing them to a tight configuration with the minimum energy consumption. Although our results hold for any dimension, for simplicity of exposition, the numerical experiments depicted are in the two dimensional case.**

***Index Terms*—time-difference of arrival localization, Fisher information matrix, finite frames, tight frames.**

## I. Introduction

The problem of localizing a source given range or range-difference measurements taken using a network of passive sensors using least squares estimation has been extensively studied in the past [1]. In this context, given a sensor network, the problem of placing the sensors in order to improve the estimation accuracy of the network given time-difference of arrival measurements [2] is of central importance.

The problem has been studied for some time now and under different assumptions many solutions were proposed in the vast literature. Broadly speaking, we can distinguish two approaches based on ideas from control theory [3], [4], [5] and methods from estimation theory [6]–[16], respectively.

Previous work in the literature has already considered the use of tight frames for optimal sensor placement [5], [14]. These papers provided an optimal placement strategy given the number of anchor points or proposed heuristics to construct frames whose properties maximize localization accuracy. Constructing tight frames that also obey additional desirable properties is quite hard in general. Consider for example the spectral Tetris algorithm [17], that constructs unit norm tight frames, which suffers from the drawback that in many instances vectors are repeated throughout the frame.

In this paper we do not deal directly with the problem of sensor placement. Instead, we consider a scenario where the sensor locations are already given but they can be changed. Imagine, for example, a scenario where the sensors are located on vehicles, unmanned aerial vehicles (UAVs) or even people that are not stationary. Given this initial configuration we ask how to change it with the minimum effort in order to improve the theoretical average localization accuracy. We show how this can be achieved using tight frames to describe the coordinates of the sensors and then we provide numerical results that validate the approach. Our proposed sensor allocation technique is computationally simple since it only involves a single singular value decomposition [18].

Before we present our main results, we provide a quick overview of the localization problem from time-difference of arrival measurements and of the main properties of tight frames that we use in this paper.

## II. Primer on the time-difference of arrival localization

Given a sensor network composed of $m$ devices in the $n$ dimensional space, we introduce the sensor positions matrix

$$\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_m\end{bmatrix} \in \mathbb{R}^{n\times m}. \quad (1)$$

In this paper we assume that the rank of $\mathbf{A}$ is $n$. We denote by $\mathbf{x} \in \mathbb{R}^n$ the source coordinate vector to be estimated. The time-difference of arrival localization problem can be written as a system of linear equalities

$$\mathbf{\Phi y} = \mathbf{b},\ \mathbf{y} = \begin{bmatrix}\|\mathbf{x}\|_2 \\ \mathbf{x}\end{bmatrix} \in \mathbb{R}^{n+1}, \quad (2)$$

where

$$\mathbf{\Phi} = \begin{bmatrix} d_1 & \mathbf{a}_1^T \\ d_2 & \mathbf{a}_2^T \\ \vdots & \vdots \\ d_n & \mathbf{a}_m^T \end{bmatrix} \in \mathbb{R}^{m\times(n+1)},\ \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m, \quad (3)$$

with the source's range-difference between sensor $i$ and a reference at the origin $d_i = \|\mathbf{a}_i - \mathbf{x}\|_2 - \|\mathbf{x}\|_2 + n_i$ and $b_i = \frac{1}{2}(\|\mathbf{a}_i\|_2^2 - d_i^2)$ for $i = 1, \dots, m$ and where $\mathbf{n} \in \mathbb{R}^m$ is the zero mean i.i.d. Gaussian noise with variance $\sigma^2$. An estimated position of the source $\hat{\mathbf{x}}$ is given by the unconstrained least squares estimate of (2)

$$\hat{\mathbf{x}} = \mathbf{W}(\mathbf{\Phi}^T\mathbf{\Phi})^{-1}\mathbf{\Phi}^T\mathbf{b},\ \mathbf{W} = \begin{bmatrix}\mathbf{0}^T \\ \mathbf{I}\end{bmatrix}. \quad (4)$$

To evaluate the localization accuracy the Cramer-Rao bound (CRB) is often used. Defined as the inverse Fisher information

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matrix, the CRB provides a lower bound on the variance of an unbiased estimator. The Fisher information matrix for time-difference of arrival for the sensor network whose positions are described by $\mathbf{A}$ was first given in [19] and it was shown in [20] that it can be expressed as:

$$\begin{aligned}\mathbf{F_A} =&\frac{1}{\sigma^2}\left(\sum_{i=1}^{m}\mathbf{a}_i\mathbf{a}_i^T - m\tilde{\mathbf{a}}\tilde{\mathbf{a}}^T\right)\\ =&\frac{1}{\sigma^2}\left(\mathbf{A}\mathbf{A}^T - m^{-1}\mathbf{A}\mathbf{1}\mathbf{1}^T\mathbf{A}^T\right),\end{aligned} \tag{5}$$

where $\tilde{\mathbf{a}} = m^{-1}\sum_{i=1}^{m}\mathbf{a}_i = m^{-1}\mathbf{A}\mathbf{1}$ is the average of the vectors $\mathbf{a}_i$ and $\mathbf{1}$ is the all-ones vector.

In order to minimize the CRB and therefore provide accurate estimates of the source our goal will be to maximize the determinant of the Fisher information matrix.

## III. Primer on tight frames

A family of vectors $\mathbf{a}_i \in \mathbb{R}^n,\ i = 1,\dots,m$ is called a frame for $\mathbb{R}^n$ if there exists constants $0 < \alpha \le \beta < \infty$ such that

$$\alpha\|\mathbf{x}\|_2^2 \le \sum_{i=1}^{m}|\mathbf{x}^T\mathbf{a}_i|^2 \le \beta\|\mathbf{x}\|_2^2,\ \text{for all } \mathbf{x}\in\mathbb{R}^n. \tag{6}$$

The matrix $\mathbf{A}$ is called the synthesis operator of the frame while the frame operator $\mathbf{A}\mathbf{A}^T$ obeys $\alpha\mathbf{I} \le \mathbf{A}\mathbf{A}^T \le \beta\mathbf{I}$. When $\alpha = \beta$ we call the frame tight (or $\alpha$-tight). We will refer equivalently to the frame or its synthesis operator.

Let $\mathbf{A}\in\mathbb{R}^{n\times m}$ have the singular value decomposition $\mathbf{A} = \mathbf{U}\mathbf{\Sigma_A}\mathbf{V}^T$. It was shown in Theorem 2 of [21] for example that, with respect to the Frobenius norm, the closest $\alpha$-tight frame to $\mathbf{A}$ is given by

$$\mathbf{B} = \mathbf{U}\begin{bmatrix}\alpha\mathbf{I} & \mathbf{0}\end{bmatrix}\mathbf{V}^T. \tag{7}$$

We note that tight frames have been used in the past to improve least squares estimation from noisy measurements. For example, in wireless communications they improve channel estimation accuracy [22] and in sparse recovery problems they reduce the mean squared error [23] [24], on average over all sparsity levels.

Consider now that $\mathbf{A} = \mathbf{U}\mathbf{\Sigma_A}\mathbf{V}^T$ is normalized such that $\|\mathbf{A}\|_F^2 = m$. If we denote by $\mathbf{B}$ the closest tight frame to $\mathbf{A}$ such that $\|\mathbf{B}\|_F^2 = \|\mathbf{A}\|_F^2$ we have that

$$\begin{aligned}&\mathbf{B} = \mathbf{U}\mathbf{\Sigma_B}\mathbf{V}^T = \mathbf{U}\begin{bmatrix}s\mathbf{I} & \mathbf{0}_{n\times(m-n)}\end{bmatrix}\mathbf{V}^T,\\ &\text{with } s = \sqrt{\frac{\sum_{i=1}^{n}\sigma_i^2}{n}},\end{aligned} \tag{8}$$

where the scalars $\sigma_i$ are the singular values of $\mathbf{A}$. We call $\mathbf{B}$ an $s$-tight frame. The distance between the initial frame $\mathbf{A}$ and the $s$-tight frame $\mathbf{B}$ is given by

$$\begin{aligned}\|\mathbf{A}-\mathbf{B}\|_F^2 =&\|\mathbf{U}\mathbf{\Sigma_A}\mathbf{V}^T - \mathbf{U}\mathbf{\Sigma_B}\mathbf{V}^T\|_F^2\\ =&\|\mathbf{U}(\mathbf{\Sigma_A}-\mathbf{\Sigma_B})\mathbf{V}^T\|_F^2 = \|\mathbf{\Sigma_A}-\mathbf{\Sigma_B}\|_F^2\\ =&\sum_{i=1}^{n}(\sigma_i - s)^2.\end{aligned} \tag{9}$$

Since $\mathbf{B}$ is an $s$-tight frame with $\|\mathbf{B}\|_F^2 = \|\mathbf{A}\|_F^2 = m$ an important relationship that we will use is

$$\|\mathbf{B}\|_F^2 = \|\mathbf{\Sigma_B}\|_F^2 = ns^2 = m \Rightarrow s^2 = \frac{m}{n}. \tag{10}$$

We now move to show the role of tight frames in the time-difference of arrival localization problem.

## IV. Sensor placement via tight frames

In this section we explore the consequences of placing the network sensors in a configuration that describes a tight frame. For simplicity of exposition we discuss the two-dimensional case, i.e., $n = 2$, but similar results hold in general.

Given that a sensor network is placed in a two-dimensional space according to a $s$-tight frame $\mathbf{B}$ we have the determinant of the Fisher information matrix using (5):

$$\begin{aligned}\det(\mathbf{F_B}) =& \frac{1}{\sigma^4}\det(\mathbf{B}\mathbf{B}^T - m^{-1}\mathbf{B}\mathbf{1}\mathbf{1}^T\mathbf{B}^T)\\ =&\frac{1}{\sigma^4}\det(s^2\mathbf{I} - m^{-1}\mathbf{B}\mathbf{1}\mathbf{1}^T\mathbf{B}^T)\\ =&\frac{1}{\sigma^4}\det(s^2\mathbf{I})\det(1 - m^{-1}\mathbf{1}^T\mathbf{B}^T(s^2\mathbf{I})^{-1}\mathbf{B}\mathbf{1})\\ =&\frac{s^4}{\sigma^4}\left(1 - \frac{1}{ms^2}\mathbf{1}^T\mathbf{B}^T\mathbf{B}\mathbf{1}\right)\\ =&\frac{m^2}{4\sigma^4}\left(1 - \frac{2}{m^2}\mathbf{1}^T\mathbf{B}^T\mathbf{B}\mathbf{1}\right)\\ =&\frac{m^2}{4\sigma^4}\gamma,\end{aligned} \tag{11}$$

where $\sigma^2$ is the noise variance and we have used the fact that $\det(\alpha\mathbf{X}) = \alpha^2\det(\mathbf{X})$ for a $2\times 2$ matrix and the identity

$$\det(\mathbf{X}+\mathbf{A}\mathbf{B}) = \det(\mathbf{X})\det(\mathbf{I}+\mathbf{B}\mathbf{X}^{-1}\mathbf{A}). \tag{12}$$

In order to maximize the expression (11) we need to minimize $\mathbf{1}^T\mathbf{B}^T\mathbf{B}\mathbf{1}$. Given that $\mathbf{B}$ is a tight frame we know that the spectrum of $\mathbf{B}^T\mathbf{B}$ is limited to $\Lambda(\mathbf{B}^T\mathbf{B}) = \{s^2, 0\} = \{m/2, 0\}$ where the eigenvalues have multiplicities 2 and $m-2$ respectively. In the worst case scenario if $(m^{-1/2}\mathbf{1})$ is an eigenvector of $\mathbf{B}^T\mathbf{B}$ with eigenvalue $m/2$ we have the minimum value $\gamma = 1 - \frac{2}{m^2}\mathbf{1}^T\mathbf{B}^T\mathbf{B}\mathbf{1} = 1 - \frac{2}{m^2}\sqrt{m}\frac{m}{2}\sqrt{m} = 0$. In the best case scenario we have the maximum $\gamma = 1$ when $\mathbf{1}^T\mathbf{B}^T\mathbf{B}\mathbf{1} = 0$. In fact, due to the spectral properties of $\mathbf{B}^T\mathbf{B}$ we have that $\gamma\in[0,1]$.

Since $\mathbf{B}^T\mathbf{B}$ has a null space of dimension $(m-2)$ we expect that, for large enough $m$, with high probability the constant vector $\mathbf{1}$ will have on average, for a randomly generated frame, a large contribution in the null space. Therefore we expect $\mathbf{1}^T\mathbf{B}^T\mathbf{B}\mathbf{1} \ll m^2/2$ in general. For random $s$-tight frames $\mathbf{B}$ that also obey $\|\mathbf{B}\|_F^2 = m$ we have

$$\begin{aligned}\mathbb{E}[\gamma] =&1 - \frac{2}{m^2}\mathbb{E}[\mathbf{1}^T\mathbf{B}^T\mathbf{B}\mathbf{1}]\\ =&1 - \frac{2}{m^2}\mathbb{E}[\|\mathbf{B}\mathbf{1}\|_2^2]\\ =&1 - \frac{4s^2}{m^2} = 1 - \frac{2}{m}.\end{aligned} \tag{13}$$

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With the normalization $\|\mathbf{B}\|_F^2 = m$ the maximum value of the determinant in (11) matches the value achieved via the splay configurations proposed in [20]. The symmetry property discussed in [20] still needs to hold to reach the maximum $\det(\mathbf{F_B}) = m^2/(4\sigma^4)$ but we are able to show that symmetric tight frames in general are able to reach this maximum.

For certain numbers of sensors $m$ in the network it is trivial to construct tight frames $\mathbf{B}$ that have in their null space the vector $\mathbf{1}$. Consider for example the following: given a tight frame $\mathbf{B} \in \mathbb{R}^{2\times m}$ we construct an extended frame $\mathbf{C} \in \mathbb{R}^{2\times 4m}$ in three steps

$$\begin{aligned} \mathbf{C} &\leftarrow \begin{bmatrix} \mathbf{B} & \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{B} \end{bmatrix} \in \mathbb{R}^{2\times 2m}, \\ \mathbf{C} &\leftarrow \begin{bmatrix} \mathbf{C} & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{C} \end{bmatrix} \in \mathbb{R}^{2\times 4m}, \\ \mathbf{C} &\leftarrow \frac{1}{2}\mathbf{C}. \end{aligned} \tag{14}$$

The resulting $\mathbf{C} \in \mathbb{R}^{2\times 4m}$ is tight, has the same Frobenius norm as $\mathbf{B}$ and $\mathbf{C1} = \mathbf{0}$. The result is tight because we concatenate tight matrices (flipping the signs of the rows in the frame preserves tightness) while the fact that $\mathbf{C1} = \mathbf{0}$ is true by construction (we first concatenate a reflection about the $y$-axis ensuring that the $x$ coordinates sum to zero and then we reflect about the $x$-axis ensuring that the $y$ coordinates sum to zero too). The last update to $\mathbf{C}$ guarantees that the resulting tight frame has the same Frobenius norm as $\mathbf{B}$ (each concatenation increases the Frobenius norm by $\sqrt{2}$).

Given the sensor positions $\mathbf{A}$ the work to be done in order to convert it to a tight frame $\mathbf{B}$ is given from (9) to be

$$\mathcal{W} = (\sigma_1 - s)^2 + (\sigma_2 - s)^2. \tag{15}$$

The choice in (8) keeps equality between the Frobenius norms of $\mathbf{A}$ and $\mathbf{B}$ – meaning that on average the squared sensor distance with regards to a reference origin point is kept constant. If the goal is to minimize the effort to relocate the sensors then the choice $s_{\text{min}} = \frac{\sigma_1+\sigma_2}{2}$ leads to the minimum work $\mathcal{W}_{\text{min}} = \frac{(\sigma_1-\sigma_2)^2}{2}$. We call the $s_{\text{min}}$-tight frame that achieves this minimum

$$\mathbf{B}_{\text{min}} = \mathbf{U} \begin{bmatrix} s_{\text{min}}\mathbf{I} & \mathbf{0}_{2\times(m-2)} \end{bmatrix} \mathbf{V}^T. \tag{16}$$

Therefore, in scenarios where the amount of energy spent by the sensor network is relevant (assuming a mobile network, for example an UAV scenario) the choice of the nearest tight frame for positioning may be different from (8). The difference lies in the Frobenius norms of these frames. Lastly, we mention that displacement of the sensors in a network from an $\alpha$-tight frame at time $t = 0$ to a $\beta$-tight configuration at time $t = 1$ can be achieved exclusively through $\delta$-tight frames where $\delta = t\alpha + (1-t)\beta$ for $t \in [0, 1]$. Therefore, transitions from one tight configuration to another are possible while preserving the average localization accuracy of the sensor network.

## V. Experimental Results

In this section we provide numerical experiments to show how sensor placement via tight frames can provide lower estimation error. To evaluate performance we consider the root mean squared error between the true source location $\mathbf{x}$ and its estimate $\hat{\mathbf{x}}$

$$\text{RMSE}(\mathbf{x}, \hat{\mathbf{x}}) = \sqrt{\frac{1}{2}\|\mathbf{x} - \hat{\mathbf{x}}\|_2^2}. \tag{17}$$

For simplicity of exposition we assume a two-dimensional scenario.

In Fig. 1 we show the initial sensors placement given by a random matrix $\mathbf{A}$, the closest tight frame next to it with the same Frobenius norm denoted by $\mathbf{B}$ (8) and the closest tight frame $\mathbf{B}_{\text{min}}$ (16). Notice how in the initial configuration the sensors are approximately concentrated along the bisector of

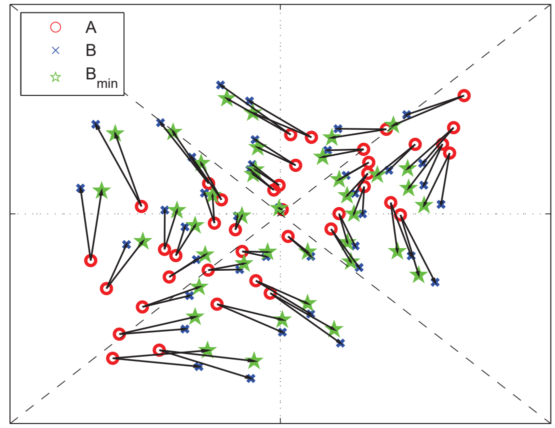


Figure 1: A two-dimensional sensor network with $m = 40$ sensors positioned by a random frame $\mathbf{A}$ and two tight frames $\mathbf{B}$ (8) and $\mathbf{B}_{\text{min}}$ (16) computed from $\mathbf{A}$. The displacements of the sensors in the initial network, due to the tightness constraints, are depicted by arrows. The sensors are placed in an area of size 1km×1km.

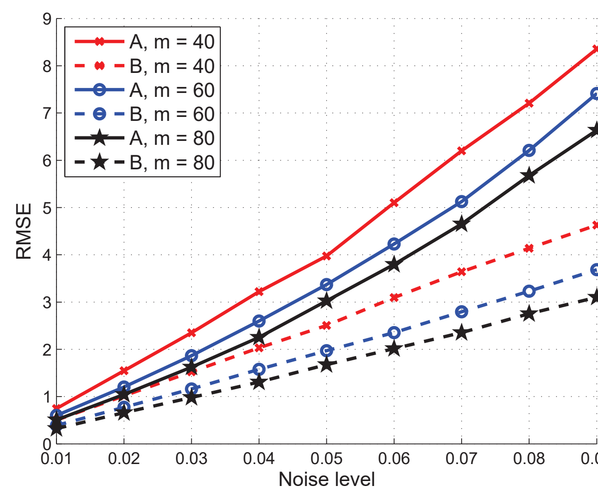


Figure 2: Estimation accuracy evolution for increasing noise levels and fixed number of sensors $m \in \{40, 60, 80\}$ achieved by sensor networks whose positioning is given either by the random frame $\mathbf{A}$ or by the closest $s$-tight frame $\mathbf{B}$.

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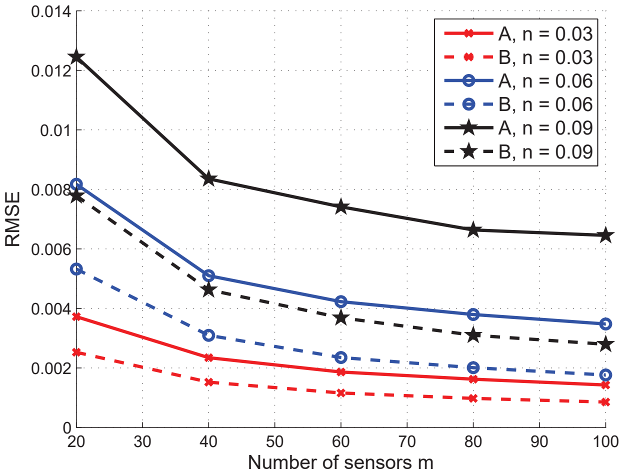


Figure 3: Similar experimental setup to Fig. 2 for noise levels $\{0.03, 0.06, 0.09\}$ and varying number of sensors $m$ in the network.

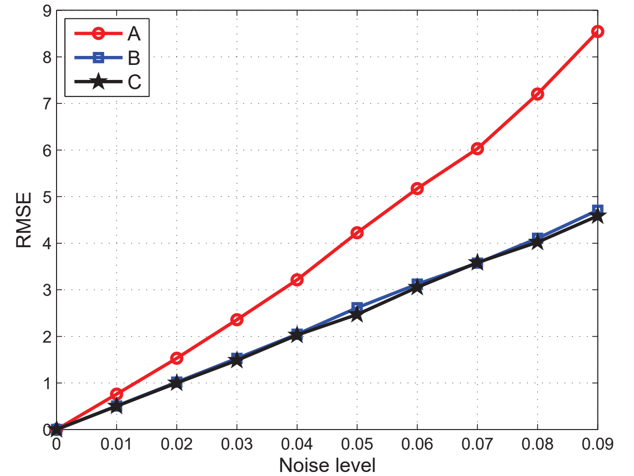


Figure 4: Estimation accuracy evolution for increasing noise levels and fixed number of sensors $m = 40$ achieved by sensor networks whose positioning is given either by the random frame $\mathbf{A}$ or by the closest $s$-tight and symmetric $s$-tight frames $\mathbf{B}$ and $\mathbf{C}$, respectively.

the plane. The two tight frames proceed to push the sensors in similar directions perpendicular to this bisector. Sensors placed further from the origin in the initial random configuration are pushed towards the center in order to keep the Frobenius norm of $\mathbf{B}$ under control.

In Figs. 2, 3 and 4 we show the RMSE estimation performance of sensor networks whose sensor placement is defined by different frames. All numerical experiments follows the same setup: we randomly generate via a Gaussian distribution the initial sensor positions in the frame $\mathbf{A}$ for which we fix $\sigma_1 = 3$ and $\sigma_2 = 1$ and when we calculate the new sensor positions in a tight frame $\mathbf{B}$ built via (8) and a symmetric tight frame $\mathbf{C}$ via (14). The results we show are averaged in the following way: we generate 100 instances of $\mathbf{A}$ (and consequently $\mathbf{B}$ and/or $\mathbf{C}$) and then for each instant we proceed to estimate 1000 randomly generated sources $\mathbf{x}$ from noisy distance measurements. The noise is i.i.d. Gaussian with zero mean and variance $1/\sqrt{m}$.

Figs. 2 and 3 show the effect of the noise and the number of measurements on the RMSE of the estimation. We compare the estimation accuracy of sensor networks positioned randomly via a frame $\mathbf{A}$ with sensor networks defined via $s$-tight frames $\mathbf{B}$ (8). We notice that in all cases the sensors positioned via $s$-tight frames always perform better than their random counterparts. Of course, lower noise levels and higher number of sensors lead to better estimation performance. The performance gap increases with the noise level and with the number of sensors in the network $m$.

Last, in Fig. 4 we compare the random initial positioning with the positioning given by the $s$-tight frame $\mathbf{B}$ (8) closest to the random configuration and the symmetric $s$-tight frame $\mathbf{C}$ (14). The frame $\mathbf{C}$ is created starting from the positions of first $m/4$ sensor from $\mathbf{A}$. The estimation accuracy given by positioning the sensor network via $\mathbf{B}$ or $\mathbf{C}$ is nearly identical and much better than the random configuration. As previously discussed, for a relative large number of sensors $m$ in the network the symmetry constraint imposed to maximize the determinant of the Fisher information matrix does not seem to play a crucial role in the design. A large enough random $s$-tight frame performs similarly on average without the explicit requirement for symmetry.

## VI. Conclusions

In this paper we show experimentally that sensor networks whose sensors are positioned according to random tight frames provide near optimal average localization accuracy from time-difference of arrival measurements. We base the result on an analysis of the Fisher information matrix and the maximization of its determinant. In the experimental setting we do not study directly the problem of optimal sensor placement but instead, given an initial configuration, we tackle the problem of moving the sensors with the minimum amount of energy required such that the network achieves near optimal localization accuracy.

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