Generalized Conditional Maximum Likelihood Estimators in the Large Sample Regime

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Abstract—**In modern array processing or spectral analysis, mostly two different signal models are considered: the conditional signal model (CSM) and the unconditional signal model. The discussed signal models are Gaussian and the signal sources parameters are connected either with the expectation value in the conditional case or with the covariance matrix in the unconditional one. We focus on the CSM resulting from several observations of partially coherent signal sources whose amplitudes undergo a Gaussian random walk between observations. In the proposed generalized CSM, the signal sources parameters become connected with both the expectation value and the covariance matrix. Even though an analytical expression of the associated generalized conditional maximum likelihood estimators (GCMLEs) can be easily exhibited, it does not allow computation of GCMLEs in the large sample regime. As a main contribution, we introduce a recursive form of the GCMLEs which allows their computation whatever the number of observations combined. This recursive form paves the way to assess the effect of partially coherent amplitudes on GCMLEs mean-squared error in the large sample regime. Interestingly, we exhibit non consistent GMLEs in the large sample regime.**

I. INTRODUCTION

In many practical problems of interest (radar, sonar, communication, ...) dealing with deterministic parameters estimation, the observations consists of a complex circular vector [1]. In this instance, one of the most studied estimation problem is that of identifying the components of a vector of observations $\mathbf{y}_1$[1] formed from a linear superposition of P signal sources $\mathbf{x}_1$ to noisy data $\mathbf{v}_1$ [2][3][4][5]

$$\mathbf{y}_1 = \mathbf{H}_1(\boldsymbol{\theta})\,\mathbf{x}_1 + \mathbf{v}_1,\ \mathbf{y}_1, \mathbf{v}_1 \in \mathbb{C}^{N_1}, \mathbf{H}_1(\boldsymbol{\theta}) \in \mathbb{C}^{N_1 \times P}, \quad \text{(1a)}$$

where the mixing matrix depends on an unknown deterministic parameter vector $\boldsymbol{\theta} \in \mathbb{R}^Q$. This problem has received considerable attention during the last fifty years, both for time series analysis [4] and array processing [5]. In the first case, one usually has to estimate the frequencies of complex sine waves from a single experiment data. In the second case, one looks for the directions of arrival (or spatial frequencies) of multiple plane waves impinging on a narrow-band array of sensors using multiple snapshots. Theses two problems have been merged into the framework of modern array processing [1][5] where mostly two different signal models are considered: the conditional signal model (CSM) and the unconditional signal model [3]. The discussed signal models are Gaussian and the angular/frequency dependency is given by parameters which are connected with the expectation value in the conditional case and with the covariance matrix in the unconditional one.

In this paper, we focus on the CSM where k $(k \geq 2)$ observations are available, that is $1 \leq l \leq k$:

$$\mathbf{y}_l = \mathbf{H}_l(\boldsymbol{\theta})\,\mathbf{x}_1 + \mathbf{v}_l,\ \mathbf{y}_l, \mathbf{v}_l \in \mathbb{C}^{N_l}, \mathbf{H}_l(\boldsymbol{\theta}) \in \mathbb{C}^{N_l \times P}, \quad \text{(1b)}$$

and the Gaussian measurement noise sequence $\{\mathbf{v}_l\}_{l=1}^{k}$ is temporally and spatially white: $\mathbf{v}_l \sim \mathcal{CN}\left(\mathbf{0}, \sigma_v^2 \mathbf{I}\right)$. In the standard CSM, a perfectly coherent signal sources scenario amounts to assume that the individual signals $\mathbf{x}_1$ remain perfectly constant during the k observations. If one concatenates the observation vectors $\mathbf{y}_l$ on a horizon of k observations from the first observation, i.e. $\overline{\mathbf{y}}_k^T = \left(\mathbf{y}_1^T, \ldots, \mathbf{y}_k^T\right)$, then one obtains the following augmented CSM:

$$\overline{\mathbf{y}}_k = \overline{\mathbf{H}}_k(\boldsymbol{\theta})\,\mathbf{x}_1 + \overline{\mathbf{v}}_k, \quad \overline{\mathbf{y}}_k \sim \mathcal{CN}\left(\overline{\mathbf{H}}_k(\boldsymbol{\theta})\,\mathbf{x}_1, \sigma_v^2 \mathbf{I}\right), \quad \text{(1c)}$$

where $\overline{\mathbf{y}}_k, \overline{\mathbf{v}}_k \in \mathbb{C}^{\mathcal{N}_k}$, $\overline{\mathbf{H}}_k(\boldsymbol{\theta}) \in \mathbb{C}^{\mathcal{N}_k \times P}$, $\mathcal{N}_k = \sum_{l=1}^{k} N_l$.

However, in a real-life experiment some experimental factors may prevent from observing perfectly coherent signal sources (see Section IV). In that perspective, we address the case where the signal sources are partially coherent because their amplitudes undergo a Gaussian random walk between observations:

$$\mathbf{x}_l = \mathbf{F}_{l-1}\mathbf{x}_{l-1} + \mathbf{w}_{l-1}, \quad 2 \leq l \leq k, \quad \text{(2a)}$$

$$\mathbf{y}_l = \mathbf{H}_l(\boldsymbol{\theta})\,\mathbf{x}_l + \mathbf{v}_l, \quad 1 \leq l \leq k, \quad \text{(2b)}$$

where $\mathbf{x}_l, \mathbf{w}_{l-1} \in \mathbb{C}^P$, $\mathbf{F}_{l-1} \in \mathbb{C}^{P \times P}$, and the fluctuation noise sequence $\{\mathbf{w}_l\}_{l=1}^{k-1}$ is Gaussian, temporally white and uncorrelated with the noise sequence $\{\mathbf{v}_l\}_{l=1}^{k}$. The Gaussian random walk (2a) introduces a more general class of CSM for which (1c) becomes:

$$\overline{\mathbf{y}}_k = \overline{\mathbf{A}}_k(\boldsymbol{\theta})\,\mathbf{x}_1 + \overline{\mathbf{n}}_k(\boldsymbol{\theta}), \qquad \overline{\mathbf{y}}_k \sim \mathcal{CN}\left(\overline{\mathbf{A}}_k(\boldsymbol{\theta})\,\mathbf{x}_1, \mathbf{C}_{\overline{\mathbf{n}}_k}(\boldsymbol{\theta})\right), \quad \text{(2c)}$$

where $\overline{\mathbf{n}}_k(\boldsymbol{\theta}) \in \mathbb{C}^{\mathcal{N}_k}$ and $\overline{\mathbf{A}}_k(\boldsymbol{\theta}) \in \mathbb{C}^{\mathcal{N}_k \times P}$ are detailed hereinafter in Section II. The most noteworthy point introduced

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[1]Hereinafter, scalars, vectors and matrices are represented, respectively, by italic, bold lowercase and bold uppercase characters. $[\mathbf{A}\ \mathbf{B}]$ and $\left[\begin{smallmatrix}\mathbf{A}\\ \mathbf{B}\end{smallmatrix}\right]$ denote the matrix resulting from the horizontal and the vertical concatenation of $\mathbf{A}$ and $\mathbf{B}$, respectively. The matrix resulting from the vertical concatenation of k matrices $\mathbf{A}_1$, ... , $\mathbf{A}_k$ of same column number is denoted $\overline{\mathbf{A}}_k$. $\mathbf{I}$ is the identity matrix. $E[\cdot]$ denotes the expectation operator. If $\mathbf{x}$ and $\mathbf{y}$ are two complex random vectors: $\mathbf{C}_\mathbf{x}$, $\mathbf{C}_\mathbf{y}$ and $\mathbf{C}_{\mathbf{x},\mathbf{y}}$ are respectively the covariance matrices of $\mathbf{x}$, of $\mathbf{y}$ and the cross-covariance matrix of $\mathbf{x}$ and $\mathbf{y}$.

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by the proposed generalized CSM (2c), is that the parameters $\boldsymbol{\theta}$ are now connected with both the expectation value and the covariance matrix, which is a significant change in comparison with the usual CSM (1c). As shown in Section II, even in the simplest case where the set of unknown parameters is restricted to $\mathbf{x}_1$ and $\boldsymbol{\theta}$, the MLEs of $\mathbf{x}_1$ and $\boldsymbol{\theta}$ based on $\overline{\mathbf{y}}_k$ (2c), so-called in the following the generalized CMLEs (GCMLEs) of $\mathbf{x}_1$ and $\boldsymbol{\theta}$, are the solution of the maximization of a risk $\mathcal{R}_k(\boldsymbol{\theta})$ (6b) involving the computation of $\mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}(\boldsymbol{\theta})$ and $|\mathbf{C}_{\overline{\mathbf{n}}_k}(\boldsymbol{\theta})|$, where $\mathbf{C}_{\overline{\mathbf{n}}_k}(\boldsymbol{\theta})$ is not block diagonal (except if $\mathbf{C}_{\mathbf{w}_l} = \mathbf{0}$, $1 \leq l \leq k-1$). Therefore, the computation of the GCMLE of $\boldsymbol{\theta}$ from the batch-form of $\mathcal{R}_k(\boldsymbol{\theta})$ becomes computationally intractable as the number of observations k increases. It is likely the reason why the study of the performance of this class of Gaussian observation model in the large sample regime ($k \to \infty$) has been somewhat overlooked in the open literature [1][5].

Interestingly enough, the observation model of interest (2a-2b) belongs to the general class of linear discrete state-space (LDSS) models [6] represented with the state (2a) and measurement (2b) equations. By exploiting some new results on linear minimum variance distortionless response (LMVDR) filters for LDSS models [7][8], we show that the GCMLEs of $\mathbf{x}_1$ and $\boldsymbol{\theta}$ can be recursively computed without the need to compute at each new observation $\mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}(\boldsymbol{\theta})$ nor $|\mathbf{C}_{\overline{\mathbf{n}}_k}(\boldsymbol{\theta})|$ where $\mathbf{C}_{\overline{\mathbf{n}}_k}(\boldsymbol{\theta})$ is a $\mathcal{N}_k$-by-$\mathcal{N}_k$ matrix, but only the inverse or the determinant of a few N_k-by-N_k matrices. The immediate benefit brought by these recursive forms is the computability of the GCMLEs of $\mathbf{x}_1$ and $\boldsymbol{\theta}$ for any value of k. A second benefit is the capability to assess their mean-squared error (MSE) in the large sample regime ($k \to \infty$), at least by Monte-Carlo simulations. The example given in Section IV exemplifies the non negligible impact of an amplitude fluctuation which introduces a lower limit in the achievable MSE of GCMLEs of $\mathbf{x}_1$ and $\boldsymbol{\theta}$. From a practical point of view, the existence of this lower limit shows that, when signal sources become partially coherent, there exists an optimal number of observations that can be combined in order to estimate their amplitudes and other unknown associated parameters with the minimum (or almost minimum) achievable MSE.

II. Analytical expression of GCMLEs

Since the Gaussian random walk (2a) of the individual signals $\mathbf{x}_1$ can be recast as, $2 \leq l \leq k$:

$$\mathbf{x}_l = \mathbf{B}_{l,1}\mathbf{x}_1 + \mathbf{G}_l\overline{\mathbf{w}}_{l-1}, \;\; \mathbf{G}_l\overline{\mathbf{w}}_{l-1} = \sum_{i=1}^{l-1} \mathbf{B}_{l,i+1}\mathbf{w}_i, \qquad (3a)$$

$$\mathbf{G}_l \in \mathbb{C}^{P\times(l-1)P}, \;\; \mathbf{B}_{l,i} = \left| \begin{array}{ll} \mathbf{F}_{l-1}\mathbf{F}_{l-2}...\mathbf{F}_i, & l > i \\ \mathbf{I} & ,l = i \\ \mathbf{0} & ,l < i \end{array} \right. ,$$

an equivalent form of (2b) is:

$$\mathbf{y}_l = \mathbf{A}_l(\boldsymbol{\theta})\mathbf{x}_1 + \mathbf{n}_l(\boldsymbol{\theta}), \quad \mathbf{A}_l(\boldsymbol{\theta}) = \mathbf{H}_l(\boldsymbol{\theta})\mathbf{B}_{l,1},$$
$$\left| \begin{array}{l} \mathbf{n}_1 = \mathbf{v}_1 \\ \mathbf{n}_l(\boldsymbol{\theta}) = \mathbf{v}_l + \mathbf{H}_l(\boldsymbol{\theta})\mathbf{G}_l\overline{\mathbf{w}}_{l-1}, \quad 2 \leq l \leq k \end{array} \right. , \qquad (3b)$$

leading to (2c):

$$\overline{\mathbf{y}}_k = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_k \end{pmatrix} = \begin{bmatrix} \mathbf{H}_1(\boldsymbol{\theta}) \\ \mathbf{A}_2(\boldsymbol{\theta}) \\ \vdots \\ \mathbf{A}_k(\boldsymbol{\theta}) \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{n}_2(\boldsymbol{\theta}) \\ \vdots \\ \mathbf{n}_k(\boldsymbol{\theta}) \end{bmatrix} = \overline{\mathbf{A}}_k(\boldsymbol{\theta})\mathbf{x}_1 + \overline{\mathbf{n}}_k(\boldsymbol{\theta}). \qquad (3c)$$

For the sake of simplicity, we assume that σ_v^2, $\{\mathbf{F}_l\}_{l=1}^{k-1}$, $\{\mathbf{C}_{\mathbf{w}_l}\}_{l=1}^{k-1}$ are known. Thus the set of unknown parameters is restricted to $\mathbf{x}_1$ and $\boldsymbol{\theta}$.

Since $\overline{\mathbf{y}}_k \sim \mathcal{CN}\left(\overline{\mathbf{A}}_k(\boldsymbol{\theta})\mathbf{x}_1, \mathbf{C}_{\overline{\mathbf{n}}_k}(\boldsymbol{\theta})\right)$ (3c), the log likelihood function is, up to a constant value, defined as [4][5][1]:

$$L(\overline{\mathbf{y}}_k; \boldsymbol{\theta}, \mathbf{x}_1) = -\ln|\mathbf{C}_{\overline{\mathbf{n}}_k}(\boldsymbol{\theta})| - \left(\overline{\mathbf{y}}_k - \overline{\mathbf{A}}_k(\boldsymbol{\theta})\mathbf{x}_1\right)^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}(\boldsymbol{\theta})\left(\overline{\mathbf{y}}_k - \overline{\mathbf{A}}_k(\boldsymbol{\theta})\mathbf{x}_1\right), \qquad (4)$$

and the GCMLEs of $\mathbf{x}_1$ and $\boldsymbol{\theta}$ based on $\overline{\mathbf{y}}_k$ are given by:

$$\left(\widehat{\mathbf{x}}_{1|k}, \widehat{\boldsymbol{\theta}}_k\right) = \arg\max_{\mathbf{x}_1,\boldsymbol{\theta}} \left\{L(\overline{\mathbf{y}}_k; \boldsymbol{\theta}, \mathbf{x}_1)\right\}. \qquad (5)$$

It is then well known [4][5][1] that $\widehat{\mathbf{x}}_{1|k} = \mathbf{x}_{1|k}\left(\widehat{\boldsymbol{\theta}}_k\right)$ where:

$$\mathbf{x}_{1|k}(\boldsymbol{\theta}) = \left(\overline{\mathbf{A}}_k^H(\boldsymbol{\theta})\mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}(\boldsymbol{\theta})\overline{\mathbf{A}}_k(\boldsymbol{\theta})\right)^{-1}\overline{\mathbf{A}}_k^H(\boldsymbol{\theta})\mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}(\boldsymbol{\theta})\overline{\mathbf{y}}_k, \qquad (6a)$$

$$\widehat{\boldsymbol{\theta}}_k = \arg\max_{\boldsymbol{\theta}} \left\{\mathcal{R}_k(\boldsymbol{\theta})\right\}, \;\; \mathcal{R}_k(\boldsymbol{\theta}) = L\left(\overline{\mathbf{y}}_k; \boldsymbol{\theta}, \mathbf{x}_{1|k}(\boldsymbol{\theta})\right) \qquad (6b)$$

or equivalently:

$$\widehat{\boldsymbol{\theta}}_k = \arg\max_{\boldsymbol{\theta}} \left\{\mathcal{I}_k(\boldsymbol{\theta}) - \mathcal{J}_k(\boldsymbol{\theta})\right\} \qquad (7a)$$

$$\mathcal{I}_k(\boldsymbol{\theta}) = \left\|\Pi_{\overline{\mathbf{A}}_k(\boldsymbol{\theta})}^{\mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}(\boldsymbol{\theta})}\overline{\mathbf{y}}_k\right\|^2_{\mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}(\boldsymbol{\theta})}, \qquad (7b)$$

$$\mathcal{J}_k(\boldsymbol{\theta}) = \ln|\mathbf{C}_{\overline{\mathbf{n}}_k}(\boldsymbol{\theta})| \qquad (7c)$$

where $\mathbf{\Pi}_{\mathbf{A}}^{\mathbf{C}} = \mathbf{A}\left(\mathbf{A}^H\mathbf{C}\mathbf{A}\right)^{-1}\mathbf{A}^H\mathbf{C}$ and $\|\mathbf{u}\|_{\mathbf{C}}$ denote, respectively, the oblique projection matrix on $span\{\mathbf{A}\}$ and the norm of vector $\mathbf{u}$ for the Hermitian inner product defined by the Hermitian positive-definite matrix $\mathbf{C}$.

According to (7a-c), the GCMLE of $\boldsymbol{\theta}$ is the solution of the maximization of a non-linear multidimensional optimization problem involving the computation of $\mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}(\boldsymbol{\theta})$ and $|\mathbf{C}_{\overline{\mathbf{n}}_k}(\boldsymbol{\theta})|$, where $\mathbf{C}_{\overline{\mathbf{n}}_k}(\boldsymbol{\theta})$ is not block diagonal (except if $\mathbf{C}_{\mathbf{w}_l} = \mathbf{0}$, $1 \leq l \leq k-1$).

As a consequence, if one resorts to a grid search approach to solve the maximization problem, for each selected value $\boldsymbol{\theta}^i$ of the grid, the evaluation of $\mathcal{I}_k\left(\boldsymbol{\theta}^i\right)$ and $\mathcal{J}_k\left(\boldsymbol{\theta}^i\right)$ request $O\left(\mathcal{N}_k^3\right)$ multiplications, where $\mathcal{N}_k = \sum_{l=1}^k N_l$, which becomes rapidly computationally prohibitive as the number of observations k increases.

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III. Recursive form of GCMLEs

In this section, we consider the computation of $\mathbf{x}_{1|k}(\boldsymbol{\theta})$ (6a) and $\{\mathcal{I}_k(\boldsymbol{\theta}), \mathcal{J}_k(\boldsymbol{\theta})\}$ (7b-7c) for a selected value $\boldsymbol{\theta}$ of the parameter space. We show that $\{\mathbf{x}_{1|k}(\boldsymbol{\theta}), \mathcal{I}_k(\boldsymbol{\theta})\}$ and $\mathcal{J}_k(\boldsymbol{\theta})$ can be computed recursively by means of two distinct recursions; a first one associated with a LMVDR filter and a second one associated with a Kalman Filter (KF).
For legibility, we omit the dependency of $\mathbf{H}_l$ and $\mathbf{n}_l$ on $\boldsymbol{\theta}$; $\mathbf{H}_l(\boldsymbol{\theta})$ and $\mathbf{n}_l(\boldsymbol{\theta})$ are simply denoted $\mathbf{H}_l$ and $\mathbf{n}_l$. The same applies to $\mathbf{C}_{\overline{\mathbf{n}}_k}(\boldsymbol{\theta})$, $\mathbf{x}_{1|k}(\boldsymbol{\theta})$, $\mathcal{I}_k(\boldsymbol{\theta})$ and $\mathcal{J}_k(\boldsymbol{\theta})$ simply denoted $\mathbf{C}_{\overline{\mathbf{n}}_k}$, $\mathbf{x}_{1|k}$, $\mathcal{I}_k$ and $\mathcal{J}_k$.

A. Recursive form of $\mathbf{x}_{1|k}(\boldsymbol{\theta})$ and $\mathcal{I}_k(\boldsymbol{\theta})$

By noticing that:

$$\Pi_{\overline{\mathbf{A}}_k}^{\mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}} \overline{\mathbf{y}}_k = \overline{\mathbf{A}}_k \left(\overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{A}}_k\right)^{-1} \overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{y}}_k = \overline{\mathbf{A}}_k \mathbf{x}_{1|k},$$

$\mathcal{I}_k$ can be rewritten as:

$$\mathcal{I}_k = \mathbf{x}_{1|k}^H \mathbf{P}_{1|k}^{-1} \mathbf{x}_{1|k}, \quad \mathbf{P}_{1|k} = \left(\overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{A}}_k\right)^{-1}. \tag{8}$$

While at first glance estimators for LDSS models may not seem to relate directly with the problem at hand, it turns out that by exploiting some new results on LMVDR filters [7][8], one can show that $\mathbf{x}_{1|k}$ and $\mathcal{I}_k$ can be recursively computed from observation to observation, without the need to compute at each new observation $\mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}$ as in (6a) and (8). The proof is obtained as follows.

Step1: one builds from (2a-b) an auxiliary LDSS model consisting of the same observations associated with an augmented state for $l \geq 2$:

$$l = 1: \quad \mathbf{y}_1 = \mathbf{H}_1 \mathbf{x}_1 + \mathbf{v}_1$$

$$l = 2: \begin{cases} \begin{pmatrix} \mathbf{x}_2 \\ \varkappa_2 \end{pmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{I} \end{bmatrix} \mathbf{x}_1 + \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{0} \end{pmatrix} \\ \mathbf{y}_2 = [\mathbf{H}_2 \ \mathbf{0}] \begin{pmatrix} \mathbf{x}_2 \\ \varkappa_2 \end{pmatrix} + \mathbf{v}_2 \end{cases}$$

$$l \geq 3: \begin{cases} \begin{pmatrix} \mathbf{x}_l \\ \varkappa_l \end{pmatrix} = \begin{bmatrix} \mathbf{F}_{l-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{x}_{l-1} \\ \varkappa_{l-1} \end{pmatrix} + \begin{pmatrix} \mathbf{w}_{l-1} \\ \mathbf{0} \end{pmatrix} \\ \mathbf{y}_l = [\mathbf{H}_l \ \mathbf{0}] \begin{pmatrix} \mathbf{x}_l \\ \varkappa_l \end{pmatrix} + \mathbf{v}_l \end{cases}$$

which can be recast as:

$$\left| \begin{array}{l} \mathbf{x}'_1 = \mathbf{x}_1 \\ \mathbf{y}_1 = \mathbf{H}'_1 \mathbf{x}'_1 + \mathbf{v}_1 \end{array} \right., \ l \geq 2: \left| \begin{array}{l} \mathbf{x}'_l = \mathbf{F}'_{l-1} \mathbf{x}'_{l-1} + \mathbf{w}'_{l-1} \\ \mathbf{y}_l = \mathbf{H}'_l \mathbf{x}'_l + \mathbf{v}_l \end{array} \right., \tag{9}$$

provided that $\mathbf{H}'_1 = \mathbf{H}_1$, $\mathbf{F}'_1 = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{I} \end{bmatrix}$, and $\forall l \geq 2$:

$$\mathbf{x}'_l = \begin{pmatrix} \mathbf{x}_l \\ \varkappa_l \end{pmatrix}, \mathbf{F}'_l = \begin{bmatrix} \mathbf{F}_l & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathbf{w}'_{l-1} = \begin{pmatrix} \mathbf{w}_{l-1} \\ \mathbf{0} \end{pmatrix}, \mathbf{H}'_l = [\mathbf{H}_l \ \mathbf{0}].$$

Then (3b) becomes:

$$\mathbf{y}_l = \mathbf{A}'_l \mathbf{x}_1 + \mathbf{n}'_l, \quad \mathbf{A}'_l = \mathbf{H}'_l \mathbf{B}'_{l,1}, \quad \left| \begin{array}{l} \mathbf{n}'_1 = \mathbf{v}_1 \\ \mathbf{n}'_l = \mathbf{v}_l + \mathbf{H}'_l \mathbf{G}'_l \overline{\mathbf{w}}'_{l-1}, \quad 2 \leq l \leq k \end{array} \right. . \tag{10}$$

Note that by definition, $\forall l \geq 1$:

$$\mathbf{B}'_{l,1} = \mathbf{F}'_{l-1} \ldots \mathbf{F}'_2 \mathbf{F}'_1 = \begin{bmatrix} \mathbf{F}_{l-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \ldots \begin{bmatrix} \mathbf{F}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{I} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{B}_{l,2} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{l,1} \\ \mathbf{I} \end{bmatrix},$$
$$\mathbf{A}'_l = \mathbf{H}'_l \mathbf{B}'_{l,1} = [\mathbf{H}_l \ \mathbf{0}] \begin{bmatrix} \mathbf{B}_{l,1} \\ \mathbf{I} \end{bmatrix} = \mathbf{H}_l \mathbf{B}_{l,1} = \mathbf{A}_l,$$

and $\forall l \geq 2$:

$$\mathbf{G}'_l \overline{\mathbf{w}}'_{l-1} = \sum_{i=1}^{l-1} \mathbf{B}'_{l,i+1} \mathbf{w}'_l = \sum_{i=1}^{l-1} \begin{bmatrix} \mathbf{B}_{l,i+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{w}_q \\ \mathbf{0} \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{i=1}^{l-1} \mathbf{B}_{l,i+1} \mathbf{w}_q \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{G}_l \overline{\mathbf{w}}_{l-1} \\ \mathbf{0} \end{pmatrix}.$$

Therefore, $\forall l \geq 2$:

$$\mathbf{n}'_l = \mathbf{v}_l + \mathbf{H}'_l \mathbf{G}'_l \overline{\mathbf{w}'}_{l-1} = \mathbf{v}_l + \mathbf{H}_l \mathbf{G}_l \overline{\mathbf{w}}_{l-1} = \mathbf{n}_l. \tag{11}$$

Step2: since $\mathbf{H}'_1 = \mathbf{H}_1$ and $\mathbf{C}_{\mathbf{v}_1}$ have full rank, if we consider the LDSS model (9), the LMVDR filter of $\mathbf{x}'_k$ exists [7][8] and is defined by[2]:

$$\overline{\mathbf{W}}_k^b = \arg\min_{\overline{\mathbf{W}}_k} \left\{\mathbf{P}_{k|k}\left(\overline{\mathbf{W}}_k\right)\right\} \text{ s.t. } \overline{\mathbf{W}}_k^H \overline{\mathbf{A}}'_k = \mathbf{B}'_{k,1}, \tag{12a}$$

where $\mathbf{P}_{k|k}\left(\overline{\mathbf{W}}_k\right) = E\left[\left(\overline{\mathbf{W}}_k^H \overline{\mathbf{y}}_k - \mathbf{x}'_k\right)\left(\overline{\mathbf{W}}_k^H \overline{\mathbf{y}}_k - \mathbf{x}'_k\right)^H\right]$, which is equivalent to [7][8]:

$$\overline{\mathbf{W}}_k^b = \arg\min_{\overline{\mathbf{W}}_k} \left\{E\left[\widehat{\mathbf{r}}_k \widehat{\mathbf{r}}_k^H\right]\right\} \text{ s.t. } \overline{\mathbf{W}}_k^H \overline{\mathbf{A}'}_k = \mathbf{B}'_{k,1},$$
$$\widehat{\mathbf{r}}_k = \overline{\mathbf{W}}_k^H \overline{\mathbf{n}}'_k - \mathbf{G}'_k \overline{\mathbf{w}'}_{k-1}, \ \overline{\mathbf{W}}_k = \left[\overline{\mathbf{W}}_k^{\mathbf{x}} \ \overline{\mathbf{W}}_k^{\varkappa}\right]. \tag{12b}$$

Since $\overline{\mathbf{W}}_k^b$ is analogous to a linearly constrained Wiener filter [9, §2.5], its batch form is given by [9, §2]:

$$\mathbf{C}_{\overline{\mathbf{n}}'_k} \overline{\mathbf{W}}_k^b = \overline{\mathbf{A}'}_k \left(\overline{\mathbf{A}'}_k^H \mathbf{C}_{\overline{\mathbf{n}}'_k}^{-1} \overline{\mathbf{A}'}_k\right)^{-1} \left(\mathbf{B}'_{k,1}\right)^H + \tag{13a}$$
$$\left(\mathbf{I} - \overline{\mathbf{A}'}_k \left(\overline{\mathbf{A}'}_k^H \mathbf{C}_{\overline{\mathbf{n}}'_k}^{-1} \overline{\mathbf{A}'}_k\right)^{-1} \overline{\mathbf{A}'}_k^H \mathbf{C}_{\overline{\mathbf{n}}'_k}^{-1}\right) \mathbf{C}_{\overline{\mathbf{n}}'_k, \mathbf{G}'_k \overline{\mathbf{w}}'_{k-1}},$$

that is, since $\overline{\mathbf{n}}'_k = \overline{\mathbf{n}}_k$ and $\overline{\mathbf{A}}'_k = \overline{\mathbf{A}}_k$:

$$\mathbf{C}_{\overline{\mathbf{n}}_k} \overline{\mathbf{W}}_k^b = \overline{\mathbf{A}}_k \left(\overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{A}}_k\right)^{-1} \left[\mathbf{B}_{k,1}^H \ \mathbf{I}\right] + \tag{13b}$$
$$\left(\mathbf{I} - \overline{\mathbf{A}}_k \left(\overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{A}}_k\right)^{-1} \overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}\right) \left[\mathbf{C}_{\overline{\mathbf{n}}_k, \mathbf{G}_k \overline{\mathbf{w}}_{k-1}} \ \mathbf{0}\right].$$

Therefore, (12a-12b) yields the following separable solutions:

$$\left(\overline{\mathbf{W}}_k^{\mathbf{x}}\right)^b = \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{A}}_k \left(\overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{A}}_k\right)^{-1} \mathbf{B}_{k,1}^H +$$
$$\mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \left(\mathbf{I} - \overline{\mathbf{A}}_k \left(\overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{A}}_k\right)^{-1} \overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}\right) \mathbf{C}_{\overline{\mathbf{n}}_k, \mathbf{G}_k \overline{\mathbf{w}}_{k-1}}$$
$$\left(\overline{\mathbf{W}}_k^{\varkappa}\right)^b = \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{A}}_k \left(\overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{A}}_k\right)^{-1}, \tag{14a}$$

[2]The superscript ^{b} is used to remind the reader that the value under consideration is the "best" one according to a given criterion.

leading to:

$$\mathbf{x}_{1|k} = \left(\overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{A}}_k\right)^{-1} \overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{y}}_k = \widehat{\varkappa}_{k|k}^b \quad (14b)$$

$$\mathbf{P}_{1|k} = \left(\overline{\mathbf{A}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\mathbf{A}}_k\right)^{-1} = E\left[\left(\widehat{\varkappa}_{k|k}^b - \widehat{\varkappa}_{k|k}\right)\left(\widehat{\varkappa}_{k|k}^b - \widehat{\varkappa}_{k|k}\right)^H\right]. \quad (14c)$$

Thus, there is a connection between the LMVDR filter of $\mathbf{x}'_k$ from (9) and $\mathbf{x}_{1|k}$ in (6a) and $\mathbf{P}_{1|k}$ in (8).

Step3: the auxiliary LDSS model (9) satisfies the usual uncorrelation conditions associated with the KF: a) the noise sequences $\{\mathbf{w}'_l\}$ and $\{\mathbf{v}_l\}$ are zero-mean, white, uncorrelated with known covariances $\mathbf{C}_{\mathbf{w}'_l}$ and $\mathbf{C}_{\mathbf{v}_l}$, b) $\mathbf{x}'_1 = \mathbf{x}_1$ is uncorrelated with $\{\mathbf{w}'_l, \mathbf{v}_l\}$. Based on these facts, the solution of (12a-12b) can also be computed recursively, since then the LMVDR filter shares the same recursion as the KF, except for the initialization [7][8].

Finally, $\mathbf{x}_{1|k}$ (6a) and $\mathcal{I}_k$ (7b)(8) can be computed recursively as follows:

$$\mathcal{I}_k = \mathbf{x}_{1|k}^H \mathbf{P}_{1|k}^{-1} \mathbf{x}_{1|k}, \quad \begin{cases} \mathbf{x}_{1|k} = [\mathbf{0}\ \mathbf{I}]\, \widehat{\mathbf{x}'}_{k|k}^b \\ \mathbf{P}_{1|k} = [\mathbf{0}\ \mathbf{I}]\, \mathbf{P}_{k|k}^b \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \end{cases}, \quad (15a)$$

where $\widehat{\mathbf{x}'}_{k|k}^b$ and $\mathbf{P}_{k|k}^b$ follow the KF recursion [7][8]:

$$\widehat{\mathbf{x}'}_{k|k}^b = \left(\mathbf{I} - \mathbf{W}_k^{bH}\mathbf{H}'_k\right)\mathbf{F}'_{k-1}\widehat{\mathbf{x}'}_{k-1|k-1}^b + \mathbf{W}_k^{bH}\mathbf{y}_k, \quad (15b)$$

$$\mathbf{P}_{k|k-1}^b = \mathbf{F}'_{k-1}\mathbf{P}_{k-1|k-1}^b\mathbf{F}'^H_{k-1} + \mathbf{C}_{\mathbf{w}'_{k-1}} \quad (15c)$$

$$\mathbf{W}_k^b = \left(\mathbf{H}'_k\mathbf{P}_{k|k-1}^b\mathbf{H}'^H_k + \mathbf{C}_{\mathbf{v}_k}\right)^{-1}\mathbf{H}'_k\mathbf{P}_{k|k-1}^b \quad (15d)$$

$$\mathbf{P}_{k|k}^b = \left(\mathbf{I} - \mathbf{W}_k^{bH}\mathbf{H}'_k\right)\mathbf{P}_{k|k-1}^b, \quad (15e)$$

except at time $k = 1$ where:

$$\mathbf{x}_{1|1} = \mathbf{P}_{1|1}^b\mathbf{H}_1^H\mathbf{C}_{\mathbf{v}_1}^{-1}\mathbf{y}_1, \ \mathbf{P}_{1|1}^b = \left(\mathbf{H}_1^H\mathbf{C}_{\mathbf{v}_1}^{-1}\mathbf{H}_1\right)^{-1}. \quad (15f)$$

B. Recursive form of $\mathcal{J}_k(\boldsymbol{\theta})$

Firstly, according to [10, 14.17]:

$$|\mathbf{C}_{\overline{\mathbf{n}}_k}| = \left|\begin{bmatrix} \mathbf{C}_{\overline{\mathbf{n}}_{k-1}} & \mathbf{C}_{\overline{\mathbf{n}}_{k-1},\mathbf{n}_k} \\ \mathbf{C}_{\overline{\mathbf{n}}_{k-1},\mathbf{n}_k}^H & \mathbf{C}_{\mathbf{n}_k} \end{bmatrix}\right| = \left|\mathbf{C}_{\mathbf{n}_k|\overline{\mathbf{n}}_{k-1}}\right|\left|\mathbf{C}_{\overline{\mathbf{n}}_{k-1}}\right|,$$

$$\mathbf{C}_{\mathbf{n}_k|\overline{\mathbf{n}}_{k-1}} = \mathbf{C}_{\mathbf{n}_k} - \mathbf{C}_{\overline{\mathbf{n}}_{k-1},\mathbf{n}_k}^H\mathbf{C}_{\overline{\mathbf{n}}_{k-1}}^{-1}\mathbf{C}_{\overline{\mathbf{n}}_{k-1},\mathbf{n}_k}.$$

Secondly, according to (2c): $\mathbf{C}_{\overline{\mathbf{n}}_k} \triangleq \mathbf{C}_{\overline{\mathbf{y}}_k}$ and $\mathbf{C}_{\mathbf{n}_k|\overline{\mathbf{n}}_{k-1}} \triangleq \mathbf{C}_{\mathbf{y}_k|\overline{\mathbf{y}}_{k-1}}$. Therefore $\mathbf{C}_{\mathbf{n}_k|\overline{\mathbf{n}}_{k-1}} \triangleq \mathbf{C}_{\mathbf{y}_k|\overline{\mathbf{y}}_{k-1}}$ can be computed by the KF recursion associated to the LDSS model resulting from the addition to (2a-2b) of the following initial state equation: $\mathbf{x}_1 = \mathbf{F}_0\mathbf{x}_0 + \mathbf{w}_0$, $\mathbf{C}_{\mathbf{x}_0} = \mathbf{0}$, $\mathbf{F}_0 = \mathbf{I}$, $\mathbf{C}_{\mathbf{w}_0} = \mathbf{0}$. Indeed then $\mathbf{C}_{\mathbf{n}_k|\overline{\mathbf{n}}_{k-1}} \triangleq \mathbf{C}_{\mathbf{y}_k|\overline{\mathbf{y}}_{k-1}} \triangleq \mathbf{S}_{k|k-1}^b$ [6] where:

$$\mathbf{P}_{k|k-1}^b = \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}^b\mathbf{F}_{k-1}^H + \mathbf{C}_{\mathbf{w}_{k-1}} \quad (16a)$$

$$\mathbf{S}_{k|k-1}^b = \mathbf{H}_k\mathbf{P}_{k|k-1}^b\mathbf{H}_k^H + \mathbf{C}_{\mathbf{v}_k} \quad (16b)$$

$$\mathbf{K}_k^b = \mathbf{P}_{k|k-1}^b\mathbf{H}_k^H\left(\mathbf{S}_{k|k-1}^b\right)^{-1}, \quad (16c)$$

$$\mathbf{P}_{k|k}^b = \left(\mathbf{I} - \mathbf{K}_k^b\mathbf{H}_k\right)\mathbf{P}_{k|k-1}^b. \quad (16d)$$

Finally $\mathcal{J}_k$ can be computed recursively as:

$$\mathcal{J}_k = \ln\left|\mathbf{S}_{k|k-1}^b\right| + \mathcal{J}_{k-1}. \quad (16e)$$

IV. An example of GCSM

A. Measurement of the backscattering coefficient of a target

Let us consider a radar system consisting of a 1-element antenna array receiving scaled, time-delayed, and Doppler-shifted echoes of a known complex bandpass signal. The antenna receives a pulse train (burst) of N pulses with a pulse repetition interval T, backscattered by a "slow" moving target [11] (no range migration during the burst). The target is assumed to have a radial motion towards the radar with an imposed constant radial speed ω and a constant aspect angle, which leads to a constant complex backscattering coefficient ρ along the trajectory. At time t_l, a simplified observation model at the output of the range matched filter is given by [11]:

$$\mathbf{y}_l = \mathbf{h}_l(\theta)\,\beta\frac{\rho}{r_l^2} + \mathbf{v}_l, \ \mathbf{h}_l^T(\theta) = \left(1, \ldots, e^{j2\pi\theta(N-1)}\right)$$

where $\theta = -2\omega T/\lambda_c$, $-0.5 \leq \theta \leq 0.5$, is the (normalized) Doppler frequency of the target, λ_c is the radar wavelength, r_l is the range of the target at time t_l, β represents the complex factor including transmission power, antenna gain and signal processing gains, and $\mathbf{v}_l$ is a temporally white thermal noise with known power σ_v^2. Indeed, in a radar system, the thermal noise power is accurately estimated from snapshots obtained while the transmitter is turned off. In order to increase the precision of the measurement of ρ, k observations are made along the trajectory. For the sake of illustration, the time t_l, $1 \leq l \leq k$, are set such that $r_l^2 = r_1^2/f^{l-1}$, which leads to the desired observation model:

$$\begin{aligned} \mathbf{y}_l &= \mathbf{h}_l(\theta)\,\beta f^{l-1}\frac{\rho}{r_1^2} + \mathbf{v}_l \\ &\Updownarrow \\ \mathbf{y}_l &= \mathbf{h}_l(\theta)\,\beta x_l + \mathbf{v}_l, \ x_1 = \frac{\rho}{r_1^2}, \ x_l = f x_{l-1}. \end{aligned} \quad (17a)$$

However, in a real-life experiment some experimental factors generally prevent from having a constant backscattering coefficient. For instance, it may be difficult for a target to keep a strictly constant radial trajectory, or fluctuation of the propagation medium are sometime unavoidable during the whole observation time interval. All these factors can be taken into account globally by introducing a random fluctuation from observation to observation, which leads to a more realistic observation model:

$$\mathbf{y}_l = \mathbf{h}_l(\theta)\,\beta x_l + \mathbf{v}_l, \ x_1 = \frac{\rho}{r_1^2}, \ x_l = f x_{l-1} + w_{l-1}. \quad (17b)$$

Secondly, due to adverse wind conditions, the true velocity of the target may differ from the desired one; hence the Doppler frequency θ must be estimated as well. In this setting, the joint estimation of (x_1, θ) in the ML sense based on k observations leads to the GCMLEs $\left(\widehat{x}_{1|k}, \widehat{\theta}_k\right)$ (6a-7a).

B. A case study

Both CMLEs of (θ, x_1) deriving from the CSM (17a) and GCMLEs of (θ, x_1) deriving from the GCSM (17b) are displayed on Fig. 1. and Fig. 2. in the following case study: $N = 10$, $\theta = 0.1$, $x_1 = (1+j)/\left(2\sqrt{2}\right)$, $\sigma_v^2 = 1$, and $f = 1.01$, which means that the range of the target changes significantly as the number of observations increases $(1 \leq k \leq 250)$. Both CMLEs and GCMLEs of (θ, x_1) are obtained via the recursive form of $\mathbf{x}_{1|k}(\theta)$, $\mathcal{I}_k(\theta)$ and $\mathcal{J}_k(\theta)$ computed over a discretization of $]-0.5, 0.5[$ with a step of $1/4096$. The empirical MSEs are assessed with 10^4 Monte-Carlo trials. In order to highlight the impact of a target random fluctuation (17b) on the MSE of the MLEs, we consider three cases with small fluctuations $\left(\sigma_{w_l}^2 = \sigma_w^2 \in \left\{10^{-5}, 10^{-4}, 10^{-3}\right\}\right)$. We also provide the conditional Cramér-Rao bound (CCRB) for θ and x_1 [12].

As expected, in the case of a pure CSM (17a), the CMLEs of θ and x_1 based on $\overline{\mathbf{y}}_k$ are asymptotically consistent and efficient in the large sample regime where the size of $\overline{\mathbf{h}}_k$, that is $N \times k$, increases indefinitely [12]. What is more surprising is the behavior of the MSE of the GCMLEs of θ and x_1 which saturates in the large sample regime. Since the recursive form of $\mathbf{x}_{1|k}(\theta)$, $\mathcal{I}_k(\theta)$ and $\mathcal{J}_k(\theta)$ only involves matrices of size $N \times N$, i.e. 10×10 in the present case study, the saturation highlighted does not result from numerical issues involved in inverse or determinant computation. The correctness of the results displayed is also supported by the fact that the CMLEs computed with the same algorithm (in the particular case where $\sigma_{w_l}^2 = \sigma_w^2 = 0$) behaves as expected [12].

Thus, Fig. 1. and Fig. 2. exemplify the impact of a target fluctuation on the MLEs performance in the large sample regime, which introduces a lower limit in the achievable MSE. Practically speaking, this lower limit shows that, when the amplitude of a target becomes partially coherent, there exists an optimal number of observations that can be combined in order to estimate its parameters with a nearly minimum achievable MSE.

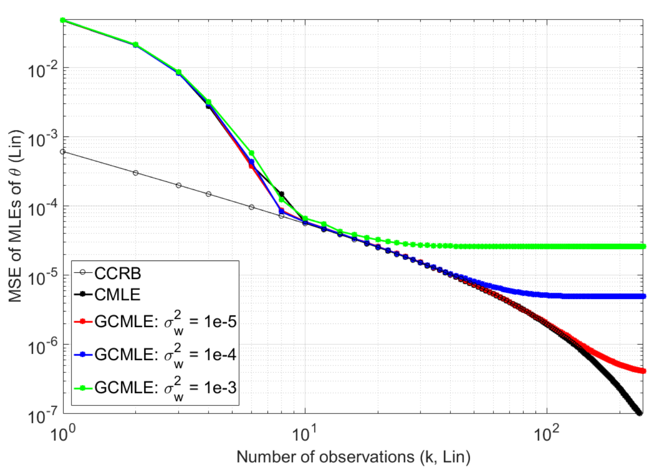

Fig. 1. MSE of the CMLE and GCMLE of θ versus k

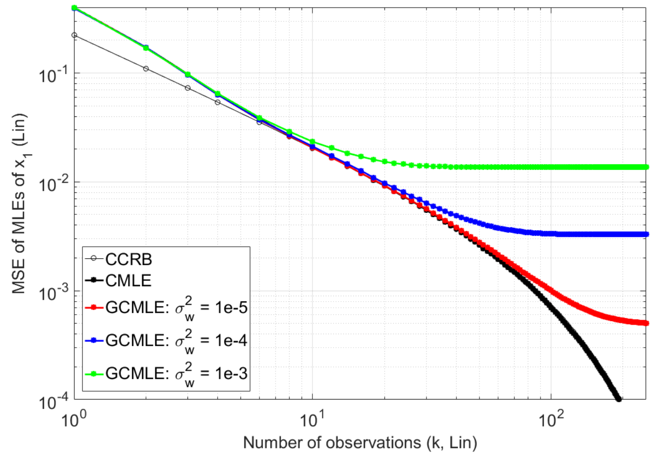

Fig. 2. MSE of the CMLE and GCMLE of x_1 versus k

V. CONCLUSION

The introduction of a numerically stable recursive form of the GCMLEs allows computation of the GCMLEs in the large sample regime and paves the way to assess their MSEs, at least by Monte-Carlo simulations. By relying on the simple example introduced, there is every reason to believe that the GCMLEs are non consistent MLEs in the large sample regime, which highlights both the consequence of partially coherent signal sources and the consequence of combining (even slightly) dependent observations. The numerical evaluation of the associated CRBs (see [5, (8.34)]) will allow to determine if the GCMLEs are efficient or not in the large sample regime. However since the CRBs depends on $\mathbf{C}_{\overline{\mathbf{n}}_k}^{-1}(\boldsymbol{\theta})$, a recursive form of the CRBs is required to obtain reliable numerical evaluations, which is the topic of on-going research.

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