Minimum Length Solution for One-Dimensional Discrete Phase Retrieval Problem

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***Abstract*—Recently it has been shown that the one-dimensional discrete phase retrieval problem may not have always a causal solution for certain input magnitude data, but it has been proved that the extended form of the one-dimensional discrete phase retrieval problem has always a causal solution within the same conditions. In this work we are looking for the minimum length solution for one-dimensional discrete phase retrieval problem. The Non-uniform Discrete Fourier Transform based approach is introduced and experimental results are also presented.**

I. Phase retrieval problems

One classical signal recovery task is the reconstruction of a Fourier transform pair from data on either or both domains. There are many examples of reconstruction of Fourier transform pair in optics, electrical engineering, quantum physics, and astronomy [1]. Within this class, the problem of phase retrieval is to reconstruct the signal from only magnitude measurements [2]:

Problem 1: Given $\tilde{X} \geq 0$, find x such that its Fourier transform $X = \mathcal{F}\{x\}$ satisfies $|X| = \tilde{X}$.

In practice one deals with sampled data. In case of one-dimensional sequences the Discrete-Time Fourier Transform is used:

Problem 2: Given $\tilde{X}(\omega_k) \geq 0$, $k \in \mathbb{K} \subset \mathbb{Z}$, find $x(n)$ such that:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

satisfies $|X(\omega_k)| = \tilde{X}(\omega_k)$.

If we assume that $x(n)$ is causal and of finite length, the Discrete Fourier Transform (DFT) is typically implemented. The main one dimensional discrete phase retrieval (1-D DPhR) problem can be stated as follows:

Problem 3: Let $\tilde{X}(k)$, $k = 0, 1, \ldots, N-1$ a sequence of nonnegative numbers, which will be called the input magnitude data. A solution of 1-D DPhR problem is a complex signal of length M ($M \leq N$) $x(n)$, $n = 0, 1, \ldots, N-1$, with $x(n) = 0$ for $n = M, M+1, \ldots, N-1$, such that its Fourier transform

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}, \quad k = 0, 1, \ldots, N-1. \tag{1}$$

satisfies

$$|X(k)| = \tilde{X}(k) \tag{2}$$

for all $k = 0, 1, \ldots, N-1$.

Note that input magnitude data $\tilde{X}(k)$ correspond to $\omega_k = \frac{2\pi k}{N}$, where $k = 0, 1, \ldots, N-1$. Also the methods using autocorrelation from circular autocorrelation require $M \leq (N-1)/2$ [3], [4].

One can obtain a solution to 1-D DPhR problem by finding the zeros of z-transform of autocorrelation, Hilbert transform, computation of cepstral coefficients [5]–[7], but the most common approaches are iterative transform algorithms, which alternate between time and frequency domains [4].

Depending on the given input magnitude data $\tilde{X}(k)$ and the signal length M, the problem may or may not have a solution. Indeed, a solution to the 1-D DPhR problem exists if certain conditions are satisfied by the input magnitude data, namely the corresponding trigonometric polynomial must be nonnegative [3].

Even though we cannot find (or there does not exist) a solution satisfying (2), we can use optimization methods to search that best approximates (2) in some sense. The following least-squares problem or empirical risk minimization [8] is the most well known:

Problem 4: Find $x(n)$, a discrete signal of length M ($M \leq N$), such that to minimize

$$\min_{x(n)} \sum_{k=0}^{N-1} [\tilde{X}^2(k) - |X(k)|^2]^2 \tag{3}$$

where $\tilde{X}^2(k)$ are called the measurements, and $X(k)$ are given by (1) for all $k = 0, 1, \ldots, N-1$.

Note that Problem 4 has always solution, but Problem 3 may have solution or not. When Problem 3 has a solution, this verifies also Problem 4. Solutions to both Problem 3 and Problem 4 are subject to ambiguities [9].

Nevertheless, a solution to Problem 4 which is not a solution to Problem 3 gives the magnitudes of the DFT of the solution that are different from the input magnitude data. Unfortunately, small changes in input magnitude data can sometimes provide large changes in the phase of the solution of 1-D DPhR problem [10]. Consequently we may not have always strong arguments that the solution obtained to Problem 4 is indeed

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the one we are looking to solve 1-D PhR problem, as stated by Problem 3.

Another doubt is related to the length of the solution of Problems 3 and 4 respectively. Problem 4 has always a solution of length M ($2M-1 \leq N$); Problem 3 may have a solution only of length larger than M; this situation may appear because the Fourier transform has not been sampled according to the length of the sequence and time-domain aliasing is present in the input magnitude data [11].

We have reformulated the 1-D DPhR problem into an extended form of the 1-D Discrete Phase Retrieval problem. In the suggested approach, the DFT has been computed in a number of points twice the number of input magnitude data. We recall that oversampling of Fourier transform magnitude by 2 has been proposed and analyzed in [12].

In the extended form of the 1-D Discrete Phase Retrieval problem the input magnitude data represent the specified values of the DFT's magnitude for even indexes, and the values of the DFT's magnitude for odd indexes have to be found. This approach does not change the input magnitude data of Fourier transform; indeed it preserves the input magnitude data at given frequencies $\omega_k = \frac{2\pi k}{N}$. Moreover, this approach avoids the time-domain aliasing [13]. Recent work has proved that extended form of the 1-D Discrete Phase Retrieval problem has always a solution that can be obtained using the error-reduction algorithm [14].

It follows that we can obtain a solution to 1-D PhR problem by keeping fixed the input magnitude data; this solution has a finite length and the length may vary according to the input magnitude data. The goal of this work is to state the issue of the minimum length solution for one-dimensional discrete phase retrieval problem and to introduce an approach to find this minimum length solution.

The paper is organized as follows. Section II discusses the minimum length solution of 1-D PhR problem, then the NDFT based approach is proposed in Section III. Experimental results are presented in Section IV. Section V concludes our achievements.

II. Minimum length solution of 1-D PhR problem

In this section we begin with an example, then we define the minimum length solution of 1-D PhR problem. We recall that $x(n)$ needs to be zero for $n \geq N/2$ in order to avoid aliasing in computation of $|X(k)|^2$ [15].

Example 1: [13]
Let $N = 5$ and

$$\tilde{X}(k) = \begin{cases} 3, & k = 0; \\ 1, & k = 1, 2, 3, 4. \end{cases} \tag{4}$$

Note that input magnitude data $\tilde{X}(k)$ correspond to $\omega_k = \frac{2\pi k}{5}$, where $k = 0, 1, 2, 3, 4$.

Any attempt to solve 1-D DPhR problem for these input magnitude data is unsuccessful [16].

The solution of the extended form of the 1-D Discrete Phase Retrieval problem [13] obtained after convergence is:

$$x(n) = \begin{cases} 1.4 & n = 0; \\ 0.4 & n = 1, 2, 3, 4; \\ 0 & n = 5, 6, 7, 8, 9, \end{cases}$$

It follows that we do not have any solution of length 3, but we have a solution of length 5. The question for these input magnitude data (4) is whether the minimum length of the solution is 5 or 4.

For the general case we have:

Definition 1: Let $\tilde{X}(\omega_k)$, $k = 0, 1, \ldots, N-1$ a sequence of positive numbers, which will be called the input magnitude data. A minimum length solution of 1-D DPhR problem is the smallest integer L and an associated discrete causal signal $x(n)$ of length L for which its Fourier Transform :

$$X(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n} \tag{5}$$

satisfies

$$|X(\omega_k)| = \tilde{X}(\omega_k) \tag{6}$$

for all $k = 0, 1, \ldots, N-1$.

As one can see, in relation with Problem 2, here

$$\mathbb{K} = \{\omega_k = 2\pi k/N | k = \overline{0, N-1}\}.$$

Two main issues can be distinguished in relation with minimum length solution of 1-D PhR problem:

1) selection of L;
2) L has being set, how to verify whether there is a solution of length L to 1-D PhR problem?

There are many search algorithms that may help the selection of L. In the following we shall focus on the second issue.

III. NDFT based approach

Difficulties may appear when one tries to evaluate the Fourier transform or its inverse. This is needed for the work therein when one would like to make use of certain iterative algorithms. The most widely used algorithms for phase retrieval are iterative alternating projection algorithms pioneered by Gerchberg-Saxton [17] and Fienup [18].

Generally, the Fourier transform is computed using DFT in almost all numerical applications. Suppose that DFT is calculated on certain P points. A first requirement is $P \geq 2L-1$ due to causality constraint [4]. Second, the DFTs on P points usually do not give the magnitude for $\omega_k = \frac{2\pi k}{N}$, except for the situation when N divides P; this is seldom the case. Thus for computational reasons, the DFT calculation to get the spectrum must be replaced by NDFT (Nonuniform Discrete Fourier Transform) evaluation [19].

The Fourier transform can be computed for any length of a sequence and there are no special constraints on the length of the sequence for the computation of the inverse of the Fourier transform. This is not anymore the case of DFT and NDFT. For computing of the DFT or of the NDFT and of their inverses, we need to have the same number of samples: for the sequence and for the transform. It follows that we have to add new supplementary frequencies for the computation of the NDFT, besides the settled frequencies $\omega_k = \frac{2\pi k}{N}$, $(k = 0, 1, \ldots, N-1)$ for which the input magnitude data are known.

The selection of the supplementary frequencies is a major issue of the minimum length solution NDFT based approach for the 1-D DPhR problem. Note that such supplementary frequencies have been used in the case of extended form of 1-D DPhR problem. However, there it has been natural to choose as supplementary frequencies for $\omega_k = \frac{2\pi(2k-1)}{2N}$, where $k = 0, 1, \ldots, N-1$.

In the case of the minimum length solution NDFT based approach for the 1-D DPhR problem, we propose to set the supplementary frequencies by dividing the fundamental interval $[0, 2\pi)$ in equal parts.

In the very rare cases, when the supplementary frequencies may overlap the settled frequencies, we have to add a certain fixed frequency after division of the fundamental interval. This modification of the supplementary frequencies avoids the singularity of NDFT matrix. However, when we select the supplementary frequencies, if possible, the computational load of the NDFT and of its inverse should be not considered. Our suggestions to change the supplementary frequencies are the following:

1) add $\pi/2$;
2) add the mean of the first nonzero settled and supplementary frequencies.

Our experiments show that the second suggestion works better, at the expense of the computation complexity.

The NDFT based approach for 1-D DPhR problem is described in Figure 1. This algorithm is almost a standard iterative procedure for phase retrieval, having several particularities:

- Fourier transform has been substituted by NDFT:
$$X(z_k) = \sum_{n=0}^{P-1} x(n) z_k^{-n}, \tag{7}$$
where $z_k = e^{j\omega_k}$, $k = 0, 1, \ldots, P-1$;
- the initialization of Fourier transform magnitudes with input magnitude data is done only for settled frequencies;
- at every iteration, after NDFT computation and only for settled frequencies, the Fourier transform magnitudes are substituted by the input magnitude data;
- for supplementary frequencies the Fourier transform magnitudes are modified only by NDFT computation.

To verify the convergence of the iterative procedure, we modified the error function used by error-reduction algorithm such that the error is computed between the specified input magnitude data and the corresponding NDFT magnitudes:

$$E_p^{\mathrm{NDFT}} = \frac{1}{N} \sum_{k=0}^{N-1} \left| \tilde{X}(\omega_k) - |\hat{X}_{p+1}(k)| \right|^2. \tag{8}$$

The following example will help to understand the mechanism of the NDFT based approach for the 1-D DPhR problem.

```
Input:       I (number of iterations);
             N (number of input magnitude data);
             X~(ω_k), k = 0,N-1 (input magnitude data);
             ω_k = 2πk/N, k = 0,N-1 (settled frequencies);
             L (length of the signal x(n));
             P = 2L-1, (length of the NDFT);
Set:         ω_k, k = N,P-1 (supplementary frequencies);
             X~(ω_k) = 0, k = N,P-1
             (initial magnitude - supplementary frequencies);
             ∠X^_1(ω_k) = 0, k = 0,P-1;
             (the initial estimate of the phase);
Compute:     X_1(k) = X~(ω_k), k = 0,P-1 ;
for p = 1,I  x_p(n) = NDFT_P^{-1}{X_p(k)}, n = 0,P-1;
             x^_{p+1}(n) = x_p(n), n = 0,L-1;
             x^_{p+1}(n) = 0, n = L,P-1;
             X^_{p+1}(k) = NDFT_P{x^_{p+1}(n)}, k = 0,P-1;
             ∠X^_{p+1}(k) = arg X^_{p+1}(k); k = 0,P-1
             (the new estimate of the phase)
             X_{p+1}(2k) = X~(ω_k) exp[∠X^_{p+1}(2k)],
             k = 0,L-1;
             X_{p+1}(2k+1) = X^_{p+1}(k), k = L,P-1;
             E_{p+1}^NDFT = 1/N Σ_{k=0}^{N-1} |X~(ω_k) - |X^_{p+1}(k)||^2 ;
end;
Output:      ∠X^_{I+1}(k) - (final estimate of the phase).
```

Fig. 1. NDFT based approach for 1-D DPhR problem

Example 2: Example 1 revisited

In the following we shall look for a solution $x(n)$ of length 4 such that

$$X(\omega) = \sum_{n=0}^{3} x(n) e^{-j\omega n} \tag{9}$$

satisfies

$$|X(\omega_k)| = \begin{cases} 3, & \omega_k = 0; \\ 1, & \omega_k = 2\pi k/5, k = \overline{1,4}. \end{cases} \tag{10}$$

Since the length of the NDFT must be $P = 2L - 1 = 7$, we need 2 supplementary frequencies. We select the following supplementary frequencies: $\omega_5 = 2\pi/3$, and $\omega_6 = 4\pi/3$, by dividing the interval $[0, 2\pi]$ in 3 equal parts. Thus the points z_k used to compute the NDFT are as follows: $z_k = e^{j2\pi k/5}, k = \overline{0,4}$, $z_5 = e^{j2\pi/3}$, and $z_6 = e^{j4\pi/3}$. Consequently the NDFT matrix is not singular.

Our experiments show that we get a solution for the 1-D DPhR problem with the NDFT based approach for the

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input magnitude data given by (10) of minimum length 4. The solution is:

$$x(n) = \begin{cases} 0.2702 + 0.5592j, & n = 0; \\ 0.0723 + 0.3332j, & n = 1; \\ 0.2393 + 0.9865j, & n = 2; \\ 0.2829 + 0.9938j, & n = 3; \\ 0, & n = 4, 5, 6. \end{cases}$$

Thus we answer to the question raised at the end of the Example 1: for the input magnitude data given by (4), the minimum length of the solution is 4.

IV. Experimental results

The solution to the Problem 3 exists when certain conditions are satisfied [3]. For instance, it has been shown there that if the mean of input magnitude data is rather close to zero, there are really small chances to get a solution to Problem 3. If the input magnitude data are severely biased toward positive values, then the direct method of the 1-D DPhR has a solution almost always. For this reason, the minimum length solution is more needed when the input magnitude data has a mean rather close to zero than severely biased toward positive values.

In this section, the input magnitude data will be randomly uniform selected between [0,1]. Also we wait 1000 iterations for E_p^{NDFT} to reach the convergence level of -100 dB. From our experience we found that for small lengths and in almost all cases, 1000 iterations and convergence level of -100 dB are enough to distinguish whether the error-reduction algorithm converges.

A. Minor variation of Example 2

The first set of experiments is a minor variation of Example 2. Our goal is to see what happens:

1) when the input magnitude data are varying;
2) when the supplementary frequencies are not following the recommendations from Section III.

We have followed the same framework as in the case of Example 2, however the input magnitude data have been randomly selected. We have generated 1000 sequences of input magnitude data with length 7. For every one of these magnitude data we run the NDFT based approach for the 1-D DPhR problem. We wait 1000 iterations for E_p^{NDFT} to reach the convergence level of -100 dB.

The first goal was to test if there is a minimal solution of length 5 or not. In this part of the experiment we kept the supplementary frequencies as $\omega_5 = 2\pi/3$ and $\omega_6 = 4\pi/3$. We have found solutions of length 4 for the 1-D DPhR problem for around 90% of the trials. For some of input magnitude data, the minimum length solution might be 3, for other might be 4. However, the minimum length solution was 5 for about 10% of the trials. Consequently, the minimum length solution varies according to the input magnitude data.

In the second part of the experiment we modified randomly supplementary frequencies and the input magnitude data. We have found that the the associated discrete causal signal $x(n)$ to the minimum length solution differs when the supplementary frequencies are changed. Moreover, the convergence or the divergence depend on the choice of the supplementary frequencies. Indeed, for some supplementary frequencies and input magnitude data, the error function may not converge. When it does not converge, E_p^{NDFT} may have oscillations.

To conclude, the selection of supplementary frequencies is a sensitive issue and in the following we shall strictly follow the recommendations from Section III.

TABLE I
Number (No) of minimum length solutions for different L ($N = 5$)

L	3	4	5
No	288	630	82

TABLE II
Number (No) of minimum length solutions for different L ($N = 7$)

L	4	5	6	7
No	197	569	259	5

B. Number of minimum length solutions

In this set of experiments the goal was to evaluate the number of minimum length solutions for different number of input magnitude data. We have generated 1000 sequences of input magnitude data with length N, where N was equal to 5, 7, 9, 11, 13, and 15.

For every one of these input magnitude data we have first verified whether they have solution to the Problem 3 using the DFT criterion from [3]. If they have solution to the Problem 3, then we automatically set M as the minimum length solution.

If the input magnitude data have no solution to the Problem 3, then we run the NDFT based approach for these input magnitude data and for $L = M + 1$. We wait 1000 iterations for E_p^{NDFT} to reach the convergence level of -100 dB. If the convergence level is reached, then the minimum length solution is set $M + 1$.

If the input magnitude data have no solution to the Problem 3 and no minimum length solution of $M+1$, then we run the NDFT based approach for these input magnitude data and for $L = M + 2$. We wait 1000 iterations for E_p^{NDFT} to reach the convergence level of -100 dB. If the convergence level is reached, then the minimum length solution is set $M + 2$.

The same procedure was applied for $L = M + 2$ to $L = N$.

In this way we found the number of minimum length solutions for every N and for every input magnitude data.

The results are presented in Tables I to VI. The selection of supplementary frequencies has followed the recommendations from Section III. We just note that situations when adding frequency is needed can be easily detected, by verifying whether a settled frequency and a supplementary frequency are equal.

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TABLE III
NUMBER (NO) OF MINIMUM LENGTH SOLUTIONS FOR DIFFERENT L ($N = 9$)

L	5	6	7	8	9
No	125	573	258	14	30

TABLE IV
NUMBER (NO) OF MINIMUM LENGTH SOLUTIONS FOR DIFFERENT L ($N = 11$)

L	6	7	8	9	10	11
No	68	399	369	140	24	0

TABLE V
NUMBER (NO) OF MINIMUM LENGTH SOLUTIONS FOR DIFFERENT L ($N = 13$)

L	7	8	9	10	11	12	13
No	36	242	362	332	31	1	0

TABLE VI
NUMBER (NO) OF MINIMUM LENGTH SOLUTIONS FOR DIFFERENT L ($N = 15$)

L	8	9	10	11	12	13	14	15
No	22	192	446	287	48	5	0	0

To conclude this second set of experiments, the main one-dimensional discrete phase retrieval may have not solution of length M, but we can find a (minimum length) solution having the same input magnitude data by assuming that the solution has length larger than M.

V. CONCLUSION

The presence of time-domain aliasing in DFT is certain when the sequence has infinite length. Time-domain aliasing might be also present in DFT when the frequency sampling has not been performed adequate. Moreover, the magnitudes of the DFT contain a nonlinear processing of the DFT. Consequently, when present, the time-domain aliasing penetrates the magnitudes of the DFT in nonlinear fashion and it is difficult to distinguish.

If the number of the measurements of the DFT's magnitudes is less than the length of the sequence that generated the magnitudes, then the one-dimensional phase retrieval problem cannot provide a solution having the appropriate length. In such case, we need an approach to find first the length of the solution, and then the solution.

In this work our goal was to state the issue of the minimum length solution for one-dimensional discrete phase retrieval problem and to introduce an NDFT based approach to find this minimum length solution. We found that the one-dimensional discrete phase retrieval problem has always a solution of minimum length and we have shown how to find this solution using the NDFT approach. Unfortunately the associated solution to the minimum length may change according to the selection of the supplementary frequencies.

In this work we did not develop optimal procedures for optimal frequencies selection. We did not look for optimal selection of the minimum length parameter. These might be the goals of future work.

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