A Novel Method for Sampling Bandlimited Graph Signals

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***Abstract*—In this paper we propose a novel vertex based sampling method for k-bandlimited signals lying on arbitrary graphs, that has a reasonable computational complexity and results in low reconstruction error. Our goal is to find the smallest set of vertices that can guarantee a perfect reconstruction of any k-bandlimited signal on any connected graph. We propose to iteratively search for the vertices that yield the minimum reconstruction error, by minimizing the maximum eigenvalue of the error covariance matrix using a linear solver. We compare the performance of our method with state-of-the-art sampling strategies and random sampling on graphs. Experimental results show that our method successfully computes the smallest sample sets on arbitrary graphs without any parameter tuning. It provides a small reconstruction error, and is robust to noise.**

***Index Terms*—Graph signal processing, sampling, spectral graph theory**

I. Introduction

Graphs provide a natural way of representing signals lying on arbitrary domains, and are of practical use in many applications involving social, biological, sensor networks, large-scale data and machine learning [1], [2]. The field of graph signal processing provides means to analyze graph structures and connectivity information, and extends classical signal processing tools such as translation, spectral analysis and downsampling onto irregular domains. Sampling of graph vertices requires a different characterization than the traditional Nyquist-Shannon sampling theorem, as there is no well defined notion of every other vertex on graphs, for example. Finding an optimal set of samples that allows a perfect reconstruction of graph signals is therefore still an open problem of graph signal processing.

In this paper, we propose a novel sampling algorithm for k-bandlimited signals in Paley-Wiener spaces on arbitrary graphs. We use the spectrum of the graph to select the optimal set of vertices that has the minimum number of samples required to perfectly reconstruct the graph signal. The main challenge in our work, as in other state-of-the-art methods, is to find a good trade-off between computational efficiency and high reconstruction quality. Our method computes the optimal sampling set of smallest size with minimum reconstruction error compared to state-of-the-art methods and is robust also in the presence of noise. There is a beneficial trade-off between the computational complexity of our sampling algorithm and the accuracy of reconstruction from samples as well. Our method results in less reconstruction error compared to algorithms with lower computational complexity, and has lower computational complexity than algorithms that have comparable reconstruction errors.

The fundamentals of sampling k-bandlimited signals in Paley-Wiener spaces on arbitrary graphs are described in [3], [4], where a sufficient condition that guarantees perfect reconstruction from a set of nodes is presented. This sufficient condition, however, is not necessary, implying that sampling set dimensions may exceed the theoretical lower bound. A solution to this problem is proposed in [5]–[8] by showing that a set of size k always exists to perfectly reconstruct a k-bandlimited graph signal. The sampling set selection algorithm presented in [7] requires a singular value decomposition (SVD) solver, which increases system complexity. Improvements are presented in [9] by introducing an algorithm that relies on *graph spectral proxies* to approximate the frequency bound of graph signals and predict the optimal sampling set by maximizing this bound. The method proposed in [9] avoids the computation of frequency decomposition; however, the performance depends on a hyperparameter that has to be tuned according to the problem settings. A similar approach is presented in [10] that requires higher computational complexity, but without the necessity of parameter tuning.

A study of the uncertainty principle for graph signals was developed in [11], [12], where sampling strategies to improve recovery performances of aforementioned works are also presented in the latter. Other works on downsampling graph signals rely on vertex-domain characteristics [13], [14] and multiscale approaches [15], but are less efficient in terms of finding the optimal set of nodes for bandlimited reconstruction. Authors approximate the optimal sampling set in [16] again using a vertex-based method, however the algorithm does not scale well with large sized graphs. The randomized sampling strategy in [17] also tackles the problem of sampling k-bandlimited signals on graphs of very large size with relatively low computational complexity, but by surpassing the theoretical minimum number of samples. Compared to state-of-the-art sampling strategies, our method is able to achieve a better trade-off between computational complexity and finding the optimal sampling set of smallest size with minimum reconstruction error.

The outline of the paper is as follows. We introduce the notation and theoretical background on sampling for

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k-bandlimited graph signals in Section II. In Sections III and IV we explain our sampling strategy and algorithm, and our reconstruction method, respectively. We validate the performance of our approach in Section V and present our conclusions in Section VI.

II. Notation and Background

A graph is denoted as $\mathcal{G} = (\mathcal{V}, E)$ where $\mathcal{V}$ is the set of N vertices and E are the edges in between. The edge connecting vertices i and j has a weight w_{ij} that characterizes the strength of connection between the nodes, where $w_{ij} \in [0, 1] \subset \mathbb{R}\, \forall i, j$. The weights w_{ij} are stored in the $N \times N$ adjacency matrix W. We denote the graph Laplacian $L = D - W$ where D is the diagonal degree matrix with $d_{ii} = \sum_{j \in \mathcal{V}} w_{ij}$. The Laplacian L is a symmetric positive semi-definite matrix, with a set of orthonormal eigenvalues $q_1, ..., q_N$ associated with eigenvalues $\lambda_1 \leq ... \leq \lambda_N$. Any signal f lying on the graph can be represented as a linear combination of the Laplacian eigenvectors, as these eigenvectors form an orthonormal basis referred as the graph Fourier transform (GFT) [2]. The eigenvalues $\lambda_1 \leq ... \leq \lambda_N$ carry the notion of frequency for graph signals. A signal of bandwidth $\omega = \lambda_n$ is therefore a linear combination of only the first n eigenvectors of L. The space of λ_n-bandlimited signals is called a Paley-Wiener space, and is denoted as $PW_{\lambda_n}(\mathcal{G}) \subset \mathbb{R}^N$.

Definition 2.1 (Uniqueness set): A subset of vertices $\mathcal{S} \subset \mathcal{V}$ is a uniqueness set [3] for signals in $PW_\omega(\mathcal{G})$ if $\forall f, g \in PW_\omega(\mathcal{G})$, $f(\mathcal{S}) = g(\mathcal{S}) \rightarrow f = g$.

Definition 2.1 implies that for any signal $f \in PW_\omega(\mathcal{G})$ can be reconstructed from its samples provided that the sample set is a uniqueness set $\mathcal{S}$ for signals in $PW_\omega(\mathcal{G})$. This definition gives way to the following lemma and theorem [5]:

Lemma 2.1: A set of nodes $\mathcal{S}$ is a uniqueness set for signals in $PW_\omega(\mathcal{G})$ if and only if $PW_\omega(\mathcal{G}) \cap L_2(\mathcal{S}^c) = \{0\}$ with $L_2(\mathcal{S}^c)$ being the space of all signals ϕ that are zero in $\mathcal{S}$ and nonzero on at least one node of $\mathcal{S}^c$.

Theorem 2.1: A bandlimited signal $f \in PW_\omega(\mathcal{G})$ can be perfectly reconstructed from its samples in $\mathcal{S}$ if and only if $\omega < \inf_{\phi \in L_2(\mathcal{S}^c)} \omega(\phi) \triangleq \omega_c(\mathcal{S})$ where ϕ is a signal on $\mathcal{G}$ with a bandwidth of $\omega(\phi)$.

These observations then lead to the following corollary and proposition [9]:

Corollary 2.1: A set of vertices $\mathcal{S}$ is a uniqueness set for all signals $f \in PW_{\lambda_n}(\mathcal{G})$ if and only if $q_1(\mathcal{S}), ..., q_n(\mathcal{S})$ are linearly independent, where λ_n is the n^{th} smallest eigenvalue of L and $q_i(\mathcal{S})$ is the reduced eigenvector that corresponds to the i^{th} smallest eigenvalue. The term reduced implies that $q_i(\mathcal{S}) \in \mathbb{R}^{|\mathcal{S}|}$ with rows corresponding to the indices of the sampling set $\mathcal{S}$.

Proposition 2.1: For any frequency λ_n, the smallest uniqueness set $\mathcal{S}_{opt}$ for signals $f \in PW_{\lambda_n}(\mathcal{G})$ has a size $|\mathcal{S}_{opt}| = n$. We will refer to such sets as minimum uniqueness set.

Corollary 2.1 gives us a guideline to compute any uniqueness set for signals $f \in PW_{\lambda_n}(\mathcal{G})$ while Proposition 2.1 indicates that the smallest sampling set for such signals have a dimension of n. Equipped with all aforementioned statements, we introduce a novel proposition that allows to construct minimum uniqueness sets iteratively:

Proposition 2.2: For any minimum uniqueness set $\mathcal{S}$ of size n for signals in $PW_{\lambda_n}(\mathcal{G})$, there is always at least one node $\mathcal{S}_i \notin \mathcal{S}$ such that $\mathcal{S} \cup \mathcal{S}_i$ is a uniqueness set of size $n+1$ for signals in $PW_{\lambda_{n+1}}(\mathcal{G})$.

Using Corollary 2.1 and Propositions 2.1 and 2.2, we now move forward with our algorithm to solve our sampling problem on graphs.

III. Proposed Sampling Method

For a given bandlimit frequency λ_n we would like to find the minimum uniqueness set $\mathcal{S}_{opt}$ for the space $PW_{\lambda_n}(\mathcal{G})$ that guarantees a perfect reconstruction for any signal $f \in PW_{\lambda_n}(\mathcal{G})$. Proposition 2.1 points out that there is a correspondence between the size of the minimum uniqueness set $\mathcal{S}_{opt}$ for signals in $PW_{\lambda_n}(\mathcal{G})$ and their bandwidth λ_n, i.e. $|\mathcal{S}_{opt}| = n$. Corollary 2.1 states that in order to find the vertices that belong to the minimum uniqueness set $\mathcal{S}_{opt}$, we have to find n linearly independent rows from the matrix composed of the first n eigenvectors of L. From Proposition 2.2, we see that we can iteratively compute a minimum uniqueness set $\mathcal{S}$ of any size by adding a sample to a previously computed minimum uniqueness set, provided that Corollary 2.1 holds for the new minimum uniqueness set.

Algorithm 1: Sampling algorithm

Input : Sampling set size n, first n eigenvectors of L, i.e. Q_n
Output : Minimum uniqueness set $\mathcal{S}_{opt}$
Initialize: $\mathcal{S} = \emptyset$

$\mathcal{S} \leftarrow \mathcal{V}_i$, where i is the index of any nonzero element of first eigenvector q_1
for $m = 2$ *to* n **do**
  create matrix $Q_m(\mathcal{S})$
  compute $x = \text{null}(Q_m(\mathcal{S}))$
  compute $b = Q_m(\mathcal{S}^c)x$
  $i \leftarrow \arg\max_i |b(i)|$
  $\mathcal{S} \leftarrow \mathcal{S} \cup \mathcal{S}^c(i)$
end
$\mathcal{S}_{opt} \leftarrow \mathcal{S}$

We denote Q_n as the $N \times n$ matrix composed of the first n eigenvectors of L as its columns arranged according to increasing eigenvalues. Our goal is to find n linearly independent rows of Q_n, where the independent row indices correspond to the indices of the nodes within the minimum uniqueness set. We start with an empty set and at each iteration add one node to our sampling set $\mathcal{S}$ such that it always remains a minimum uniqueness set. The first node to add

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to $\mathcal{S}$ is the one with the row index corresponding to any nonzero element of the first eigenvector of L. At iteration $m < n$, we have computed a subset of the required minimum uniqueness set $\mathcal{S}_m$ that is a minimum uniqueness set for the space $PW_{\lambda_m}(\mathcal{G})$. At iteration $m+1$ we add the node $\mathcal{V}_i$ to $\mathcal{S}_m$ such that $\mathcal{S}_{m+1} = \mathcal{S}_m \cup \mathcal{V}_i$ is a minimum uniqueness set for signals in $PW_{\lambda_{m+1}}(\mathcal{G})$. We adopt the notation $Q_k(\mathcal{S})$ to denote any $|\mathcal{S}| \times k$ matrix that is composed of the first k *reduced* eigenvectors of L, i.e. $Q_k(\mathcal{S}) = [q_1(\mathcal{S}) \quad ... \quad q_k(\mathcal{S})]$. Then, we create the $m \times (m+1)$ matrix $Q_{m+1}(\mathcal{S}_m)$, which has a rank equal to m since $Q_m(\mathcal{S}_m)$ has full rank. The nullspace of $Q_{m+1}(\mathcal{S}_m)$ gives us the unique unit vector x that is orthogonal to all rows of $Q_{m+1}(\mathcal{S}_m)$. We then build the $|\mathcal{S}_m^c| \times (m+1)$ matrix $Q_{m+1}(\mathcal{S}_m^c)$ and we look for the row that is the most linearly dependent to x. The index of this row, i, corresponds to the absolute maximum element index of the product vector $b = Q_{m+1}(\mathcal{S}_m^c)x$, and the i^{th} row of $Q_{m+1}(\mathcal{S}_m^c)$ is linearly independent from all rows of $Q_{m+1}(\mathcal{S}_m)$. It must be noted that normalization of the rows of $Q_{m+1}(\mathcal{S}_m^c)$ prior to computation of $b = Q_{m+1}(\mathcal{S}_m^c)x$ improves the performance of our algorithm. We then add the vertex $\mathcal{S}_m^c(i)$ to set $\mathcal{S}_m$ and obtain $\mathcal{S}_{m+1}$. Since the matrix $Q_{m+1}(\mathcal{S}_{m+1})$ is now of rank $m+1$, the set $\mathcal{S}_{m+1}$ is a minimum uniqueness set for the space $PW_{\lambda_{m+1}}(\mathcal{G})$ following Corollary 2.1 and Proposition 2.1. Also from Proposition 2.2 we know that we will always find at least one row of $Q_{m+1}(\mathcal{S}_m^c)$ that is not orthogonal to x. Therefore we continue adding nodes to our sampling set until we reach $|\mathcal{S}| = n$.

IV. Reconstruction

After we build the minimum uniqueness set, we try to reconstruct the bandlimited signal f from its samples $f(\mathcal{S}_{opt})$ using the least squares reconstruction method described in [18]. This method can be used for any uniqueness set $\mathcal{S}$ for signals in a given $PW_{\lambda_n}(\mathcal{G})$ and has the form:

$$f(\mathcal{S}^c) = Q_n(\mathcal{S}^c)(Q_n^T(\mathcal{S})Q_n(\mathcal{S}))^{-1}(Q_n^T(\mathcal{S})f(\mathcal{S})) \quad (1)$$

The formulation in (1) can be further simplified for our case with $\mathcal{S} = \mathcal{S}_{opt}$. Since $Q_n(\mathcal{S}_{opt})$ is a square invertible matrix, we have:

$$f(\mathcal{S}_{opt}^c) = Q_n(\mathcal{S}_{opt}^c)Q_n^{-1}(\mathcal{S}_{opt})f(\mathcal{S}_{opt}) \quad (2)$$

We express the reconstructed signal as $\hat{f} = f(\mathcal{S}_{opt}) \cup f(\mathcal{S}_{opt}^c)$.

V. Experiments and Results

We now evaluate the performance of the proposed algorithm by comparing the reconstruction errors of bandlimited signals on different graphs, using our method and other sampling set selection algorithms. Namely, we compare our algorithm (M1) with the vertex sampling algorithm in [10] (M2), the vertex sampling algorithm in [9] (M3) with parameters $k = 2, 10, 50$, random sampling (M4), the probabilistic sampling approach in [17] (M5) and the vertex sampling framework in [7] (M6). We have generated different types of graphs each with $N = 300$ nodes using [19] as listed below:

$\mathcal{G}_1$: Erdös-Renyi random graph (unweighted), connection probability 0.5.

$\mathcal{G}_2$: Erdös-Renyi random graph (unweighted), connection probability 0.05.

$\mathcal{G}_3$: 6-nearest neighbor random sensor graph.

$\mathcal{G}_4$: 290-nearest neighbor random sensor graph.

A. Sampling random signals of a single bandwidth

We generated random bandlimited graph signals of fixed bandwidth to sample and reconstruct, with and without additive noise, as described below:

f_{11}: Noise free signal with $n = \dim PW_\omega(\mathcal{G}) = 140$. Nonzero GFT coefficients are randomly generated from distribution $\mathcal{N}(1, 0.5^2)$.

f_{21}: Signal f_{11} is contaminated by noise resulting in 20dB SNR, where the noise has zero mean and GFT coefficients of noise are randomly generated from standard uniform distribution on the open interval (0,1). The additive noise is normalized to have unit norm.

We then sampled these signals and tried to reconstruct the original signal f_{11} from samples. We report the results in terms of the logarithm of mean squared reconstruction error, i.e., log(MSE), with respect to the number of samples used for each method. Each experiment is repeated 10 times using random signals and the average results are depicted in Fig.1 for the noise free and noisy cases.

We see from Fig.1 that our method and method M6 yield the lowest reconstruction error for number of samples equal to the signal bandwidth, and reach a stable reconstruction error earlier than others on all types of graphs tested. The performance of our algorithm and M6 are nearly indifferent and are both superior to other tested methods in terms of reconstruction error. Our method, however, has an overall lower computational complexity compared to M6. To sample $|\mathcal{S}|$ nodes from a graph of N nodes, both our method and M6 have to find and store the first $|\mathcal{S}|$ eigenvectors. The complexities of sampling set search algorithms of our method and M6 are $\mathcal{O}(|\mathcal{S}|^4 + N|\mathcal{S}|^2)$ and $\mathcal{O}(N|\mathcal{S}|^4)$, respectively. M6 has a greedy sampling set search algorithm that requires a singular value decomposition in each iteration. Our method is computationally more efficient compared to M6 for searching for the sampling set, especially with large graphs.

The bottom row of Fig. 1 shows the reconstruction error for the noisy signal f_{21} on all graphs for each tested method. We see that our sampling algorithm is robust against noise. Similar to the noise free case, the reconstruction error of our method converges to a stable minimum earlier than other methods except for M6 on $\mathcal{G}_3$ and $\mathcal{G}_4$. The performance of our method and M6 are again comparable on all graphs, with a slightly lower reconstruction error for M6 than our method when the number of samples is less than or equal to the signal bandwidth on $\mathcal{G}_2$ and $\mathcal{G}_4$.

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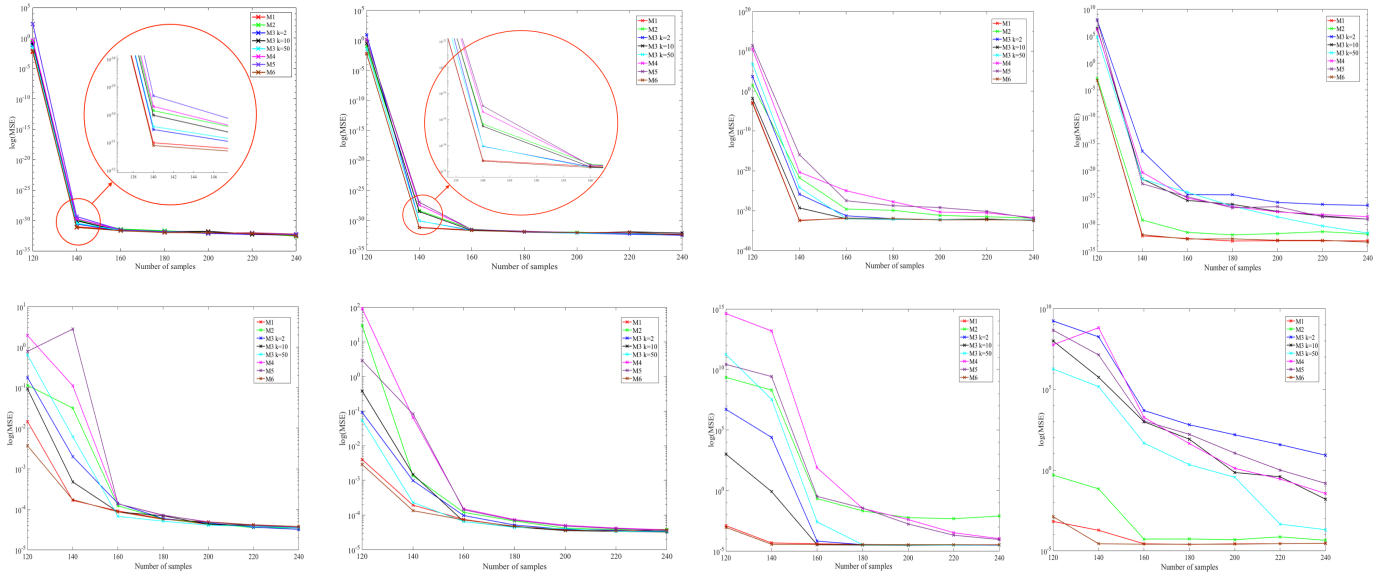
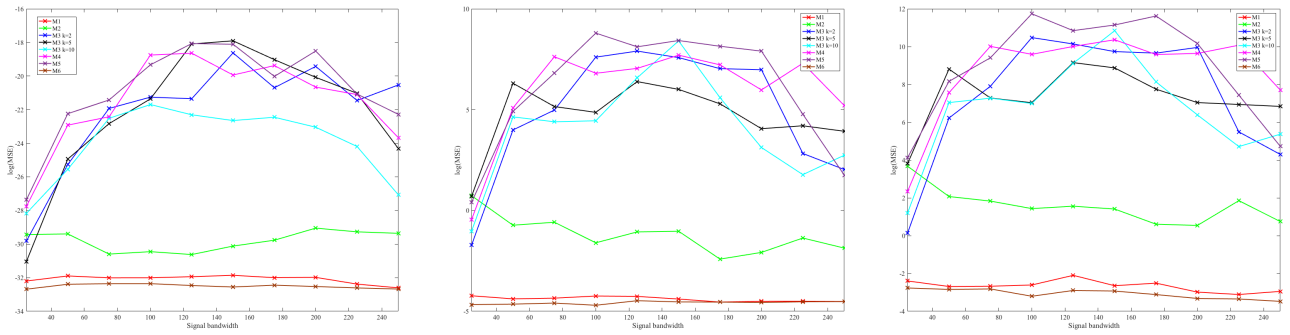


(a) (b) (c) (d)

Fig. 1: Reconstruction MSE in log-scale vs. number of samples on (a) $\mathcal{G}_1$, (b) $\mathcal{G}_2$, (c) $\mathcal{G}_3$, (d) $\mathcal{G}_4$ for noise free signal f_{11} (top row) and noisy signal f_{12} (bottom row).



(a) (b) (c)

Fig. 2: Reconstruction MSE in log-scale vs. bandwidths of graph signals for noise free signal (a), noisy signal (b) and theoretical bound for log(MSE) (c) on $\mathcal{G}_4$.

B. Sampling random signals of multiple bandwidths

We then generated random bandlimited graph signals of varying bandwidths to sample and reconstruct, with and without additive noise, as described below:

f_{12_n}: Noise free signal with $n \in [25, 250]$. Nonzero GFT coefficients are randomly generated from distribution $\mathcal{N}(1, 0.5^2)$.

f_{22_n}: Signal f_{12_n} is contaminated by noise resulting in 20dB SNR, where the noise has zero mean and GFT coefficients of noise are randomly generated from standard uniform distribution on the open interval (0,1). The additive noise is normalized to have unit norm.

We then sampled these signals using numbers of samples equal to the bandwidth of f_{12_n} for each n and tried to reconstruct the original signal f_{12_n} using the samples. Each experiment is again averaged over 10 runs and results on $\mathcal{G}_4$, which has been arbitrarily chosen for demonstration of results, are depicted in Fig. 2.

In Fig. 2, (a) and (b) we present the reconstruction MSE in log-scale vs. the bandwidths of noise free and noisy graph signals we have reconstructed in our experiments, respectively, on $\mathcal{G}_4$. Our method performs well for signals of all tested bandwidths on $\mathcal{G}_4$, and is again robust to noise. Moreover, our results and also the results of M6 are more stable compared to others. For the signals contaminated by additive noise of unit norm, we can compute a theoretical upper bound for MSE, that corresponds to the maximum eigenvalue of the error covariance matrix in our method so that the reconstruction

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error is minimized. The theoretical upper bounds for MSE for each noisy signal f_{22_n} of different bandwidths are shown on Fig. 2(c). Our method and M6 both have relatively small and steady theoretical upper bounds for MSE, while for other methods this bound is fluctuating, as shown in the bottom row of Fig. 2. In terms of computational complexity, M3 does not require a full eigendecomposition of the graph Laplacian, unlike M2, but the complexity increases as the value of k is increased. Our method does not have this trade-off as it does not require any parameter tuning. Besides the random sampling method M4, method M5 has the best computational efficiency since it does not require any eigendecomposition of the graph Laplacian. Although our method needs the first n eigenvectors of L, our reconstruction errors are significantly smaller compared to M4 and M5 throughout the experiments. The theoretical error bound and experimental errors of our method are very similar to those of M6, and in most cases our method yields to slightly more reconstruction error than M6 for all signals of different bandwidths on all graphs. The experimental errors between our method and M6 differ by a maximum of 0.46dB on $\mathcal{G}_4$, averaged over all tested signals with different bandwidths. However, our method is computationally more efficient than M6 in all of our experiments. The trade-off between our method and M6 is small in terms of the reconstruction error, but much larger in our method's advantage in terms of computational complexity.

VI. Conclusion

In this work we presented a sampling set selection method for k-bandlimited graph signals lying on any type of connected graph. Our method finds the optimal set of vertices that guarantees a perfect reconstruction of the signal, does not require parameter tuning and needs to compute and store only the first k eigenvectors of the graph Laplacian. We are able to find a small and stable theoretical error bound for the reconstruction of noisy signals of different bandwidths. Experimental results over different graphs and signals of different bandwidths show that our method leads to reconstruction error comparable to state-of-the-art methods while being robust to noise. Our method has the advantage of having considerably lower computational efficiency compared to methods that yield slightly less error with much higher computational complexity. Future work consists of lowering the overall computational complexity of our method further by using an efficient approximation for our eigendecomposition computations.

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