Effect of Random Sampling on Noisy Nonsparse Signals in Time-Frequency Analysis

Isidora Stanković
GIPSA Lab/Faculty of Electrical Engineering
University Grenoble Alpes/University of Montenegro
Grenoble, France/Podgorica, Montenegro
isidoras@ac.me

Miloš Brajović, Miloš Daković
Faculty of Electrical Engineering
University of Montenegro
Podgorica, Montenegro
{milosb,milos}@ac.me

Cornel Ioana
GIPSA Lab
University of Grenoble Alpes
Grenoble, France
cornel.ioana@gipsa-lab.grenoble-inp.fr

***Abstract*—The paper examines the exact error of randomly sampled reconstructed nonsparse signals having a sparsity constraint. When signal is randomly sampled, it looses the property of sparsity. It is considered that the signal is reconstructed as sparse in the joint time-frequency domain. Under this assumption, the signal can be reconstructed by a reduced set of measurements. It is shown that the error can be calculated from the unavailable samples and assumed sparsity. Unavailable samples degrade the sparsity constraint. The error is examined on nonstationary signals, with the short-time Fourier transform acting as a representative domain of signal sparsity. The presented theory is verified on numerical examples.**

***Index Terms*—compressive sensing, nonsparse signals, random sampling, time-frequency analysis**

I. Introduction

Nonstationary signals are dense in both time and frequency, when considered separately. They can be localized in the time-frequency domains. However, they could be located within much smaller regions in the joint domain using appropriate representations [1]–[6], with the short-time Fourier transform (STFT) being the basic transformation. The signals are sparse in the time-frequency domain if the number of nonzero coefficients in this domain is much smaller than the total number of coefficients.

According to the compressive sensing (CS) theory, sparse signals can be reconstructed using less samples/measurements than required by the sampling theorem [7]–[12]. Reducing the number of measurements will introduce noise in the analysis of the signals. The properties of the noise from [13], [14] will be used to define reconstruction properties in the case of randomly sampled STFT. If a nonsparse signal is reconstructed with a reduced set of available samples, then the noise due to the missing samples of nonreconstructed coefficients will be considered as an additive input noise in the reconstructed signal.

Because of its nonstationary nature, signals in time-frequency domain are usually approximately sparse or nonsparse. In the CS literature, only the general bounds for the reconstruction error for nonsparse signals (reconstructed with the sparsity assumption) are derived [9], [15]–[17]. In this paper, we present an exact relation for the expected squared error, reconstructed from a reduced set of signal samples, under the sparsity constraint. The error depends on the number of available samples and the assumed sparsity. In order to be more compatible with the practical problems, we will consider that the signals are randomly sampled, i.e. not on the grid. Also, signals with additive noise will be considered. Since the signal is not on the grid, it looses the property of sparsity in the transformation domain. The properties of uniform sampling without noise are examined in [18]. The effects of random sampling and noise are illustrated and checked on examples.

The paper is organized as follows. The theory of random sampling in time-frequency analysis using the compressive sensing framework will be explained in Section II. The influence of nonsparsity in randomly sampled signal will be shown in Section III. Examples will be given in Section IV and the conclusions are presented in Section V.

II. Theoretical Background

A. Random sampling

Consider a general form of a multicomponent signal

$$x(t) = \sum_{l=1}^{C} x_l(t), \tag{1}$$

with C non-stationary components $x_l(t)$, $l = 1, 2, \ldots, C$. The signal is of a time-varying nature. Although not sparse in the Fourier transform (FT) domain, it may be sparse in the joint time-frequency domain. In this paper, we assume that the signal is sparse in the STFT domain, which is defined as

$$S_N(t,\Omega) = \int_{-\infty}^{\infty} x(t+\tau)w(\tau)e^{-j\Omega\tau}d\tau, \tag{2}$$

where $w(\tau)$ is the window function with duration T, centered at point t. The periodic extension of the product $x(t+\tau)w(\tau)$, for a given t can be expanded in Fourier series as follows

$$x(t+\tau)w(\tau) = \frac{1}{N}\sum_{k=0}^{N-1} X_t(k)e^{j2\pi k(\tau-T/2)/T}, \tag{3}$$

with series coefficients $X_t(k)$ being equal to the discrete FT (DFT) coefficients if $x(n+m)w(m)$ is used to denote $x(n\Delta t+$

ISBN 978-90-827970-1-5 © EURASIP 2018

$m\Delta t)w(m\Delta t)$, and the specific sampling interval $\Delta t = T/N$ is assumed. In that case we may write

$$X_t(k) = \frac{N}{T}\int_0^T x(t+\tau)w(\tau)e^{-j2\pi k(\tau - T/2)/T}d\tau$$
$$= \sum_{m=0}^{N-1} x(n+m)w(m)e^{-j\frac{2\pi}{N}km}.$$

Note that $X_t(k)$ is calculated for fixed t, and it can be interpreted as the STFT of the discrete-time signal $x(n)$, that is, $X_n(k) = S_N(n,k)$. The function $w(m)$ is the discrete window of length N. In matrix form, the STFT is

$$\mathbf{S}_N(n) = \mathbf{W}_N\mathbf{H}_N\mathbf{x}(n), \tag{4}$$

where $\mathbf{S}_N(n) = [S_N(n,0), S_N(n,1), \dots, S_N(n,N-1)]^T$, $\mathbf{x}(n)$ is the vector of the original signal samples within the considered window, $\mathbf{W}_N$ is the DFT matrix of size $N \times N$ with coefficients $W(m,k) = \exp(-j2\pi km/N)$ and $\mathbf{H}_N$ is the matrix with the window values at its diagonal.

The windowed signal $\hat{x}(n,m) = x(n+m)w(m)$, which is K-sparse in the STFT domain, can be written as

$$\hat{x}(n,m) = \sum_{i=1}^{K} A_i(n)e^{j2\pi mk_i/N}, \tag{5}$$

where $A_i(n)$ denotes the amplitude of ith signal component. Now we will assume that the signal is sensed at random positions. The new signal will be $\hat{x}(t_n,\tau_m)$ with $0 \le \tau_m < T$. The special case of uniform sampling, when sampling instants t_n, τ_m are integer multiples of Δt, is considered in [18]. The signal is written as

$$\hat{x}(t_n,\tau_m) = \sum_{i=1}^{K} A_i(t_n)e^{j\frac{2\pi}{T}\tau_m k_i}. \tag{6}$$

with STFT vector being $\mathbf{S}_N(t_n) = S_N(t_n,k)$.

B. Reduced set of samples

With the assumption of signal sparsity in the STFT domain, it can be reconstructed with a reduced set of measurements [7]–[12]. The reduced set of available samples is $N_A \ll N$. The signal samples are positioned at $t_n + \tau_m \in \mathbb{N}_A = \{t_n+\tau_1, t_n+\tau_2, \dots, t_n+\tau_{N_A}\} \in \{t_1, t_2, \dots t_{N_t}\}$. The STFT coefficients $S_N(t_n,k)$ are reconstructed from the available set of samples under the assumption that the signal is K-sparse, i.e., that the vector $\mathbf{S}_N(t_n)$ has only K nonzero elements.

The available samples of the windowed signal are

$$\mathbf{y}_n = [x(t_n+\tau_1)w(\tau_1), ..., x(t_n+\tau_{N_A})w(\tau_{N_A})]^T \tag{7}$$

or in vector form

$$\mathbf{y}_n = \mathbf{A}\,\mathbf{S}_N(t_n),$$

where $\mathbf{A}$ is the $N_A \times N$ partial inverse DFT matrix with rows corresponding to the available samples positions.

A general goal of CS is reconstructing the missing samples of the original sparse signal by minimizing the sparsity using the available samples

$$\min \|\mathbf{S}_N(t_n)\|_0 \text{ subject to } \mathbf{y}_n = \mathbf{A}\,\mathbf{S}_N(t_n).$$

The initial STFT is calculated using only the available samples

$$S_{N0}(t_n,k) = \sum_{m=1}^{N_A} \hat{x}(t_n,\tau_m)e^{-j\frac{2\pi}{T}\tau_m k} \tag{8}$$

or in vector form $\mathbf{S}_{N0}(t_n) = N\mathbf{A}^H\mathbf{y}_n$.

The mean and the variance of STFT when sampled randomly are

$$\mathrm{E}\{S_{N0}(t_n,k)\} = \sum_{i=1}^{K}\frac{N_A}{N}A_i(t_n)\delta(k-k_i), \tag{9}$$

$$\mathrm{var}\{S_{N0}(t_n,k)\} = \frac{N_A}{N^2}\sum_{i=1}^{K}|A_i(t_n)|^2(1-\delta(k-k_i)). \tag{10}$$

where $\delta(k) = 1$ only for $k = 0$ and $\delta(k) = 0$, elsewhere.

Time-varying signals are usually not strictly sparse in the STFT domain. Also, randomness in sampling destroys the sparsity in the signal. They are usually approximately sparse on nonsparse. A signal is approximately sparse if the coefficients at positions $k \in \mathbb{K} = \{k_1, k_2, \dots, k_K\}$ are significantly larger than the other coefficients, and it is considered as nonsparse if the coefficients at $k \notin \mathbb{K}$ are of the same order as the coefficients at $k \in \mathbb{K}$.

C. Reconstruction algorithm

The reconstruction is done in an iterative way using the orthogonal matching pursuit (OMP) algorithm [11]. The reconstruction is based on estimating the positions of the nonzero components and calculating the signal amplitudes at these positions using the known measurements. The procedure is described in a pseudo-code as follows

$$
\begin{aligned}
&\mathbb{K} = \emptyset, \quad \mathbf{y}_r = \mathbf{y}_n\\
&\textbf{for}\ \ i = 1:K\\
&\quad \mathbf{S}_{N0}(t_n) = N\mathbf{A}^H\mathbf{y}_r\\
&\quad k = \arg\{\max_k |\mathbf{S}_{N0}(t_n)|\}\\
&\quad \mathbb{K} = \mathbb{K} \cup \{k\}\\
&\quad \mathbf{A}_K = \mathbf{A}(:,\mathbb{K})\\
&\quad \mathbf{S}_{NK}(t_n) = (\mathbf{A}_K^H\mathbf{A}_K)^{-1}\mathbf{A}_K^H\mathbf{y}_n\\
&\quad \mathbf{s}_r = \mathbf{A}_K\mathbf{S}_{NK}(t_n)\\
&\quad \mathbf{y}_r = \mathbf{y}_n - \mathbf{s}_r\\
&\textbf{end}\\
&\mathbf{S}_{NR}(t_n) = \mathbf{S}_{NK}(t_n).
\end{aligned}
$$

Note that the error calculation does not depend on the reconstruction algorithm used. The important fact is that, for a successful reconstruction, the measurement matrix must satisfy the conditions of the CS theory, which we assume in this paper.

III. Influence of Nonsparsity

In this paper, the assumption is that the signal is randomly sampled. It is approximately sparse or nonsparse in time-frequency transformation domain and it is reconstructed under the assumption that it is K-sparse. That means that K

ISBN 978-90-827970-1-5

components will be reconstructed and the remaining $N - K$ components will not be reconstructed. The nonreconstructed part will behave as noise in the initial estimate. The noise, influenced by the nonreconstructed part, will appear in the reconstructed components with mean (9) and variance (10).

The reconstruction algorithm works as an amplifier by a factor of N/N_A to the original signal components in the initial estimate at positions $k \in \mathbb{K}$. It also eliminates the influence of other components. Components, which are not reconstructed, contribute to the noise in the initial estimate with variance $(N_A/N^2)|S_N(t_n, k_i)|^2$. In the reconstruction process, this variance is scaled by a factor of $(N/N_A)^2$, which means that the final variance is then $|S_N(t_n, k_i)|^2/N_A$.

Since K components are reconstructed, the total energy of noise in all reconstructed components is K times greater than the variance in one reconstructed component. There are $N - K$ nonreconstructed components, which means that the total energy of error is

$$\|\mathbf{S}_{NK}(t_n) - \mathbf{S}_{NR}(t_n)\|_2^2 = \frac{K}{N_A} \sum_{i=K+1}^{N} |S_N(t_n, k_i)|^2 \quad (11)$$

where $\mathbf{S}_{NK}(t_n)$ is the signal with the K highest components in the original signal $\mathbf{S}_N(t_n)$, i.e. $\mathbf{S}_{NK}(t_n) = [S_N(t_n, k_1), S_N(t_n, k_2), \ldots, S_N(t_n, k_K)]^T$. The signal $\mathbf{S}_{NR}(t_n)$ represents the K reconstructed components.

Obviously, the error energy in the nonreconstructed coefficients is the sum of its energies, i.e.,

$$\|\mathbf{S}_N(t_n) - \mathbf{S}_{NK0}(t_n)\|_2^2 = \sum_{i=K+1}^{N} |S_N(t_n, k_i)|^2 \quad (12)$$

where $\mathbf{S}_{NK0}(t_n)$ is the signal $\mathbf{S}_{NK}(t_n)$ zero-padded to length N such that $\mathbf{S}_{NK0}(t_n) = \mathbf{S}_{NK}(t_n)$ at $k \in \mathbb{K}$ and $\mathbf{S}_{NK0}(t_n) = 0$ at $k \notin \mathbb{K}$. Following these results, the energy of error in the reconstruction is

$$\|\mathbf{S}_{NK}(t_n) - \mathbf{S}_{NR}(t_n)\|_2^2 = \frac{K}{N_A} \|\mathbf{S}_N(t_n) - \mathbf{S}_{NK0}(t_n)\|_2^2 .$$

A. Additive Noise Influence on Nonsparse Signals

Let assume that we have not only a reduced set of measurements but also that they are received as noisy

$$\mathbf{y}_n + \boldsymbol{\varepsilon}_n = \mathbf{A}\mathbf{S}_N(t_n).$$

We will assume the variance of noise ε to be σ_ε^2. Noisy measurements will result in a noisy initial estimate $S_{N0}(t_n, k)$. Variance in $S_{N0}(t_n, k)$, caused by the measurements input noise, is $\sigma^2_{S_{N0}(t_n,k)} = N_A \sigma_\varepsilon^2$. Since the initial estimate is multiplied by N/N_A in the reconstruction, the noise variance in the reconstructed component is

$$\text{var}\{S_{NR}(t_n, k)\} = N_A \sigma_\varepsilon^2 \Big(\frac{N}{N_A}\Big)^2 = \frac{N^2}{N_A} \sigma_\varepsilon^2.$$

Since the noise is the same in each reconstructed coefficient, the total mean squared error (MSE) in K reconstructed coefficients is [14]

$$\|\mathbf{S}_{NR}(t_n) - \mathbf{S}_{NK}(t_n)\|_2^2 = K \frac{N^2}{N_A} {\sigma_\varepsilon}^2. \quad (13)$$

TABLE I
TOTAL ERROR OF NON-NOISY STFT, WITH VARIOUS VALUES OF AVAILABLE SAMPLES N_A AND SPARSITY LEVELS K

	$N_A = N/2$		$2N/3$		$3N/4$	
K	**Stat.**	**Theory**	**Stat.**	**Theory**	**Stat.**	**Theory**
8	0.997	1.139	0.927	1.040	0.905	1.007
16	0.089	0.288	0.027	0.140	0.012	0.095
32	−0.541	−0.234	−0.586	−0.469	−0.594	−0.547
48	−0.599	−0.193	−0.639	−0.510	−0.647	−0.617

The error will then be calculated as

$$\|\mathbf{S}_{NK}(t_n) - \mathbf{S}_{NR}(t_n)\|_2^2 = \frac{K}{N_A} \|\mathbf{S}_N(t_n) - \mathbf{S}_{NK0}(t_n)\|_2^2 + K \frac{N^2}{N_A} \sigma_\varepsilon^2. \quad (14)$$

Numerical validation of presented error expression will be done in the next section.

IV. Examples

Example 1: Let us consider a non-noisy signal with three linear frequency modulated (LFM) components

$$\begin{aligned} x(t_n) = 1.5 \exp\left(j130\pi t_n/N + j36\pi t_n^2/N^2 + j\varphi_1\right) \\ + \exp\left(j30\pi t_n/N + j16\pi t_n^2/N^2 + j\varphi_2\right) \\ + \exp\left(j380\pi t_n/N - j16\pi t_n^2/N^2 + j\varphi_3\right) \end{aligned} \quad (15)$$

sampled at 1280 random instances $0 \leq t_1 < t_2 < \ldots < t_{1280} \leq 1280$. The STFT is calculated using a Hann window of length $N = 256$ with a step in time of 32. Note that the signal is not sparse in the DFT domain since its components occupy almost the whole frequency range. The phases φ_1, φ_2 and φ_3 are random between 0 and 2π.

For the comparison, we will use total error energy. That is, we will consider the whole nonsparse signal, not only the error in the K components. The total energies of errors are then calculated as

$$E_s = 10 \log \left(\|\mathbf{S}_N(t_n) - \mathbf{S}_{NR}(t_n)\|_2^2 \right) \quad (16)$$

$$E_t = 10 \log \Big(\big(\frac{K}{N_A} + 1\big) \|\mathbf{S}_N(t_n) - \mathbf{S}_{NK0}(t_n)\|_2^2 \Big), \quad (17)$$

where E_s is the statistical error, and E_t is the derived (theoretical) error. Table I represents the average values in 100 realizations of error with varying sparsity K and number of available samples N_A.

Example 2: Let us consider the signal (15) with some additive noise. The available signal samples with noise are

$$z(t_n) = x(t_n) + \varepsilon(t_n). \quad (18)$$

We assume the same parameters as in Example 1, i.e. $0 \leq t_n \leq 1280$, a Hann window of length $N = 256$ with step 32. The noise is zero-mean Gaussian noise with a standard deviation of $\sigma_\varepsilon = 0.1$. The randomly sampled STFT with all the measurements is shown in Fig. 1(left). We assume that the number of available samples per window is $N_A = 2N/3$. The

ISBN 978-90-827970-1-5

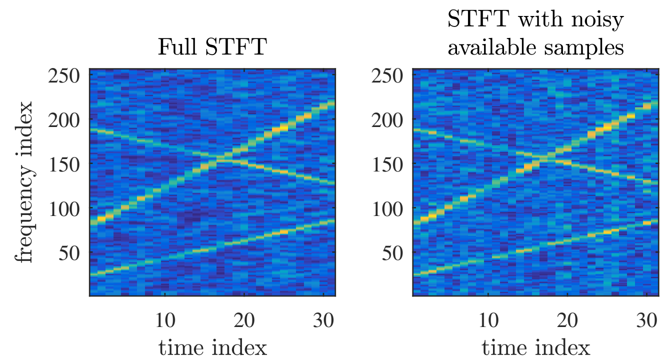

Fig. 1. Original STFT when randomly sampled (left); Initial estimate of the STFT with $2N/3$ noisy available samples per window (right)

TABLE II
TOTAL ERROR OF NOISY STFT, WITH STANDARD DEVIATION OF NOISE $\sigma_\varepsilon = 0.1$, AND VARIOUS VALUES OF N_A AND K

	$N_A = N/2$		$2N/3$		$3N/4$	
K	**Stat.**	**Theory**	**Stat.**	**Theory**	**Stat.**	**Theory**
8	1.006	1.147	0.932	1.045	0.909	1.010
16	0.103	0.298	0.034	0.145	0.017	0.100
32	−0.498	−0.210	−0.562	−0.455	−0.573	−0.533
48	−0.480	−0.148	−0.572	−0.482	−0.589	−0.592

initial estimate of the STFT with only the noisy available samples is shown in Fig. 1(right). Illustration of the reconstructed STFT with various assumed sparsities $K = [8, 16, 32, 48]$ is shown in Fig. 2. Using eq. (16) for statistical error and

$$E_t = 10\log\Big(\big(\frac{K}{N_A}+1\big)\,\|\mathbf{S}_N(t_n) - \mathbf{S}_{NK0}(t_n)\|_2^2 + K\frac{N^2}{N_A}\sigma_\varepsilon^2\Big),$$

for theoretical error, Table II examines the total error values in the reconstruction when K and N_A are varied. The values are averaged over 100 realizations.

V. Conclusions

The exact error in the reconstruction of randomly sampled nonsparse nonstationary signals is examined. Since the signal is nonstationary, the signal is assumed to be sparse in the joint time-frequency representation domain. The signal is reconstructed by a reduced set of noisy available samples defined by the compressive sensing framework. The theoretical error agrees with the statistical error calculation.

References

[1] B. Boashash, *Time Frequency Signal Analysis and Processing: A Comprehensive Reference.* 2^{nd} edition, Elsevier, 2016.
[2] L. Cohen, *Time-Frequency Analysis - Theory and Applications.* Prentice-Hall, 1995.
[3] E. Sejdić, “Time-Frequency Compressive Sensing,” in Boashash, B. (Ed.) *Time-Frequency Signal Analysis and Processing*, Academic Press, pp. 424–429, 2016.
[4] D. Malnar, V. Sučić, J. M. O’Toole, “Automatic quality assessment and optimisation of quadratic timefrequency representations,” *Electronics Letters*, vol.51, no.13, pp.1029–1031, 2015.
[5] C. Richard, “Time-frequency-based detection using discrete-time discrete-frequency Wigner distributions,” *IEEE Transactions on Signal Processing*, vol.50, no.9, pp.2170–2176, Sep 2002.
[6] L. Stanković, M. Daković, T. Thayaparan, Time-Frequency Signal Analysis with Applications, Artech House, Boston, March 2013
[7] D. L. Donoho, “Compressed sensing,” IEEE Trans. Information Theory, vol. 52, no. 4, pp. 1289–1306, 2006.
[8] E. Sejdić, A. Cam, L. F. Chaparro, C. M. Steele, T. Chau, “Compressive sampling of swallowing accelerometry signals using TF dictionaries based on modulated discrete prolate spheroidal sequences,” EURASIP Journal on Advances in Signal Processing, 2012.
[9] E. J. Candès, M. B. Wakin, “An Introduction to Compressive Sampling,” IEEE Signal Processing Magazine, vol. 21, March 2008.
[10] P. Flandrin, P. Borgnat, “Time-Frequency Energy Distributions Meet Compressed Sensing,” IEEE Transactions on Signal Processing, vol. 58, no. 6, pp. 2974–2982, 2010.
[11] D. Needell, J. A. Tropp, “CoSaMP: Iterative signal recovery from incomplete and inaccurate samples,” Applied and Computational Harmonic Analysis, vol. 26, no. 3, pp. 301–321, 2009.
[12] I. Volarić, V. Sučić, Z. Car, “A compressive sensing based method for cross-terms suppression in the time-frequency plane,” IEEE 15th International Conference on Bioinformatics and Bioengineering, pp. 1–4, 2015.
[13] L. Stanković, S. Stanković, M. G. Amin, “Missing Samples Analysis in Signals for Applications to L-Estimation and Compressive Sensing,” Signal Processing, Vol. 94, pp. 401–408, January 2014.
[14] L. Stanković, M. Daković, I. Stanković, S. Vujović, “On the Errors in Randomly Sampled Nonsparse Signals Reconstructed With a Sparsity Assumption,” *IEEE Geoscience and Remote Sensing Letters*, vol. 14, no. 12, pp. 2453–2456, December 2017.
[15] E. J. Candes, J. Romberg, T. Tao, “Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information,” IEEE Transactions on Information Theory, vol. 52, no. 2, pp. 489–509, Feb. 2006.
[16] H. Hassanieh, P. Indyk, D. Katabi, E. Price, “Simple and practical algorithm for sparse Fourier transform,’” *Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms*. Society for Industrial and Applied Mathematics, pp. 1183–1194, 2012.
[17] K. D. Ba, P. Indyk, E. Price, D. P. Woodruff, “Lower bounds for sparse recovery,” Proceedings of the twenty-first annual ACM-SIAM symposium on Discrete Algorithms. SIAM, pp. 1190–1197, 2010.
[18] I. Stanković, C. Ioana, M. Daković, “On the reconstruction of nonsparse time-frequency signals with sparsity constraint from a reduced set of samples,” Signal Processing, vol. 142, January 2018, pp. 480–484, DOI: http://dx.doi.org/10.1016/j.sigpro.2017.07.036

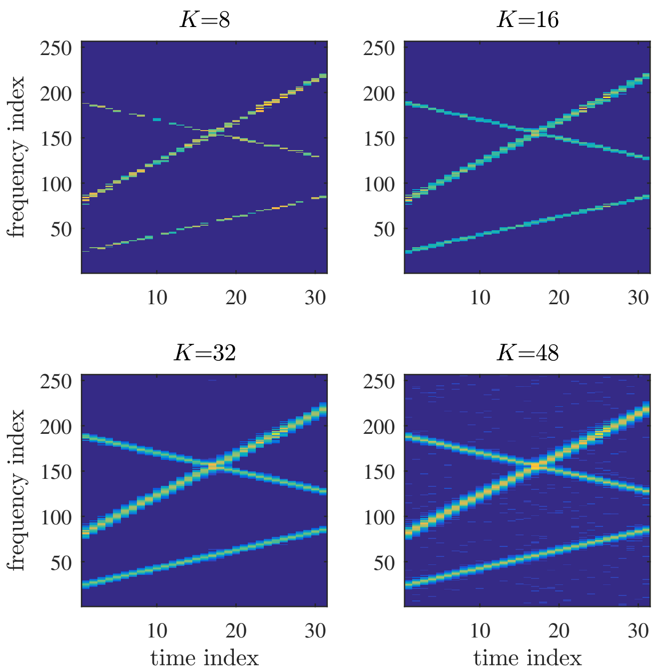

Fig. 2. Reconstruction of STFT with various values of assumed sparsity K.