A Gridless Sparse Method for Super-Resolution of Harmonics

Zai Yang

School of Automation, Nanjing University of Science and Technology, Nanjing 210094, China
Key Laboratory of System Control and Information Processing, Ministry of Education, Shanghai 200240, China

***Abstract*—As a special frequency estimation problem, harmonics estimation has applications in speech and audio processing, power systems, healthcare monitoring, etc. In this paper, we make a first attempt to propose a gridless sparse method for harmonics estimation exploiting the harmonics structure. The method uses the atomic norm with carefully designed atoms and is formulated as a convex optimization problem. Its performance is demonstrated via numerical simulations.**

***Index Terms*—Harmonics estimation, frequency estimation, atomic norm, group sparsity, gridless sparse method.**

I. Introduction

Line spectral estimation, the process of estimating the frequency components of a complex signal given its discrete samples, is a classical problem in statistical signal processing and has broad applications. Conventional approaches, e.g., maximum likelihood estimation (MLE) and subspace-based methods like MUSIC and ESPRIT, usually suffer from certain well-known limitations, e.g., the need of the model order. In contrast to this, sparse estimation methods attempt to find the solution consisting of the minimum number of frequencies and have demonstrated, mainly in the last decade, their superiority in accuracy, robustness and flexibility. A key ingredient of early sparse methods is parameter discretization/gridding, which is performed to transform approximately the original nonlinear continuous parameter estimation problem as a discrete sparse signal estimation problem given a linear system of equations, resulting in the difficulties of grid selection and performance analysis. In the past four years or so, gridless sparse methods have been developed to conquer the continuity and nonlinearity issues and meanwhile provide strong theoretical guarantees and reliable solutions based on convex optimization. A comprehensive overview of these sparse methods can be found in [1], while readers can consult [2] for details of conventional methods.

In this paper, we study the super-resolution of harmonics, a special case of line spectral estimation. A harmonic is a complex sinusoidal wave whose frequency is an integer multiple of a certain frequency, the fundamental frequency. The sinusoidal wave corresponding to the fundamental frequency is also called the first harmonic. Therefore, the frequencies of harmonics are related and our task is to locate them. In this paper we consider the case when multiple fundamental frequencies can be present.

The necessity of estimating the fundamental frequency and amplitudes of harmonics arises in various applications. In speech and audio processing, the fundamental frequency is called the pitch whose estimation is important in problems such as source separation and classification [3]–[5]. In power systems, the presence of the second and higher order harmonics represents a type of voltage and current waveform distortion and may result in power loss and damages to equipments. Their estimation is therefore important for power quality assessment [6]. In noncontact healthcare monitoring, heart-rate is a vital sign of health quality, whose estimation however is usually interfered by respiration harmonics [7].

Extensive studies have been done on this research topic. One kind of methods first estimate the frequencies without accounting for the harmonics structure using algorithms for standard line spectral estimation and then form the harmonic groups based on certain post-processing or correction steps. The other kind of methods integrate the harmonics structure into frequency estimation so that the estimation performance can be improved and the estimation result is easier to interpret. Following these ideas, algorithms inspired by conventional frequency estimation methods can be found in [3], [6], [8]–[10]. Several sparse estimation methods have also been proposed in context of different applications (see, e.g., [11]–[16]). However, these sparse methods are either based on parameter discretization or cannot exploit the harmonics structure.

In this paper, we make a first attempt to propose a gridless sparse method for harmonics estimation that exploits the harmonics structure. The proposed method consists of two key steps: 1) define a set of atoms to exploit the harmonics structure, referred to as the group sparsity, and then minimize the number of the fundamental frequencies using the atomic norm [17], and 2) relate the group-sparse atomic norm to the joint-sparse atomic norm in context of multiple measurement vectors in [18] and then provide a semidefinite program (SDP) formulation for the proposed method. Numerical simulations are provided to demonstrate the performance of the proposed atomic norm method, compared with the standard atomic norm method that does not utilize the harmonics structure.

II. Problem Formulation

Consider a complex-valued signal, $\boldsymbol{y} \in \mathbb{C}^N$, consisting of a set of K harmonically related sources with (normalized)

ISBN 978-0-9928626-7-1

fundamental frequencies f_k, $1,\ldots,K$:

$$\boldsymbol{y}=\sum_{k=1}^{K}\sum_{l=1}^{L}s_{kl}\boldsymbol{a}\left(lf_k\right), \tag{1}$$

where $s_{kl}\in\mathbb{C}$ is the coefficient for the lth harmonic with the kth fundamental frequency, and

$$\boldsymbol{a}\left(f\right)=\boldsymbol{a}\left(f,N\right)=\left[1,e^{i2\pi f},\ldots,e^{i2\pi(N-1)f}\right]^T \tag{2}$$

is a sinusoidal wave with frequency f. In this paper, we do not assume the knowledge on the number K of the fundamental frequencies but the model order L of each harmonics group. Given $\boldsymbol{y}$, our objective is to estimate $\{f_k\}$ and $\{s_{kl}\}$.

III. Proposed Gridless Sparse Solution

A. Atomic Norm Exploiting Group Sparsity

Let $\boldsymbol{S}=[s_{kl}]$ be a $K\times L$ matrix of coefficients in (1) and $\boldsymbol{s}_k$ be its kth row. Also let $c_k=\left\|\boldsymbol{s}_k\right\|_2$ and $\boldsymbol{\phi}_k=\frac{\boldsymbol{s}_k}{c_k}$, and hence $\left\|\boldsymbol{\phi}_k\right\|_2=1$. Then, (1) can be written as:

$$\boldsymbol{y}=\sum_{k=1}^{K}c_k\sum_{l=1}^{L}\phi_{kl}\boldsymbol{a}\left(lf_k\right). \tag{3}$$

Evidently, $\boldsymbol{y}$ is a linear combination of K elements in the set, $\mathcal{B}$, defined as:

$$\mathcal{B}=\left\{\boldsymbol{b}\left(f,\boldsymbol{\phi}\right)=\sum_{l=1}^{L}\phi_l\boldsymbol{a}\left(lf\right):\ \left\|\boldsymbol{\phi}\right\|_2=1\right\}, \tag{4}$$

Here we refer to any element in $\mathcal{B}$ as an atom and $\mathcal{B}$ as the set of atoms.

Following from [17], we define the atomic norm of $\boldsymbol{y}$, with respect to the set of atoms $\mathcal{B}$, as:

$$\begin{aligned}&\left\|\boldsymbol{y}\right\|_{\mathcal{B}}\\&=\inf_{f_k,\boldsymbol{\phi}_k,c_k}\left\{\sum_k c_k:\ \boldsymbol{y}=\sum_k c_k\boldsymbol{b}\left(f_k,\boldsymbol{\phi}_k\right),\ \left\|\boldsymbol{\phi}_k\right\|_2=1\right\}.\end{aligned} \tag{5}$$

We propose to estimate the fundamental frequencies in $\boldsymbol{y}$ using the above defined atomic norm. Concretely, we retrieve them from the atomic decomposition of $\boldsymbol{y}$ at which the infimum of the set in (5) is achieved. To do so, we need a computational approach to the atomic norm, which is the task of the ensuing subsection.

B. SDP Formulation for the Atomic Norm

We next provide an SDP formulation for the atomic norm in (5). This will be done based on an alternative interpretation of the atomic norm. In particular, let

$$\boldsymbol{y}_l=\sum_{k=1}^{K}s_{kl}\boldsymbol{a}\left(lf_k\right) \tag{6}$$

be the lth component of $\boldsymbol{y}$ in (1) consisting of the lth harmonics, with

$$\boldsymbol{y}=\sum_{l=1}^{L}\boldsymbol{y}_l. \tag{7}$$

A key observation is that a sinusoidal wave with frequency lf can be interpreted an l-fold undersampled version of a sinusoidal wave with frequency f. In mathematics, $\boldsymbol{a}\left(lf\right)$ is always a subvector of $\widetilde{\boldsymbol{a}}\left(f\right)=\boldsymbol{a}\left(f,NL-L+1\right)$. For $l=1,\ldots,L$, let the $N\times(NL-L+1)$ matrix $\boldsymbol{\Gamma}^{(l)}$ be such that

$$\Gamma_{nj}^{(l)}=\begin{cases}1, & \text{if } j=nl-l+1;\\ 0, & \text{otherwise.}\end{cases} \tag{8}$$

Then, we have that

$$\boldsymbol{a}\left(lf\right)=\boldsymbol{\Gamma}^{(l)}\widetilde{\boldsymbol{a}}\left(f\right). \tag{9}$$

Inserting (9) into (6), we have that

$$\boldsymbol{y}_l=\sum_{k=1}^{K}s_{kl}\boldsymbol{\Gamma}^{(l)}\widetilde{\boldsymbol{a}}\left(f_k\right)=\boldsymbol{\Gamma}^{(l)}\widetilde{\boldsymbol{y}}_l, \tag{10}$$

where

$$\widetilde{\boldsymbol{y}}_l=\sum_{k=1}^{K}s_{kl}\widetilde{\boldsymbol{a}}\left(f_k\right). \tag{11}$$

It immediately follows from (11) that $\widetilde{\boldsymbol{y}}_l$, $l=1,\ldots,L$ consist of the same frequencies, and in other words, they are joint-sparse [18]. Moreover, the shared frequencies f_k, $k=1,\ldots,K$ are exactly the fundamental frequencies of interest. Following from this observation, we form an $(NL-L+1)\times L$ matrix, $\widetilde{\boldsymbol{Y}}$, as:

$$\widetilde{\boldsymbol{Y}}=[\widetilde{\boldsymbol{y}}_1,\ldots,\widetilde{\boldsymbol{y}}_L]=\sum_{k=1}^{K}\widetilde{\boldsymbol{a}}\left(f_k\right)\boldsymbol{s}_k=\sum_{k=1}^{K}c_k\widetilde{\boldsymbol{a}}\left(f_k\right)\boldsymbol{\phi}_k. \tag{12}$$

Define a new set of atoms, $\mathcal{A}$, as:

$$\mathcal{A}=\{\widetilde{\boldsymbol{a}}\left(f\right)\boldsymbol{\phi}:\ \left\|\boldsymbol{\phi}\right\|_2=1\} \tag{13}$$

and let $\left\|\cdot\right\|_{\mathcal{A}}$ be its induced atomic norm. To relate $\left\|\widetilde{\boldsymbol{Y}}\right\|_{\mathcal{A}}$ and $\left\|\boldsymbol{y}\right\|_{\mathcal{B}}$, we have the following result.

Lemma 1: It holds that

$$\left\|\boldsymbol{y}\right\|_{\mathcal{B}}=\min_{\widetilde{\boldsymbol{Y}}}\left\|\widetilde{\boldsymbol{Y}}\right\|_{\mathcal{A}},\ \text{subject to}\ \boldsymbol{y}=\sum_{l=1}^{L}\boldsymbol{\Gamma}^{(l)}\widetilde{\boldsymbol{y}}_l. \tag{14}$$

Proof: Let F^* be the optimal value of the optimization problem in (14). We first show that $\left\|\boldsymbol{y}\right\|_{\mathcal{B}}\geq F^*$. To do so, observe that for any atomic decomposition of $\boldsymbol{y}$ as in (5), using the constructions above we can always find $\widetilde{\boldsymbol{Y}}$ and its atomic decomposition as in (12), which satisfies $\boldsymbol{y}=\sum_{l=1}^{L}\boldsymbol{\Gamma}^{(l)}\widetilde{\boldsymbol{y}}_l$ [by inserting (10) into (7)]. It follows that

$$F^*\leq\left\|\widetilde{\boldsymbol{Y}}\right\|_{\mathcal{A}}\leq\sum_{k=1}^{K}c_k, \tag{15}$$

where the second inequality follows from the definition of $\left\|\cdot\right\|_{\mathcal{A}}$. Since (15) holds for any atomic decomposition of $\boldsymbol{y}$, taking infimum at the right hand size of (15) and applying the definition of $\left\|\cdot\right\|_{\mathcal{B}}$, we have that $F^*\leq\left\|\boldsymbol{y}\right\|_{\mathcal{B}}$.

We can similarly show that $\left\|\boldsymbol{y}\right\|_{\mathcal{B}}\leq F^*$. To do so, suppose that the optimizer of the optimization problem in (14) is given by $\widetilde{\boldsymbol{Y}}^*$ and its atomic decomposition is as given in (12), where

ISBN 978-0-9928626-7-1

we replace K, c_k, f_k and $\boldsymbol{\phi}_k$ by K^*, c_k^*, f_k^* and $\boldsymbol{\phi}_k^*$ respectively, with $F^* = \sum_{k=1}^{K^*} c_k^*$. Using the identity $\boldsymbol{y} = \sum_{l=1}^{L} \boldsymbol{\Gamma}^{(l)} \widetilde{\boldsymbol{y}}_l^*$, we can obtain an atomic decomposition of $\boldsymbol{y}$ as in (3). It immediately follows that $\|\boldsymbol{y}\|_{\mathcal{B}} \leq \sum_{k=1}^{K^*} c_k^* = F^*$. So we complete the proof.

Moreover, it can be concluded from the above arguments that the atomic decomposition of $\boldsymbol{y}$ must be retrieved from the atomic decomposition of the optimizer $\widetilde{\boldsymbol{Y}}^*$. ■

Now we are ready to provide an SDP formulation for $\|\boldsymbol{y}\|_{\mathcal{B}}$. In fact, this can be readily done if an SDP formulation of the joint-sparse atomic norm $\|\cdot\|_{\mathcal{A}}$ is available. Fortunately, it has been derived in [18], to be specific,

$$\begin{aligned} \left\|\widetilde{\boldsymbol{Y}}\right\|_{\mathcal{A}} = & \min_{\boldsymbol{W},\boldsymbol{u}} \frac{1}{2\sqrt{NL-L+1}} \left[\mathrm{tr}\left(\boldsymbol{W}\right) + \mathrm{tr}\left(T\left(\boldsymbol{u}\right)\right)\right], \\ & \text{subject to } \begin{bmatrix} \boldsymbol{W} & \widetilde{\boldsymbol{Y}}^H \\ \widetilde{\boldsymbol{Y}} & T\left(\boldsymbol{u}\right) \end{bmatrix} \geq \boldsymbol{0}, \end{aligned} \tag{16}$$

where $NL-L+1$ equals the number of rows of $\widetilde{\boldsymbol{Y}}$. In (16), $T\left(\boldsymbol{u}\right)$ denotes a (Hermitian) Toeplitz matrix whose first row is given by the elements of $\boldsymbol{u}$. Inserting (16) into (14), the following SDP formulation of $\|\boldsymbol{y}\|_{\mathcal{B}}$ is obtained:

$$\begin{aligned} \|\boldsymbol{y}\|_{\mathcal{B}} = & \min_{\boldsymbol{W},\boldsymbol{u},\widetilde{\boldsymbol{Y}}} \frac{1}{2\sqrt{NL-L+1}} \left[\mathrm{tr}\left(\boldsymbol{W}\right) + \mathrm{tr}\left(T\left(\boldsymbol{u}\right)\right)\right], \\ & \text{subject to } \begin{bmatrix} \boldsymbol{W} & \widetilde{\boldsymbol{Y}}^H \\ \widetilde{\boldsymbol{Y}} & T\left(\boldsymbol{u}\right) \end{bmatrix} \geq \boldsymbol{0}, \\ & \boldsymbol{y} = \sum_{l=1}^{L} \boldsymbol{\Gamma}^{(l)} \widetilde{\boldsymbol{y}}_l. \end{aligned} \tag{17}$$

C. Atomic Decomposition Retrieval

When the SDP in (17) is solved using standard SDP solvers, the atomic decomposition of $\boldsymbol{y}$ can be retrieved from its numerical solution. Concretely, following from [18], the frequencies composing $\widetilde{\boldsymbol{Y}}^*$ can be retrieved from the following Vandermonde decomposition of $T\left(\boldsymbol{u}^*\right)$:

$$T\left(\boldsymbol{u}^*\right) = \sum_{k=1}^{K^*} p_k^* \widetilde{\boldsymbol{a}}\left(f_k^*\right) \widetilde{\boldsymbol{a}}^H\left(f_k^*\right), \tag{18}$$

where $p_k^* > 0$ and $K^* = \mathrm{rank}\left(T\left(\boldsymbol{u}^*\right)\right)$. This decomposition is unique if $T\left(\boldsymbol{u}^*\right)$ is rank-deficient and can be computed using, e.g., ESPRIT (note that a computational approach can also be found in [18] if $T\left(\boldsymbol{u}^*\right)$ has full rank). After that, c_k^* and $\boldsymbol{\phi}_k^*$ in the atomic decomposition of $\widetilde{\boldsymbol{Y}}^*$, as in (12), can be computed using a least squares method. Note that, during the same process, the atomic decomposition of $\boldsymbol{y}$, as in (3), is obtained.

D. The Noisy Case

The proposed atomic norm method can be modified and applied in the case when the observed data are corrupted by noise following a standard routine. In particular, assume that the added noise in the data vector $\boldsymbol{y}$ has bounded energy, with an upper bound η on its ℓ_2 norm. In this case we solve the following atomic norm minimization problem:

$$\min_{\boldsymbol{z}} \|\boldsymbol{z}\|_{\mathcal{B}}, \text{ subject to } \|\boldsymbol{z}-\boldsymbol{y}\|_2 \leq \eta, \tag{19}$$

where $\boldsymbol{z}$ denotes the signal part in $\boldsymbol{y}$. This means that, among all possible candidates, we find the one with the smallest atomic norm as the signal estimate, and the fundamental frequencies are retrieved from its atomic decomposition. Using the SDP formulation in (17), (19) can be readily cast as the following SDP (up to a positive scaling factor):

$$\begin{aligned} & \min_{\boldsymbol{W},\boldsymbol{u},\widetilde{\boldsymbol{Y}}} \mathrm{tr}\left(\boldsymbol{W}\right) + \mathrm{tr}\left(T\left(\boldsymbol{u}\right)\right), \\ & \text{subject to } \begin{bmatrix} \boldsymbol{W} & \widetilde{\boldsymbol{Y}}^H \\ \widetilde{\boldsymbol{Y}} & T\left(\boldsymbol{u}\right) \end{bmatrix} \geq \boldsymbol{0}, \\ & \left\| \sum_{l=1}^{L} \boldsymbol{\Gamma}^{(l)} \widetilde{\boldsymbol{y}}_l - \boldsymbol{y} \right\| \leq \eta. \end{aligned} \tag{20}$$

Note that the only difference between (20) and (17) is in the last constraint. When the SDP is solved, the fundamental frequencies and the coefficients can be retrieved following the same procedures as in the noiseless case.

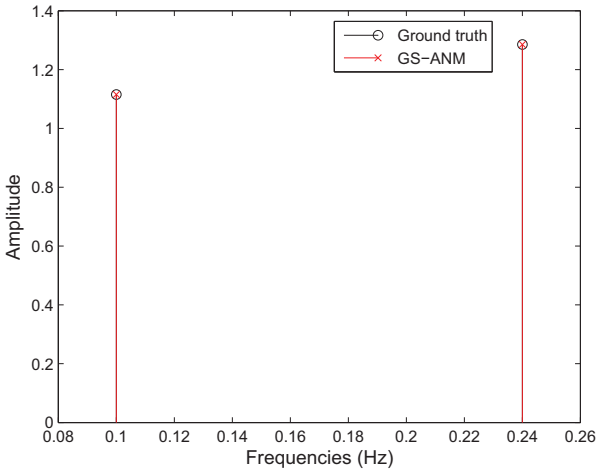

Fig. 1. Super-resolution result of the proposed atomic norm method. Fundamental frequencies and their group amplitudes (a.k.a. c_k in the atomic decomposition) are presented.

IV. Numerical Simulations

We provide numerical examples to demonstrate the performance of the proposed atomic norm method, which is referred to as GS-ANM (group sparse atomic norm minimization). For the purpose of comparison we consider the standard atomic norm method, referred to as ANM, that exploits sparsity only but does not utilize the group sparsity.

We first consider the noiseless case. In our simulation, $K = 2$ fundamental frequencies are selected as $f_1 = 0.1$ and $f_2 = 0.24$, with a number $L = 3$ of harmonics for each. The coefficients of the harmonics are randomly generated from a standard complex Gaussian distribution. $N = 20$ samples

ISBN 978-0-9928626-7-1 © EURASIP 2017

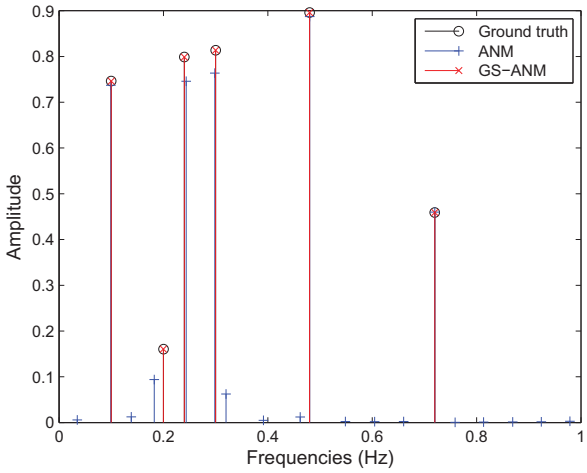

Fig. 2. Super-resolution results of the proposed and the standard atomic norm methods. All frequencies and their amplitudes are presented.

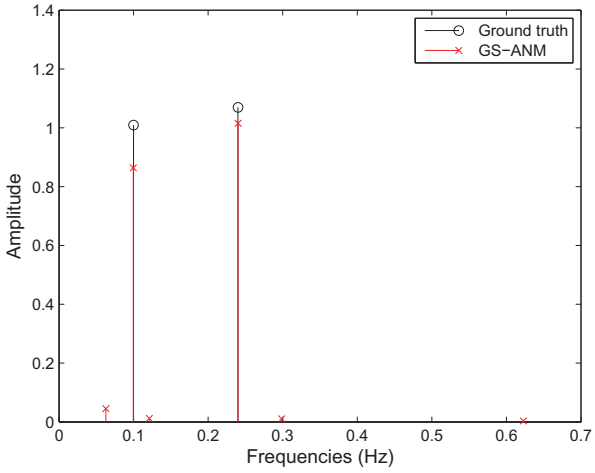

Fig. 3. Super-resolution result in the noisy case. Fundamental frequencies and their group amplitudes (a.k.a. c_k in the atomic decomposition) are presented.

are collected to super-resolve the harmonics. The estimation results are presented in Figs. 1 and 2. It can be seen from Fig. 1 that the fundamental frequencies are exactly recovered using the proposed method. In contrast to this, estimation errors exist for the standard atomic norm method. Moreover, since the group sparsity cannot be exploited in the latter, it is not evident how the estimated frequencies are harmonically related to each other.

We next consider a noisy case in which white circular complex Gaussian noise is added, resulting in a signal-to-noise ratio (SNR) of 20dB. It can be seen from Fig. 3 that the fundamental frequencies can be stably estimated from the noisy data, though small spurious peaks can be present. Compared to the standard atomic norm method, the estimation result of the proposed method is easy to interpret and appears to be closer to the ground truth (see Fig. 4).

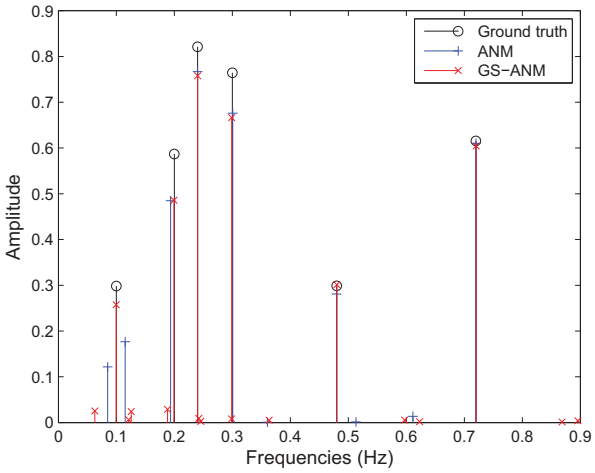

Fig. 4. Super-resolution results in the noisy case. All frequencies and their amplitudes are presented.

V. Conclusion

In this paper, the harmonics estimation problem was studied using the atomic norm, with carefully designed atoms exploiting the harmonics structure. An SDP formulation of the atomic norm was provided by relating it to the existing joint-sparse atomic norm. Its performance was illustrated via numerical simulations. While this paper makes the first step to a systematic gridless sparse approach to harmonics estimation, the proposed method can be improved in several aspects, e.g., removing the need of the model order of each harmonics group and taking into account the possible presence of non-harmonic sinusoidal waves. Besides, theoretical performance should be investigated.

Acknowledgement

The research of the project was supported by the National Natural Science Foundation of China under grant 61603187, the Natural Science Foundation of Jiangsu Province, China, under grant BK20160845, and the Foundation of Key Laboratory of System Control and Information Processing, Ministry of Education, China, under grant Scip201611.

References

[1] Z. Yang, J. Li, P. Stoica, and L. Xie, "Sparse methods for direction-of-arrival estimation," *arXiv preprint arXiv:1609.09596*, 2016.

[2] P. Stoica and R. L. Moses, *Spectral analysis of signals*. Pearson/Prentice Hall Upper Saddle River, NJ, 2005.

[3] W. Hess, *Pitch determination of speech signals: algorithms and devices*. Springer Science & Business Media, 2012, vol. 3.

[4] G. Tzanetakis and P. Cook, "Musical genre classification of audio signals," *IEEE Transactions on Speech and Audio Processing*, vol. 10, no. 5, pp. 293–302, 2002.

[5] M. Muller, D. P. Ellis, A. Klapuri, and G. Richard, "Signal processing for music analysis," *IEEE Journal of Selected Topics in Signal Processing*, vol. 5, no. 6, pp. 1088–1110, 2011.

[6] G. Singh, "Power system harmonics research: a survey," *European Transactions on Electrical Power*, vol. 19, no. 2, pp. 151–172, 2009.

ISBN 978-0-9928626-7-1 © EURASIP 2017

[7] C. Li, V. M. Lubecke, O. Boric-Lubecke, and J. Lin, "A review on recent advances in Doppler radar sensors for noncontact healthcare monitoring," *IEEE Transactions on Microwave Theory and Techniques*, vol. 61, no. 5, pp. 2046–2060, 2013.
[8] Y. Medan, E. Yair, and D. Chazan, "Super resolution pitch determination of speech signals," *IEEE Transactions on Signal Processing*, vol. 39, no. 1, pp. 40–48, 1991.
[9] S. K. Jain and S. Singh, "Harmonics estimation in emerging power system: Key issues and challenges," *Electric Power Systems Research*, vol. 81, no. 9, pp. 1754–1766, 2011.
[10] A. S. Yilmaz, A. Alkan, and M. H. Asyali, "Applications of parametric spectral estimation methods on detection of power system harmonics," *Electric Power Systems Research*, vol. 78, no. 4, pp. 683–693, 2008.
[11] C.-T. Lee, Y.-H. Yang, and H. H. Chen, "Multipitch estimation of piano music by exemplar-based sparse representation," *IEEE Transactions on Multimedia*, vol. 14, no. 3, pp. 608–618, 2012.
[12] M. Genussov and I. Cohen, "Multiple fundamental frequency estimation based on sparse representations in a structured dictionary," *Digital Signal Processing*, vol. 23, no. 1, pp. 390–400, 2013.
[13] M. Bertocco, G. Frigo, C. Narduzzi, and F. Tramarin, "Resolution enhancement by compressive sensing in power quality and phasor measurement," *IEEE Transactions on Instrumentation and Measurement*, vol. 63, no. 10, pp. 2358–2367, 2014.
[14] S. I. Adalbjörnsson, A. Jakobsson, and M. G. Christensen, "Multi-pitch estimation exploiting block sparsity," *Signal Processing*, vol. 109, pp. 236–247, 2015.
[15] L. Sun, H. Hong, Y. Li, C. Gu, F. Xi, C. Li, and X. Zhu, "Noncontact vital sign detection based on stepwise atomic norm minimization," *IEEE Signal Processing Letters*, vol. 22, no. 12, pp. 2479–2483, 2015.
[16] F. Bonavolontà, M. DApuzzo, A. Liccardo, and G. Miele, "Harmonic and interharmonic measurements through a compressed sampling approach," *Measurement*, vol. 77, pp. 1–15, 2016.
[17] V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky, "The convex geometry of linear inverse problems," *Foundations of Computational Mathematics*, vol. 12, no. 6, pp. 805–849, 2012.
[18] Z. Yang and L. Xie, "Exact joint sparse frequency recovery via optimization methods," *IEEE Transactions on Signal Processing*, vol. 64, no. 19, pp. 5145–5157, 2016.

ISBN 978-0-9928626-7-1