

Iterative Equalization Based on Expectation Propagation: a Frequency Domain Approach

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Abstract—A novel single-carrier frequency domain equalization (SC-FDE) scheme is proposed, by extending recent developments in iterative receivers that use *expectation propagation* (EP), a message passing formalism for approximate Bayesian inference. Applying EP on the family of *white* multivariate Gaussian distributions allows deriving *double-loop* frequency domain receivers with single-tap equalizers. This structure enables low computational complexity implementation via Fast-Fourier transforms (FFT). *Self-iterations* between the equalizer and the demapper and *global turbo iterations* with the channel decoder provide numerous combinations for the performance and complexity trade-off. Numerical results show that the proposed structure outperforms alternative single-tap FDEs and that its achievable rates reach the channel symmetric information rate.

I. INTRODUCTION

Frequency domain equalization for single-carrier modulation is a key technique for wireless communication in quasi-static wideband channels. It enables low complexity equalization of single-carrier (SC) or single-carrier frequency division multiple access (SC-FDMA) waveforms, which are widely used today. Examples of current and future applications are the Long Term Evolution (LTE) uplink, device-to-device and vehicle-to-vehicle communications in 4G 3GPP specification and its evolutions [1].

A long research track exists on Frequency Domain (FD) equalizers, starting from very low-complexity linear FD equalizers up to FD turbo equalizers of different kinds. In particular, FD turbo linear equalizer - interference canceller (FD LE-IC) still remains an important reference for advanced non-linear FDE, with its ability to reach the channel symmetric information rate at low spectral efficiency operating points, by using either extrinsic (EXT) [2] or a posteriori probability (APP) [3] soft feedback from the decoder. In 2015, a turbo extension of a non-linear FDE with FD feedback filter, called iterative block DFE (IBDFE) [4], was proposed by [5] which we call hereafter *self-iterated* turbo LE-IC (SILE-IC). There, an initial interference cancellation is carried out with EXT feedback from the decoder, then a second round of IC is carried out with APP feedback from the detector.

Recently, new ideas from the field of artificial intelligence, which were used for solving classification or density estimation problems, fertilized the equalization research community. This paper focuses on expectation propagation (EP), a technique for approximate Bayesian inference [6], which has already been applied to channel decoding [7] or for self-iterated MIMO detection [8] among others.

EP has been also used in the equalization domain for deriving a self-iterated time-domain turbo block LE in [9]. More recently in [10] EP is applied to the design of time-varying finite impulse response equalizers. A single-tap FD receiver based on EP was derived in [11], however, it does not include self-iterations and, due to a zero-forcing type derivation, it is severely impacted by spectral nulls. The major benefit EP brings to equalization is an alternative soft “decision” which corresponds to the *extrinsic feedback* passed from the demapper to the equalizer, which helps approaching optimal maximum a posteriori (MAP) detector’s performance.

This paper’s main contribution ensues from an EP-based message passing framework, where transmitted symbols are assumed to belong to the multivariate *white* Gaussian distribution family, which has never been used, to the authors’ knowledge, for digital receiver design. Such an approach allows deriving a novel class of single-tap FD receivers with quasi-linear computational complexity, unlike block receivers obtained from conventional EP, with general multivariate Gaussian models, which have quadratic complexity dependence on the block length. The benefits of this method are illustrated by deriving a novel frequency-domain self-iterated turbo linear equalizer using interference cancellation with EP feedback (FD SILE-EPIC). This FDE is shown to outperform other single-tap FDE with similar complexity, while being able to match the performance of exact time domain EP receivers.

The remainder of this paper is organized as follows. The system model is given in section II. The proposed receiver is derived in section III, with numerical results in section IV.

II. SYSTEM MODEL

In the following, SC transmission of a block of K symbols is considered using a bit-interleaved coded modulation (BICM) scheme. Extension to the SC-FDMA framework is straightforward. SC is considered for ease of presentation and to simplify the notations. Thus, an information block $\mathbf{b} \in \mathbb{F}_2^{K_b}$ is first encoded and then interleaved into a codeword $\mathbf{d} \in \mathbb{F}_2^{K_d}$, where \mathbb{F}_2 denotes the binary Galois field. A memoryless modulator φ maps \mathbf{d} into $\mathbf{x} \in \mathcal{X}^K$, where \mathcal{X} is the M^{th} order complex constellation with zero mean, and with average power $\sigma_x^2 = 1$, and where $K = K_d/q$, with $q \triangleq \log_2 M$. This operation maps the q -word $\mathbf{d}_k \triangleq [d_{qk}, \dots, d_{q(k+1)-1}]$ to the symbol x_k , and we use $\varphi_j^{-1}(x_k)$ to refer to d_{kq+j} .

The receiver is assumed to be ideally synchronized in time and frequency, and perfect channel state information is

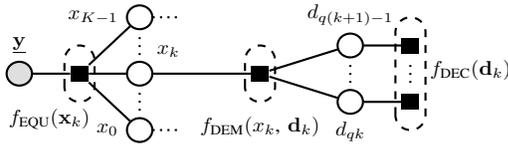


Fig. 1. Factor graph for the posterior (4) on x_k and \mathbf{d}_k .

available. The decoder and the demapper exchange extrinsic information for iterative detection. A priori, extrinsic and a posteriori log likelihood ratios (LLRs) of coded bits \mathbf{d} are respectively denoted $L_a(\cdot)$, $L_e(\cdot)$ and $L(\cdot)$. These definitions are given with respect to the soft-input soft-output receiver.

We consider a baseband circular channel model, including transceiver modules and the channel propagation

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}, \quad (1)$$

where $\mathbf{H} \in \mathbb{C}^{K \times K}$ is a circulant matrix, generated by the impulse response $\mathbf{h} = [h_0, \dots, h_{L-1}, \mathbf{0}_{K-L,1}^T]$ extended with $K-L$ zeros, $L < K$ being the channel spread. The noise at the receiver \mathbf{w} is modelled as an additive white Gaussian noise (AWGN), with $\mathcal{CN}(\mathbf{0}_K, \sigma_w^2 \mathbf{I}_K)$, i.e. a zero mean circularly symmetric Gaussian process with covariance $\sigma_w^2 \mathbf{I}_K$.

The normalized K -DFT matrix is given by its elements $[\mathcal{F}_K]_{k,l} = \exp(-2j\pi kl/K)/\sqrt{K}$, such that $\mathcal{F}_K \mathcal{F}_K^H = \mathbf{I}_K$. Then the equivalent frequency domain transmission model is

$$\underline{\mathbf{y}} = \underline{\mathbf{H}}\underline{\mathbf{x}} + \underline{\mathbf{w}}, \quad (2)$$

with $\underline{\mathbf{x}} = \mathcal{F}_K \mathbf{x}$, $\underline{\mathbf{y}} = \mathcal{F}_K \mathbf{y}$, $\underline{\mathbf{w}} = \mathcal{F}_K \mathbf{w}$, and $\underline{\mathbf{H}} = \mathcal{F}_K \mathbf{H} \mathcal{F}_K^H = \text{Diag}(\underline{\mathbf{h}})$ with the channel frequency response being

$$\underline{h}_k = \sum_{l=0}^{L-1} h_l \exp(-2j\pi kl/K), \quad k = 0, \dots, K-1, \quad (3)$$

and, thanks to DFT properties the FD noise remains AWGN $\underline{\mathbf{w}} \sim \mathcal{CN}(\mathbf{0}_K, \sigma_w^2 \mathbf{I}_K)$.

III. RECEIVER DESIGN FRAMEWORK

This section covers the approximation of the posterior probability distribution on transmitted symbols, by using an EP-based message passing with variables \mathbf{x} belonging to a multivariate white Gaussian distribution $\mathcal{CN}(\bar{\mathbf{x}}, \bar{\mathbf{v}} \mathbf{I}_K)$. The approximate distribution then allows deriving a FD receiver.

A. Factor Graph Model

The joint posterior probability density function (PDF) of data bits given FD observations is $p(\mathbf{b}, \mathbf{d}, \mathbf{x} | \underline{\mathbf{y}})$. BICM simplifies derivations by asymptotically enforcing an independence on transmitted coded and interleaved bits (i.e. by assuming sufficiently long interleavers), which yields $p(\mathbf{d}) = \prod_{j=1}^{K_d} p(d_k)$. This serves as a prior constraint provided by the decoder, from the equalizer's point of view. Any soft-input soft-output decoder can be used. Since the modulator is memoryless, the joint posterior PDF of bits \mathbf{d} and symbols \mathbf{x} is

$$p(\mathbf{d}, \mathbf{x} | \underline{\mathbf{y}}) \propto p(\underline{\mathbf{y}} | \mathbf{x}) \prod_{k=0}^{K-1} p(x_k | \mathbf{d}_k) \prod_{j=0}^{q-1} p(d_{kq+j}). \quad (4)$$

We now consider approximation of $p(\mathbf{d}, \mathbf{x} | \underline{\mathbf{y}})$ in a message passing based decoding algorithm over a factor-graph. Variable

nodes (VN) x_k and d_{kq+j} are updated iteratively, through the use of constraints imposed by factor nodes (FN) corresponding to the factorization of the posterior PDF in (4). More specifically, the factor related to FD equalization is

$$f_{\text{EQU}}(\mathbf{x}) \triangleq p(\underline{\mathbf{y}} | \mathbf{x}) \propto e^{-\underline{\mathbf{y}}^H \underline{\mathbf{y}} / \sigma_w^2 + 2\mathcal{R}(\underline{\mathbf{y}}^H \underline{\mathbf{H}} \mathcal{F}_K \mathbf{x}) / \sigma_w^2}, \quad (5)$$

where dependence on $\underline{\mathbf{y}}$ is omitted, as it is fixed during iterative detection. The factor on the mapping constraints is

$$f_{\text{DEM}}(x_k, \mathbf{d}_k) \triangleq p(x_k | \mathbf{d}_k) \propto \prod_{j=0}^{q-1} \delta(d_{kq+j} - \varphi_j^{-1}(x_k)), \quad (6)$$

where δ is the Dirac delta function, and finally, channel coding constraints are handled by the factor

$$f_{\text{DEC}}(\mathbf{d}_k) \triangleq \prod_{j=0}^{q-1} p(d_{kq+j}). \quad (7)$$

The factor graph of the BICM SC-FDE is given by Fig. 1.

B. EP Message Passing with White Gaussian Distributions

EP-based message passing algorithm is an extension of loopy belief propagation, where messages lie in the exponential distribution families of the involved VN [12]. This process allows computing a fully-factorized approximation of $p(\mathbf{d}, \mathbf{x} | \underline{\mathbf{y}})$, used to obtain marginals on variables.

Updates at a FN F connected to VNs \mathbf{v} consist of messages between VN v_i , the i^{th} component of \mathbf{v} , and F given by

$$m_{v \rightarrow F}(v_i) \triangleq \prod_{G \neq F} m_{G \rightarrow v}(v_i), \quad (8)$$

$$m_{F \rightarrow v}(v_i) \triangleq \frac{\text{proj}_{\mathcal{Q}_{v_i}} \left[\int_{\mathbf{v} \setminus v_i} f_F(\mathbf{v}) \prod_{v_j} m_{v \rightarrow F}(v_j) \mathbf{d}\mathbf{v} \setminus v_i \right]}{m_{v \rightarrow F}(v_i)}, \quad (9)$$

where $\mathbf{v} \setminus v_i$ are VNs without v_i , and $\text{proj}_{\mathcal{Q}_{v_i}}$ is the Kullback-Leibler projection towards the probability distribution \mathcal{Q}_{v_i} of VN v_i [12]. The projection operation for exponential families is equivalent to *moment matching*, which considerably simplifies the computation of messages [12].

Our simplifying assumption is that VNs \mathbf{x} lie in multivariate white Gaussian distributions. Hence, a message involving these VNs is fully characterized by a *vector mean* and a *scalar variance*. On the other hand d_{kq+j} follow a Bernoulli distribution, thus, the involved messages are characterized by bit LLRs, as in conventional loopy belief propagation algorithm.

C. Scheduling

The factor graph and the exchanged messages alone cannot yield a receiver algorithm without a scheduling scheme for the update of variable and factor nodes. Here, a robust and flexible double-loop FDE structure is proposed. The first loop refers to the exchange of extrinsic information between the decoder and the demapper in a turbo-iteration (TI), while the second loop refers to the message exchange in a self-iteration (SI) between the demapper and the equalizer.

Each TI $\tau = 0, \dots, \mathcal{T}$ consists of \mathcal{S}_τ SIs (may depend on τ), where EQU and DEM factor nodes are updated in parallel schedule, for $s = 0, \dots, \mathcal{S}_\tau$, and then the DEC factor nodes are updated with the selected soft-input soft-output decoder.

In the following, exchanged messages and their characteristic measures will be indexed by (τ, s) in the superscript.

F. Messages from DEM to DEC

After the last self-iteration of a turbo-iteration, the demapper computes extrinsic LLRs on the interleaved code bits with

$$L_e^{(\tau)}(d_{kq+j}) = \ln \frac{\sum_{\alpha \in \mathcal{X}_j^0} \mathcal{D}_k^{(\tau, \mathcal{S}_\tau)}(\alpha)}{\sum_{\alpha \in \mathcal{X}_j^1} \mathcal{D}_k^{(\tau, \mathcal{S}_\tau)}(\alpha)} - L_a^{(\tau)}(d_{kq+j}), \quad (26)$$

where \mathcal{X}_j^b is the set of $\alpha \in \mathcal{X}$ such that $\varphi_j^{-1}(\alpha) = b$.

G. Summary of the Receiver Structure

The iterative FDE derived in previous sections by applying the EP framework in the FD, with the family of white Gaussian distributions, yields the structure shown in Fig. 2. It is a low-complexity single-tap receiver structure with an internal loop.

Algorithm 1 Proposed FD SILE-EPIC Receiver

Input \underline{y} , \underline{H} , σ_w^2

- 1: Initialize decoder with $L_a^{(0)}(\mathbf{d}_k) = 0, \forall k$.
 - 2: **for** $\tau = 0$ to \mathcal{T} **do**
 - 3: Initialize equ. with $\hat{x}_k^{(\tau,0)} = 0, \forall k$ and $\sigma_\nu^{2(\tau,0)} = +\infty$.
 - 4: Use decoder's $L_a^{(\tau)}(\mathbf{d}_k)$ to compute $\mathcal{P}_k^{(\tau)}$ with (12), $\forall k$.
 - 5: **for** $s = 0$ to \mathcal{S}_τ **do**
 - 6: Use (13-15) to update demapper posteriors, $\forall k$.
 - 7: Generate soft feedback using (16)-(21), $\forall k$.
 - 8: Compute $\bar{\xi}^{(\tau,s)}$ using (23), and, $\sigma_\nu^{2(\tau,s)}$ using (25).
 - 9: Equalize using (22) and (24), for $k = 0, \dots, K - 1$.
 - 10: **end for**
 - 11: Provide extrinsic outputs $L_e^{(\tau)}(\mathbf{d}_k)$ to the decoder using (26), in order to obtain next priors $L_a^{(\tau+1)}(\mathbf{d}_k), \forall k$.
 - 12: **end for**
-

As a summary, Algorithm 1 explicitly describes the scheduling of the proposed receiver. In the following, we consider a fixed number of SI $\mathcal{S}_\tau = S$ per TI and we refer to our proposal as the S -self-iterated FD turbo receiver (*FD S-SILE-EPIC*).

IV. NUMERICAL RESULTS

A. Comparison with Single-Tap FDE in Prior Work

Transmission across the AWGN Proakis-C channel $\mathbf{h} = [1, 2, 3, 2, 1] / \sqrt{19}$, is considered with a recursive systematic convolutional (RSC) code with soft MAP decoder. Packet error rate (PER) is given by Monte-Carlo simulations. Several single-tap FD equalizers are compared to our proposal in Fig. 3: the conventional linear equalizer [2] (*LE-EXTIC*), the LE-IC with APP feedback [3] (*LE-APPIC*), and the self-iterated LE-IC of [5] (*SILE-APPIC*). Our proposal (using feature damping with $\beta_{\tau,s} = 0.7 \times 0.9^{s+\tau}$ for 8-PSK, and $\beta_{\tau,s} = 0.85^{1+s+\tau}$ for 64-QAM) brings significant improvement on the decoding threshold, that grows with the number of SIs, at all TIs. On the contrary, multiple SIs with APP feedback degrades this threshold and can end up worse than *LE-EXTIC* (not shown here due to lack of space). Without TI, 3 SIs bring respectively 9 dB and 6 dB gains for 8-PSK and 64-QAM, compared to *LE-EXTIC*, at $\text{PER} = 10^{-1}$. Performance in 64-QAM is limited at low PER without TIs, but our proposal

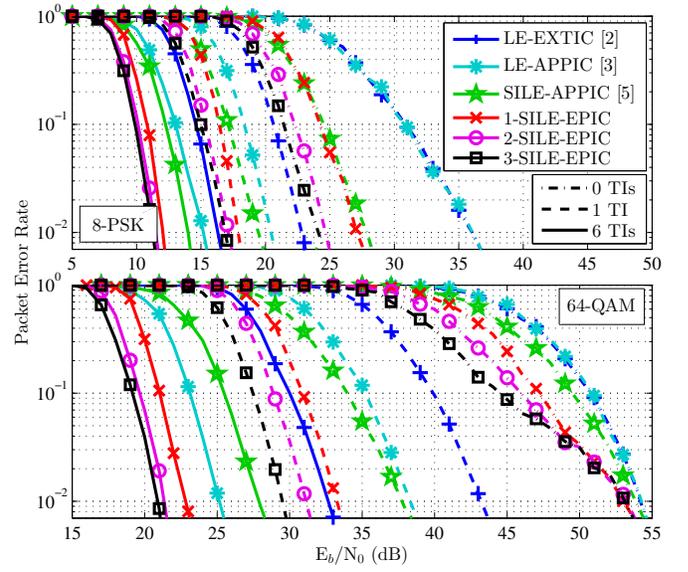


Fig. 3. PER comparison of single-tap FD equalizers in Proakis C channel, with $K = 256$ coded with rate-1/2 RSC $[1, 5/7]_s$.

with a single TI and 3 SIs reaches PER the prior work reach with 6 TIs, e.g. with six times lower decoding complexity. Besides, asymptotically (6 TIs), SIs with EP bring over 8 dB gain with respect to *SILE-APPIC*, and about 5 dB gain over *LE-APPIC*, for 64-QAM, at $\text{PER} = 10^{-2}$.

These results encourage replacing TIs with SIs as demapping complexity is often insignificant relative to decoding.

B. Comparison with EP-based Receivers in Prior Work

In Fig. 4, the proposed receiver is compared with existing EP-based equalizers from recent prior works. We consider the low-density parity check (LDPC) coded Proakis C scenario from [10], and compare our proposal to block receivers in [9] (*BEP*) and [10] (*nuBEP*), to the filter in [10] (*EP-F*) and to the single-tap FDE in [11] (*D-EP*). The latter equalizer cannot decode in Proakis-C channel, up to very high signal to noise ratios due to its sensitivity to channel nulls [11, eq. (48)]. The former structures are time-domain derivations of self-iterated EP receivers, and we see that the proposed receiver can perform as good as them, with an order of computational complexity of $(S+1)K \log_2 K$ instead of $3LK^2$ (*nuBEP*, 2 SIs), $10LK^2$ (*BEP*, 9 SIs) or $27KL^2$ (*EP-F*, 2 SIs). For small τ , our structure requires 3 SIs to reach their performance, but for $\tau > 1$, *BEP* performs worse than our structure with a single SI, despite using 9 SIs. Asymptotically ($\tau = 5$), *nuBEP* and *EP-F* have around 0.2 dB gain over *FD 3-SILE EPIC*, but they are respectively around 500 and 16 times more complex.

C. Achievable Rates of the Proposed Receiver

Achievable rates of the MAP detector, *FD LE-EXTIC* and the proposed receiver are given in Fig. 5, for the Proakis C channel with 8-PSK constellation. These values are numerically obtained using the area theorem of extrinsic information transfer charts [13]. Note that MAP detector's rate is an

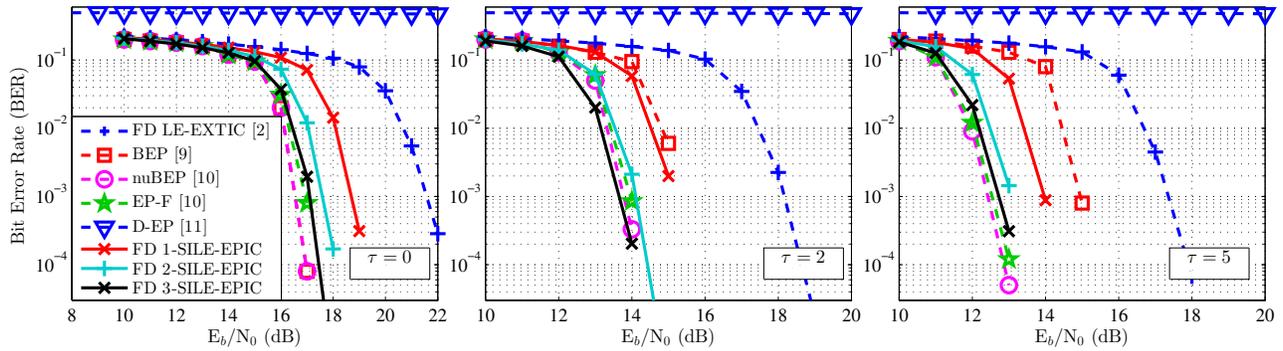


Fig. 4. BER comparison in Proakis C with 8-PSK, $K_d = 4096$ and rate-1/2 regular (3, 6) LDPC code.

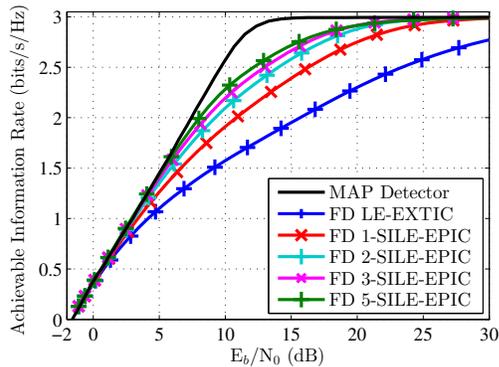


Fig. 5. Achievable rates in Proakis C with 8-PSK.

accurate approximation of the channel symmetric information rate [14], the highest possible transmission rate for practical constellations, without channel knowledge at the transmitter. While the conventional FD LE-EXTIC [2] follows this upper limit within 0.5 dB up to 0.75 bits/s/Hz, proposed EP-based self-iterations increase this range up to 2 bits/s/Hz. These rates are achievable with properly designed coding schemes.

V. CONCLUSION

Expectation propagation opens new horizons in iterative digital receiver design, with the possibility of computing novel extrinsic messages that take into account properties of exponential distributions. Although the EP framework has already been used for equalization, this is the first paper, to the authors' knowledge, that naturally extends it for low-complexity frequency domain receiver design, by constraining symbol variable nodes to lie in white Gaussian distributions. By setting the FDE design as proposed, a novel low-complexity FD turbo receiver which turns out to be an extension of conventional turbo FDE [2] was derived. EP, applied to symbol/bit variables, is hence exploited through self iterations to compute even more robust filters. This receiver is shown to outperform existing single-tap FD receivers, which have the same order of complexity (per self-iteration), with at least 3 dB gain in 8-PSK, and 5 dB in 64-QAM, in the well-known severely frequency selective Proakis-C channel. Although the proposed

receiver and exact time domain receivers have similar performance, the former has respectively one and two order of magnitude lower computational complexity than finite-impulse response and block equalization structures. In this paper, we have shown that a double-loop, low-complexity FD receiver can achieve considerable energy savings with off-the-shelf error correction codes. More details on the formal derivation of this receiver, along with further analysis, application to time-varying channel equalization, and extension to MIMO detection, will be exposed in future works.

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