Solution of the Inverse Frobenius–Perron Problem for Semi–Markov Chaotic Maps via Recursive Markov State Disaggregation

Andre McDonald
Council for Scientific and Industrial Research
Brummeria, South Africa
Email: amcdonald@csir.co.za

Michael van Wyk
School of Electrical and Information Engineering
University of the Witwatersrand
Johannesburg, South Africa

***Abstract*—A novel solution of the inverse Frobenius–Perron problem for constructing semi–Markov chaotic maps with prescribed statistical properties is presented. The proposed solution uses recursive Markov state disaggregation to construct an ergodic map with a piecewise constant invariant density function that approximates an arbitrary probability distribution over a compact interval. The solution is novel in the sense that it provides greater freedom, as compared to existing analytic solutions, in specifying the autocorrelation function of the semi–Markov map during its construction. The proposed solution is demonstrated by constructing multiple chaotic maps with invariant densities that provide an increasingly accurate approximation of the asymmetric beta probability distribution over the unit interval. It is demonstrated that normalised autocorrelation functions with components having different rates of decay and which alternate in sign between consecutive delays may be specified. It is concluded that the flexibility of the proposed solution facilitates its application towards modelling of random signals in various contexts.**

I. Introduction

Signal processing systems, in general, are required to process signals that are affected to a certain extent by random phenomena. Examples of these systems include communication systems and radar receivers that process an active signal component containing unknown information, corrupted by random signals arising from noise, interference and clutter. The accurate modelling of these random signals is a requirement for developing appropriate and effective techniques for their processing. Accurate models of random signals facilitate improved statistical inference based on the observation of signal samples, as well as more accurate prediction and tracking of signals and improved control of physical systems. Hence, the development of accurate random signal models is of fundamental importance to many signal processing applications.

Chaotic dynamical systems have been used to model random signals encountered in various contexts [1]–[3]. The trajectories of these nonlinear systems (i.e. the evolution of the system state over time) appear random due to their unpredictable and erratic nature, which is a consequence of the systems' sensitivity to the initial state. Nonlinear maps with relatively simple structure may exhibit complex dynamical behaviour and hold potential for accurate random signal modelling [4].

For ergodic chaotic maps, there exists at most one state probability distribution that is invariant under repeated application of the map. The computation of the invariant probability density function (PDF) of an ergodic map is referred to as the Frobenius–Perron (FP) problem. Whereas the solution of the FP problem is relevant to the analysis of chaotic maps, the solution of the inverse FP problem, which addresses the derivation of a chaotic map with a prescribed invariant distribution, may be used as a starting point for modelling random signals.

In constructing a chaotic dynamical system to model a discrete–time random signal, it is of interest to specify both the invariant probability distribution associated with the system state as well as the time autocorrelation function (ACF) corresponding to state trajectories. It was shown that the ACF of an ergodic chaotic map may be expressed, in general, as a linear combination of weighted component functions, where the component functions are determined by the eigenvalue spectrum of the corresponding FP operator (i.e. the operator which characterises the evolution of the state's PDF over time) [5]. Whereas typical solutions to the inverse FP problem only provide a means for constructing a chaotic map with a prescribed invariant PDF, certain analytical solutions that provide limited control over the ACF have been proposed [6]–[8]. However, to the best of the authors' knowledge, no solution that facilitates the direct specification of the component functions that comprise the ACF has yet been proposed.

This paper presents a novel analytic solution of the inverse FP problem for approximating an arbitrary probability distribution over a compact interval. The proposed solution provides greater freedom than existing solutions in specifying the ACF of the chaotic map during its construction. The solution considers the construction of piecewise linear and ergodic maps that belong to the class of semi–Markov maps. It is demonstrated that the design of a semi–Markov map with prescribed invariant PDF may be formulated as a problem of designing a Markov chain, where the transition matrix of the Markov chain partly determines the statistical properties of the resulting chaotic map. The Markov chain is designed by recursively disaggregating (or 'splitting') the states of an initial

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Markov chain and by iteratively recomputing the transition matrix in a particular fashion. The advantage of using this process is its flexibility in allowing for the selection of the transition matrix eigenvalue spectrum, thereby providing a means for specifying the component functions of the map ACF.

The flexibility of the proposed solution of the inverse FP problem is demonstrated by constructing multiple semi–Markov chaotic maps with invariant PDFs that provide an increasingly accurate approximation of the asymmetric beta probability distribution over the unit interval. It is demonstrated that, through the selection of suitable eigenvalue spectra, ACFs with varying rates of decay and which consist of both positive and oscillating component functions may be realised, while maintaining the same invariant PDF.

The remainder of this paper is set out as follows. In section II, an overview of prominent solutions to the inverse FP problem for constructing semi–Markov maps, as well as those solutions that provide control over the ACF, is provided. The proposed solution is described in section III, which is followed by a presentation of simulation results in section IV. Conclusions are drawn at the end of the paper.

II. Literature review

Gora and Boyarsky [9] considered the inverse FP problem for constructing chaotic maps with prescribed invariant PDFs that are piecewise constant. A constructive proof for the existence of piecewise linear and expanding semi–Markov maps with arbitrary piecewise constant invariant PDFs was provided. The properties of piecewise linear semi–Markov maps facilitate the representation of the FP operator as a matrix, referred to as the FP matrix. The construction of the map involves the derivation of a three–band FP matrix, and the subsequent construction of the map from the FP matrix, without taking the ACF into account during the design of the map. This solution was recently generalised to the construction of M–band matrices, where $M > 3$ [10].

Rogers et al. [3] proposed a technique for constructing semi–Markov chaotic maps with prescribed invariant PDFs that are piecewise constant. This technique constructs the map directly from an N–by–N FP matrix with a predefined structure, which is fully specified by a set of $2N$ parameters. The structure of the FP matrix leads to a simplified analytic expression for the Perron eigenvector (i.e. the eigenvector associated with the unity eigenvalue), which coincides with the invariant density of the map. It is demonstrated how a suitable choice of parameters allows for the selection of an arbitrary piecewise constant PDF. The influence of parameter selection on the rate of decay of the ACFs was investigated, but the analysis is limited to positive ACFs, and no technique for specifying individual ACF component functions is provided.

Diakonos et al. [11] proposed a stochastic algorithm for generating unimodal maps with prescribed invariant PDF and ACF. Whereas this proposed technique provides a large degree of flexibility in specifying the ACF, it is computationally intensive. Furthermore, in contrast to analytic solutions of the inverse FP problem, the technique may fail to converge or produce an accurate solution.

Baranovsky and Daems [6] considered the design of chaotic maps with prescribed invariant distributions and ACFs. The technique involves the design of an initial piecewise linear map with uniform invariant PDF and a predistorted ACF. The required map is subsequently obtained via a conjugation transformation. Whereas the technique allows for the construction of initial maps with ACFs having richer properties as compared to Markov maps, the proposed technique is limited in that it only allows for the selection of conjugate map ACFs with a restricted form (i.e. the conjugate map's normalised ACF at delays $\tau \geq 1$ is necessarily equal to the initial map's normalised ACF at delay $\tau = 1$, raised to the τth power).

Nie and Coca [7], [8] proposed a technique for constructing piecewise linear semi–Markov maps that approximate the evolution of an unknown system from a sequence of PDFs generated by the system. Whereas the proposed technique is able to capture the dynamical behaviour of the system, it requires the generation of PDFs by selecting the initial state of the system, which is not possible in certain contexts.

III. Method

In this section, several definitions and preliminary results related to the chaotic maps of interest are provided. This is followed by a description of the proposed solution of the inverse FP problem.

A. Definitions and preliminary results

Consider a nonlinear map $S : \mathcal{I} \to \mathcal{I}$, where $\mathcal{I} = [a, b]$ denotes a compact interval of the real line. Let S be measurable and nonsingular with respect to the Borel σ–algebra on $\mathcal{I}$ and the normalised Lebesgue measure. Furthermore, let X_0 denote a random variable (RV) on $\mathcal{I}$ with an absolutely continuous distribution and PDF f_0. The evaluation of the map S according to the expression $X_{i+1} = S(X_i)$, for $i \in \{0, 1, \ldots\}$, produces a sequence of RVs $\{X_1, X_2, \ldots\}$ with corresponding PDFs given by $f_{i+1}(x) = \mathcal{P}_S[f_i(x)]$. In this expression, $\mathcal{P}_S$ is the Frobenius–Perron (FP) operator associated with the map S [12]. If the PDF f_i associated with the RV X_i asymptotically converges to a unique invariant PDF $f_S^*(x)$ such that $f_S^*(x) = \mathcal{P}_S[f_S^*(x)]$, then S is ergodic.

In the remainder of this paper, ergodic chaotic maps S with unique invariant densities and that belong to the class of semi–Markov maps are considered. Semi–Markov maps [9], which constitute a superset of the class of Markov maps, are defined in what follows. Let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_N\}$ denote a partition of $\mathcal{I} = [a, b]$ into N nonoverlapping intervals, such that $Q_n = [q_{n-1}, q_n)$ for $n = 1, 2, \ldots, N-1$, $Q_N = [q_{N-1}, b]$ and $q_0 = a$. A map S belongs to the class of $\mathcal{Q}$–semi–Markov maps if there exist disjoint intervals $R_j^{(n)}$ such that, for any $n = 1, 2, \ldots N$, $Q_n = \cup_{j=1}^{k(n)} R_j^{(n)}$, $S|_{R_j^{(n)}}$ is monotonic, and $S(R_j^{(n)}) \in \mathcal{Q}$. It was proved in [9] that the invariant PDF f_S^* of a piecewise linear and expanding $\mathcal{Q}$–semi–Markov map (i.e. a map where $S|_{R_j^{(n)}}$ is linear with a slope having an absolute value greater than unity, for all $n = 1, 2, \ldots N$ and $j = 1, 2, \ldots k(n)$) is piecewise constant on the intervals of $\mathcal{Q}$.

Consider the restriction of the FP operator of a piecewise linear and expanding $\mathcal{Q}$–semi–Markov map S to the space of functions constant on the intervals of $\mathcal{Q}$. Furthermore, let PDFs with domain $\mathcal{I}$ that belong to this space be represented by row vectors $\boldsymbol{f}$ of length N, such that each vector element equals the constant value of the PDF over the corresponding interval of $\mathcal{Q}$. This restriction facilitates the representation of the FP operator $\mathcal{P}_S$ as an N–by–N matrix $\mathbf{P}_S$, which is referred to as the FP matrix of S, such that $\boldsymbol{f}_{i+1} = \boldsymbol{f}_i \mathbf{P}_S$. The invariant density $\boldsymbol{f}_S^*$ corresponding to the map S is the left eigenvector of the matrix $\mathbf{P}_S$ that corresponds to the eigenvalue of unity (this follows from the expression $\boldsymbol{f}_S^* = \boldsymbol{f}_S^* \mathbf{P}_S$). The FP matrix $\mathbf{P}_S = [P_{i,j}]_{i,j=1,2,\ldots,N}$ may be derived from its corresponding $\mathcal{Q}$–semi–Markov map by setting $P_{i,j} = |(S|_{R_k^{(i)}})'|^{-1}$ if $S(R_k^{(i)}) = Q_j$, and setting $P_{i,j} = 0$ otherwise.

Consider any N–by–N stochastic matrix $\mathbf{P}$ (i.e. a matrix with elements restricted to the interval $[0, 1]$, and with rows that sum to unity). Gora and Boyarsky [9] proved the existence of a piecewise linear and expanding semi–Markov map defined over the N–interval uniform partition $\mathcal{U}$ of the interval $\mathcal{I}$, such that the FP matrix associated with the map is equal to the stochastic matrix $\mathbf{P}$. An algorithm for constructing a $\mathcal{U}$–semi–Markov chaotic map with this property is provided in [9] (proposition 1); this algorithm is a component of the proposed solution to the inverse FP problem. For the sake of brevity, this algorithm is not repeated here.

This section is concluded with the characterisation of the ACF of a piecewise linear and expanding $\mathcal{U}$–semi–Markov map with an irreducible stochastic FP matrix $\mathbf{P}_S$ that has distinct eigenvalues and is aperiodic. The normalised ACF $\phi(\tau)$, for integer $\tau > 0$, may be approximated as

$$\phi(\tau) \approx \sum_{n=1}^{N} b_n \exp[(\ln(|\lambda_n|) + i \arg(\lambda_n))\tau], \qquad (1)$$

where b_n is a function of the eigenvectors of the FP matrix $\mathbf{P}_S$ and the midpoints of the intervals in $\mathcal{U}$, λ_n denotes the nth eigenvalue of the FP matrix $\mathbf{P}_S$, and $i \triangleq \sqrt{-1}$. In general, the FP matrix may have both real and complex eigenvalues, where $\lambda_1 = 1$ and $|\lambda_n| < 1$ for all $n = 2, \ldots, N$. Eq. 1 reveals that the normalised ACF is a linear combination of oscillating and exponentially damped component functions. The rate of decay of each component is determined by the magnitude of the corresponding eigenvalue, whereas its frequency of oscillation is determined by the argument of the corresponding eigenvalue.

B. *Proposed solution of the inverse FP problem*

The proposed technique addresses the construction of a piecewise linear and expanding semi–Markov chaotic map $S : \mathcal{I} \to \mathcal{I}$ with piecewise constant invariant PDF $\boldsymbol{f}_S^*$ that is an approximation of a prescribed PDF f_R defined over the compact interval $\mathcal{I}$. This is achieved by constructing an initial piecewise linear and expanding $\mathcal{U}$–semi–Markov chaotic map $S' : \mathcal{I} \to \mathcal{I}$ with invariant PDF that is uniform over the N equal–length intervals of $\mathcal{U}$. A piecewise linear homeomorphism H is defined, where H maps the intervals in $\mathcal{Q}$ that correspond to the N intervals between the quantiles of f_R to the corresponding uniform intervals of $\mathcal{U}$. It follows that the conjugate map S, defined as $S = H^{-1} \circ S' \circ H$, is associated with an invariant PDF that approximates f_R.

The proposed construction of the initial map S' requires the derivation of a stochastic matrix $\mathbf{P}$ with an appropriate eigenvalue spectrum and a left eigenvector that is associated with the unity eigenvalue, and that corresponds to the uniform invariant density over the interval $\mathcal{I}$. The piecewise linear and expanding $\mathcal{U}$–semi–Markov chaotic map S' is subsequently constructed using the algorithm in proposition 1 of [9], such that its FP matrix $\mathbf{P}_{S'}$ equals $\mathbf{P}$ (recall that the stochastic nature of the matrix $\mathbf{P}$ ensures the existence of an appropriate semi–Markov chaotic map S'). The stochastic matrix $\mathbf{P}$ is derived via the recursive Markov state disaggregation process, which facilitates the selection of the eigenvalues of the FP matrix $\mathbf{P}_{S'}$ of the map S'. The Markov state disaggregation process, together with the definition of the homeomorphism H, are presented in what follows.

1) Recursive Markov state disaggregation: The problem of deriving an appropriate stochastic matrix $\mathbf{P}$ may be formulated as a problem of designing a Markov chain with a transition matrix equal to $\mathbf{P}$. Markov state disaggregation (MSD) [13] is used in a recursive fashion to design a Markov chain with the required transition matrix. The original MSD process of [13] defines transition probabilities in a structured manner that facilitates control over the stationary distribution of the resultant Markov chain, as well as the eigenvalue spectrum of its transition matrix. The disaggregation process that is described and used in this paper is a special case of the original process that sacrifices some flexibility in selecting the stationary distribution for greater flexibility in selecting the eigenvalue spectrum of the transition matrix, which in turn provides greater flexibility in specifying the ACF component functions of the initial chaotic map S'.

MSD is a process whereby a specified state s_k of an existing N–state Markov chain is replaced with two new states $s_{k,1}$ and $s_{k,2}$ (i.e. the original state is disaggregated), thereby producing a resultant $(N+1)$–state Markov chain. The process defines new probabilities for transitions to and from the states. The resultant Markov chain has two properties. First, the stationary probabilities of those states that were not disaggregated remain unchanged. The new states $s_{k,1}$ and $s_{k,2}$ of the resultant Markov chain have stationary probabilities that satisfy $p(s_{k,1}) = p(s_{k,2}) = p(s_k)/2$. Second, the transition matrix of the new Markov chain has the same eigenvalues as that of the original Markov chain, in addition to a new eigenvalue λ which is prescribed at the start of disaggregation. The choice of eigenvalue is restricted to a real value such that $|\lambda| \leq P_{k,k}$, where $P_{k,k}$ is the self–transition probability of s_k.

MSD is described by defining a disaggregation operator $\mathcal{G}_{k,\lambda}$ on transition matrices of Markov chains, where k denotes

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the state that is to be disaggregated, and λ denotes the eigenvalue to be inserted into the matrix. Consider a case where the Markov chain pertaining to the transition matrix

$$\mathbf{P} = \left(\begin{array}{c|c|c} \mathbf{P}_{1,1} & c_{1,k}^T & \mathbf{P}_{1,2} \\ \hline r_{k,1} & P_{k,k} & r_{k,2} \\ \hline \mathbf{P}_{2,1} & c_{2,k}^T & \mathbf{P}_{2,2} \end{array} \right) \tag{2}$$

is to undergo disaggregation. The operator $\mathcal{G}_{k,\lambda}$ produces a new transition matrix according to the expression

$$\mathcal{G}_{k,\lambda}[\mathbf{P}] = \left(\begin{array}{c|cc|c} \mathbf{P}_{1,1} & c_{1,k}^T/2 & c_{1,k}^T/2 & \mathbf{P}_{1,2} \\ \hline r_{k,1} & \multicolumn{2}{c|}{\multirow{2}{*}{\mathbf{B}}} & r_{k,2} \\ r_{k,1} & & & r_{k,2} \\ \hline \mathbf{P}_{2,1} & c_{2,k}^T/2 & c_{2,k}^T/2 & \mathbf{P}_{2,2} \end{array} \right), \tag{3}$$

where

$$\mathbf{B} \quad = \quad \frac{P_{k,k}}{h_{\max}} \begin{pmatrix} 1/2 & h_{\max} - 1/2 \\ h_{\max} - 1/2 & 1/2 \end{pmatrix} \tag{4}$$

and $h_{\max} \triangleq P_{k,k}/(P_{k,k} + \lambda)$.

A Markov chain with a uniform stationary distribution and a prescribed transition matrix eigenvalue spectrum is derived via recursive application of MSD. Starting with the elementary single–state Markov chain, the states of the Markov chain are recursively disaggregated until a predefined number of states are obtained. Disaggregation is carried out in M rounds. During round m, where $m = 1, 2, \ldots M$, each of the 2^{m-1} states of the Markov chain at the end of round $m-1$ is disaggregated, producing a Markov chain with 2^m equiprobable states at the end of round m.

2) Derivation of the homeomorphism H: A piecewise linear homeomorphism H is derived such that the invariant PDF of the conjugate map $S = H^{-1} \circ S' \circ H$ approximates a prescribed PDF f_R over the compact interval $\mathcal{I}$. Let H map the intervals of a partition $\mathcal{Q} = \{Q_1, Q_2, \ldots Q_N\}$ of the interval $\mathcal{I}$ to the corresponding intervals of the uniform partition $\mathcal{U} = \{U_1, U_2, \ldots U_N\}$ according to the expression

$$H|_{Q_n}(x) = u_{n-1} + \left[\frac{u_n - u_{n-1}}{q_n - q_{n-1}} \right] (x - q_{n-1}). \tag{5}$$

It can be shown that, by selecting the intervals of $\mathcal{Q}$ to coincide with the N intervals between the quantiles of the prescribed distribution function F_R, a piecewise linear approximation of the prescribed distribution is obtained. Specifically, the values of q_n, $n = 1, 2, \ldots N$, are selected such that $q_n = \{q : F_R(q) = n/N\}$. Furthermore, it can be shown that the eigenvalues of FP matrices $\mathbf{P}_S$ and $\mathbf{P}_{S'}$ are equal; hence, the ACF component functions of the maps S and S' are identical (the same does not hold, in general, for the component weights and the overall ACFs).

IV. Results

The proposed solution to the inverse FP problem was used to construct chaotic maps with invariant distributions that approximate the beta(α, β) distribution with PDF $f_R(x) = x^{\alpha-1}(1-x)^{(\beta-1)}/B(\alpha, \beta)$, where $x \in [0, 1]$ and $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$. In these expressions, α and β denote the two shape parameters of the beta distribution, and Γ denotes the gamma function. Values of $\alpha = 2$ and $\beta = 3$, corresponding to an asymmetric beta distribution, were considered. The PDF $f_R(x)$ is plotted in Figs. 1 and 2.

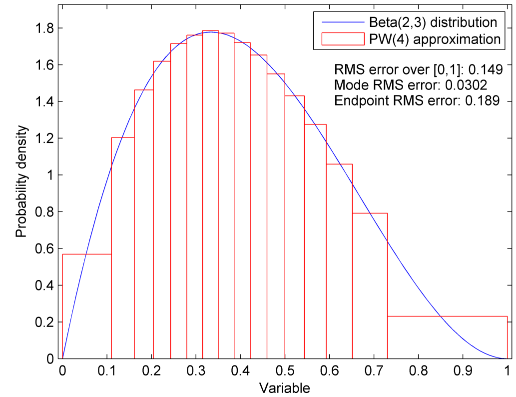

Fig. 1. PW(4) PDF approximation to the beta(2,3) distribution from set $\mathcal{V}_1$.

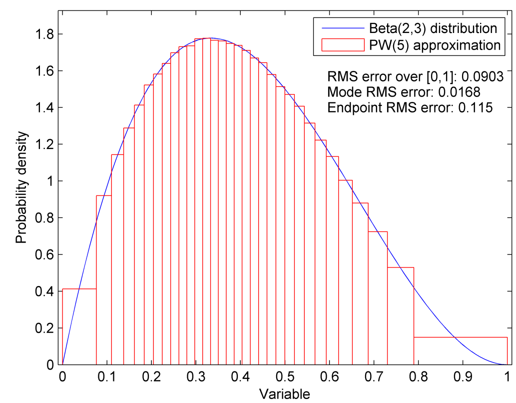

Fig. 2. PW(5) PDF approximation to the beta(2,3) distribution from set $\mathcal{V}_1$.

A set $\mathcal{V}_1$ of chaotic maps of order $M = \{3, 4, 5, 6\}$ were constructed using the proposed technique, thereby obtaining successive piecewise approximations to the beta distribution with improving accuracy (a map of order m is defined over 2^m intervals). Trajectories of 10^6 samples were generated for each map, starting with a randomly selected initial state. Figs. 1 and 2 present estimates of the piecewise constant invariant PDFs of the maps of orders 4 and 5 (denoted by PW(4) and PW(5)), as constructed from histograms. The figures reveal that the estimates provide a closer fit to the PDF of the beta distribution as the map order increases, and that the fit is closer around the mode of the distribution, as compared to the interval endpoints.

To illustrate the technique's flexibility in providing control over the ACF (and, in particular, the ability to specify the ACF component functions), three additional sets $\mathcal{V}_2$ to $\mathcal{V}_4$ of maps of order 5 were designed. The maps of these sets were designed by inserting FP matrix eigenvalues according to the expression $\lambda_{(i)} = (-1)^z \gamma P_{(i,i)}$, where $\lambda_{(i)}$ denotes the ith

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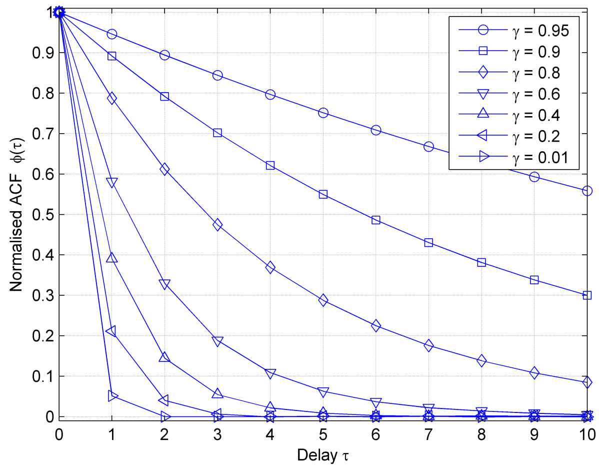

Fig. 3. ACF estimates of the PW(5) approximations to the beta(2,3) distribution from set $\mathcal{V}_2$.

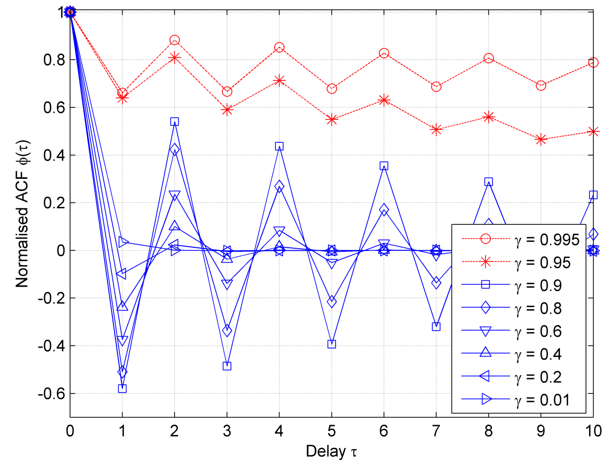

Fig. 4. ACF estimates of the PW(5) approximations to the beta(2,3) distribution from set $\mathcal{V}_3$ (solid blue lines) and $\mathcal{V}_4$ (dashed red lines).

inserted eigenvalue in a particular matrix, $P_{(i,i)}$ denotes the self–transition probability of the ith disaggregated state, and the scaling factor $\gamma \in (0, 1)$ determines the rate of decay of the corresponding ACF component (refer to eq. 1). For maps in $\mathcal{V}_2$ and $\mathcal{V}_3$, respective values of $z = 0$ and $z = i$ were selected, producing in the respective cases FP matrices with positive eigenvalues and FP matrices with both positive and negative eigenvalues. For maps in $\mathcal{V}_4$, the first two inserted eigenvalues were selected as γ and $-(\gamma - 0.1)$ respectively, in order to illustrate the effect of competing ACF components with different characteristics. The remaining eigenvalues for these maps were chosen in the same manner as for $\mathcal{V}_3$.

The empirical ACFs corresponding to the chaotic maps of $\mathcal{V}_2$ to $\mathcal{V}_4$, as obtained from the observed trajectories, are provided in Figs. 3 and 4. Fig. 3 reveals that positive ACF component functions with different rates of decay were realised. It is observed from Fig. 4 that the ACFs of maps in $\mathcal{V}_3$ alternate between positive and negative values over successive delays, due to the negative FP matrix eigenvalues. Furthermore, due to the presence of two competing components that correspond to positive and negative eigenvalues, the ACFs of maps in $\mathcal{V}_4$ oscillate around a slowly decaying mean value.

V. Conclusions

An analytic solution to the inverse FP problem for constructing piecewise linear and ergodic semi–Markov maps with invariant distributions that approximate arbitrary distributions over compact intervals was presented in this paper. The technique for constructing these maps allows for the selection of the distribution that is to be approximated, while providing greater freedom than existing techniques in specifying the ACF of trajectories generated by the map. The technique is novel in that it facilitates the selection of the ACF component functions associated with the map.

The proposed technique was demonstrated by constructing chaotic maps with piecewise constant invariant PDFs that closely approximate the asymmetric beta distribution, and with ACFs that exhibit different rates of decay and distinct behaviour, as manifested by multiple component functions with different characteristics. The accuracy and flexibility of the proposed technique allows for more accurate modelling of random signals, as encountered in signal processing systems. Improved modelling of random signals facilitates more accurate prediction and tracking of these signals. Hence, it is of importance to a wide range of signal processing applications.

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