Automatic Differentiating Wave Digital Filters with Multiple Nonlinearities

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***Abstract*—In recent years, there has been an increasing interest in real-time Virtual Analog modeling algorithms for musical audio signal processing as Wave Digital (WD) realizations, where nonlinear circuit components are identified as the source of desirable sonic characteristics. The implementation of nonlinear elements in Wave Digital Filters (WDFs) is usually restricted to just one nonlinear one-port per structure. However, recent advances led to a strictly modular realization of multiple nonlinearities by identification of contractivity properties of WDFs, allowing an assured converging iterative procedure, which depends on the presence of artificial port resistances. In this paper, an enhanced semi-modular approach based on Multi-dimensional WDFs (MDWDFs) is presented for the realization of multiple nonlinearities. Therefore, the concept of Automatic Differentiating WDFs (ADWDFs) is extended for nonlinearities as a novel approach to overcome convergence speed limitations induced by said port resistances, making the proposed method virtually independent of the same.**

***Index Terms*—Wave Digital Filter, Automatic Differentiation, Multi-Dimensional, Multiple Nonlinearities, Diode Clipper**

I. Introduction

Wave Digital Filters (WDFs) [1] have been recently widely used in the audio signal processing field for real-time Virtual Analog modeling of nonlinear Musical Circuits (e.g. Guitar distortion) containing linear and nonlinear elements such as Diode Clippers [2]–[5], Diode-based Ring Modulators [3], [6], [7], and Transistor-based circuits [8], [9].

In general, just a single one-port with a direct reflection, e.g. a static nonlinearity like a diode, can be connected to a Wave Digital (WD) structure without causing implicit relations, i.e. delay-free loops. This limits the applicability of the classic WD approach for a lot of circuit simulations, particularly nonlinear ones. There are specialized approaches to the realization of WDFs with multiple static nonlinearities [3], [8], [9], but these approaches have often in common that they require the network to be decomposed and reformulated, thus omitting the modularity property of the original WD approach [1]. Regarding iterative approaches, [2] is extended in [8] for handling multiple nonlinearities using Newton's method, whereby the linear part of the circuit is separated from the nonlinear one and delay-free loops arising in the nonlinear part are solved using Newton's method in a semi-modular fashion with a generalization for handling groups of nonlinearities presented in [4]. In [10] a fixed-point iteration method, called Scattering Iterative Method (SIM), for solving circuits with multiple nonlinearities in the WD domain is shown, whereby the speed of convergence depends on the value of port resistances. SIM has been used for Virtual Analog modeling of audio circuits with multiple nonlinearities in [6]. Later on, it has been extended to dynamic circuits with multiple nonlinearities in [11].

Our approach focuses on a Multi-Dimensional WDF (MD-WDF) realization of multiple nonlinearities, whereby an implicit correspondence that occurs by connecting a nonlinearity to a reflecting port is used to resolve the computability issue by inserting a delay element T_d of an artificial dimension $d = 2$. As has been shown in [5], [12], [13] by means of a generic MDWDF approach, if the circuit is contractive and the nonlinearity is at least non-expansive, the resulting structure does converge under iteration and, in absence of reactive components, actually approaches the correct limit value for constant input. Unfortunately, convergence speed strongly depends on impedance-matching of artificial port resistances, similar as in [10], which have to be determined beforehand for optimal operation and real-time capabilities.

In the present paper, an enhanced MDWDF approach is proposed, which considerably reduces the dependency of the system's behavior when choosing all involved artificial port resistances, and additionally converges significantly faster (quadratically) than other existing MDWDF approaches [5], [14]. Therefore, the concept of Automatic Differentiating Wave Digital Filters (ADWDFs) which we proposed in [15] is extended to determine the system's behavior for multiple nonlinearities in order to replace the fixed point iteration scheme by Newton's method. An appropriate diode model is studied as a novel application to nonlinear ADWDFs, which is verified in a concrete example.

A. Generic MDWDF model for Multiple Nonlinearities

The generic realizable MDWDF model for multiple nonlinearities from [12] is depicted in Figure 1 and will be in the forthcoming referred to as the original approach. The model comprises vector-based inputs/outputs and is generally suitable for a solution of an arbitrary number of N nonlinearities. The (prior delay-free, non-computable) looped paths $\boldsymbol{r} = [r_1, \ldots, r_N]^T$ have been pulled out of the system as well as all ξ reactive components, connected via $\boldsymbol{w} = [w_1, \ldots, w_\xi]^T$. A number of α terminated inputs and β outputs of the system are denoted as $\boldsymbol{x} = [x_1, \ldots, x_\alpha]^T$ and $\boldsymbol{y} = [y_1, \ldots, y_\beta]^T$, respectively. For a constant simulation length of K steps

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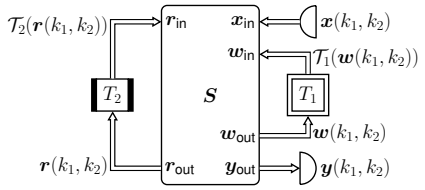


Fig. 1: Generic realizable MDWDF model with multiple nonlinearities from [12].

w.r.t. the artificial t_2-direction, whereas the time t_1 remains unbounded, inputs $\boldsymbol{x}_{\text{in}}$ as well as values $\boldsymbol{w}_{\text{in}}$ of delays in t_1-direction are held constant along the t_2-axis according to

$$\mathcal{T}_1(\boldsymbol{w}(k_1,k_2)) = \boldsymbol{w}(k_1-1,K) \quad ,k_2 = 0,\ldots,K. \tag{1}$$

Delays in t_2-direction re-iterate the WD structure, approaching a fixed point which is fed back via

$$\mathcal{T}_2(\boldsymbol{r}(k_1,k_2)) = \begin{cases} \boldsymbol{r}(k_1,k_2-1) & ,k_2 = 1,\ldots,K \\ \boldsymbol{r}(k_1-1,K) & ,k_2 = 0 \end{cases} \tag{2}$$

as a starting value to the next t_1-sample's fixed point iteration. As an extension to the generic MDWDF approach, a pseudo-secant method (PSM) has been presented in [14], utilizing a diagonal approximation of the Jacobian matrix in a damped Newton-like fashion, which converges faster (super-linearly) than the pure relaxation (fixed point iteration).

B. Solving Multiple Nonlinearities using Newton's method

Based on the original approach and throughout iteration over k_2 with $\boldsymbol{x}_{\text{in}}$ and $\boldsymbol{w}_{\text{in}}$ held constant, a corresponding system is considered describing the input/output relation of wave variables $\boldsymbol{r} = \boldsymbol{r}_{\text{in}} \in \mathbb{R}^N$ within the looped paths as

$$\boldsymbol{f}_{\boldsymbol{r}}(\boldsymbol{r}) = \boldsymbol{S}_{\boldsymbol{r}}\boldsymbol{r} + \boldsymbol{f}_{\text{NL}}(\boldsymbol{r}) = \boldsymbol{r}_{\text{out}}, \tag{3}$$

whereby $\boldsymbol{S}_{\boldsymbol{r}} \in \mathbb{R}^{N\times N}$ describes the linear part, and $\boldsymbol{f}_{\text{NL}}(\boldsymbol{r}) \in \mathbb{R}^N$ describes the nonlinear part of the system w.r.t. N connected nonlinearities. Considering an converging equilibrium as a fixed point iteration over k_2, thus

$$\boldsymbol{r}(k_2+1) = \boldsymbol{r}(k_2) \quad \Leftrightarrow \quad \boldsymbol{r}(k_2+1) - \boldsymbol{r}(k_2) = \boldsymbol{0}\,, \tag{4}$$

the system introduced in (3) becomes

$$\boldsymbol{f}_{\boldsymbol{r}}(\boldsymbol{r}) - \boldsymbol{I}_N\boldsymbol{r} = \boldsymbol{r}_{\text{out}} \tag{5}$$

with $\boldsymbol{I}_N$ as the $N \times N$ identity matrix. Thereby, fixed point iteration over k_2 is replaced by solving the system described by (5) with the solution of wave variables $\boldsymbol{r}^*$, satisfying the equality condition of (4). Since an analytical solution of (5) is not possible, an approximation of $\widetilde{\boldsymbol{r}} \approx \boldsymbol{r}^*$ is achieved by means of Newton's method iterating over k_2, thus $\boldsymbol{r}(k_2+1) = \boldsymbol{r}(k_2) + \Delta\boldsymbol{r}(k_2)$. A Newton step $\Delta\boldsymbol{r}(k_2)$ is determined by solving

$$\boldsymbol{F}_{\boldsymbol{r}}\left(\boldsymbol{r}(k_2)\right)\Delta\boldsymbol{r}(k_2) = \boldsymbol{r}_{\text{out}}(k_2) \tag{6}$$

with

$$\boldsymbol{F}_{\boldsymbol{r}}(\boldsymbol{r}) = \boldsymbol{S}_{\boldsymbol{r}} + \boldsymbol{F}_{\text{NL}}(\boldsymbol{r}) - \boldsymbol{I}_N\,, \tag{7}$$

whereby $\boldsymbol{F}_{\text{NL}}(\boldsymbol{r})$ is the Jacobian matrix of nonlinearities $\boldsymbol{f}_{\text{NL}}(\boldsymbol{r})$ and $\boldsymbol{S}_{\boldsymbol{r}}$ the constant scattering matrix of the linear part, whose determination is presented in the following section.

II. ADWDFs with Multiple Nonlinearities

Automatic Differentiating WDFs (ADWDFs) have been presented in [15] as a technique to improve convergence speeds of linear WD structures with topology-related delay-free loops, making the former independent of artificial port resistances. Moreover, it has been shown that WDFs are capable of computing first-order partial derivatives of steady-state wave variables without any additional computational blocks to determine the system's linear behavior described by a static scattering matrix $\boldsymbol{S}_{\boldsymbol{r}}$. As an extension to linear ADWDFs, a novel approach to WDFs containing multiple nonlinearities is presented in this Section. Prior, however, a brief introduction to AD and linear ADWDFs will be given.

The concept of AD is a widely used set of techniques to evaluate the derivation of a function alongside the value of the function, and represents an independent field of research. Since differentiation of functions defined by formulas is a process done according to fixed rules, it is highly suitable for automation along the same lines as function evaluation, especially if an analytical description of a function $f(x)$ is not required or unknown and only a certain derivative $f'(\hat{x})$ is needed for a given $\hat{x}$. Based on general assumptions stating that an analytical description $f'(x)$ is not available at run-time in order to compute a Newton step, AD is often not considered by authors.

Crucial to AD is the decomposition of differentials provided by the chain rule, which usually can be applied using forward or reverse accumulation. The former traverses the chain rule from inside to outside while for the latter it is done vice versa. As an example, consider the function $b : \mathbb{R}^2 \to \mathbb{R}$ with input $\boldsymbol{a} = [a_1, a_2]^T$ as a sequential step by step evaluation in forward accumulation mode given by $b(\boldsymbol{a}) = a_1a_2 + \exp(a_1+a_2) = v_1v_2 + \exp(v_1+v_2) = v_3 + \exp(v_4) = v_3 + v_5 = v_6$, whereby individual sub-expressions have been noted by variables v_i. In order to concurrently compute partial derivatives v_i' with respect to, e.g. a_1, variables v_i' are initialized to so-called seed values according to $v_1' = \partial a_1/\partial a_1 = 1$ and $v_2' = \partial a_2/\partial a_1 = 0$. In-line function evaluation and partial derivation computation steps for the given example are composed as follows:

Step	Value:	Derivative*:
1a	$v_1 = a_1$	$v_1' = 1$
1b	$v_2 = a_2$	$v_2' = 0$
2	$v_3 = v_1 \cdot v_2$	$v_3' = v_2 \cdot v_1' + v_1 \cdot v_2'$
3	$v_4 = v_1 + v_2$	$v_4' = v_1'$
4	$v_5 = \exp(v_4)$	$v_5' = \exp(v_4) \cdot v_4'$
5	$v_6 = v_3 + v_5$	$v_6' = v_3' + v_5'$
6	$b = v_6$	$\partial b/\partial a_1 = v_6'$

*Analogously for $\partial b/\partial a_2$ with seeds $v_1' = 0$, $v_2' = 1$.

From a software design perspective AD is usually implemented using operator overloading [16] by augmenting the algebra of real numbers and obtaining a new arithmetic which consists of ordered entries, elements written $\langle v, v', v'', \ldots\rangle$, with ordinary arithmetics on the first component, first-order differentiation arithmetic on the second component, and so

forth. However, this is only required for a concurrent computation of partial derivatives alongside function evaluation and would involve additional computational blocks besides the underlying WD structure. Instead, we follow a two-step approach in which the calculation of partial derivatives and function evaluation are carried out separately, utilizing the same WD structure.

A. Linear ADWDFs

WDF building blocks, especially adaptors, are composed of linear operations, namely additions and multiplications by constant values (coefficients) [1]. Obviously, any input of a block is either directly connected to a constant multiplier or a multi-port adder as first stage operational. Therefore, two scenarios of a function sub-evaluation v_i and its first-order partial derivative are possible. Summation of L independent wave variables a_l and coefficient multiplication of a_l by c w.r.t. first-order partial derivatives of wave variable a_m are given by

$$v_i = \sum_{l=1}^{L} a_l, \qquad \frac{\partial v_i}{\partial a_m} = \begin{cases} 1 & , m = l \\ 0 & , m \neq l \end{cases}, \tag{8a}$$

$$v_i = c \cdot a_l, \qquad \frac{\partial v_i}{\partial a_m} = \begin{cases} 1 \cdot c & , m = l \\ 0 & , m \neq l \end{cases}. \tag{8b}$$

According to a seed value initialization of inputs as

$$v_i' = \frac{\partial a_l}{\partial a_m} = \begin{cases} 1 & , m = l \\ 0 & , m \neq l \end{cases}, \tag{9}$$

subsequent multi-port addition or constant multiplication of partial derivatives w.r.t. a_l as described in (8a) and (8b), respectively, are properly traversed trough the WDF arithmetic blocks in forward accumulation mode and performing AD for a distinct wave variable a_l while maintaining the chain rule.

In order to obtain the full Jacobian matrix a Seed Initialization Procedure (SIP) according to (9) is applied, whereby N independent wave variables are assumed. Inputs $\boldsymbol{r}_{\text{in}}$ are set to unit vectors $\boldsymbol{e}_n$, $n \in 1, 2, \ldots, N$ as

$$\boldsymbol{e}_n = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}^T \tag{10}$$

with a value of 1 at the n-th position of $\boldsymbol{e}_n$, otherwise 0, to obtain first-order partial derivatives for $\boldsymbol{r}_{\text{out}}$ as $\boldsymbol{r}_n$ w.r.t. the n-th wave variable, whereby

$$\boldsymbol{x}_{\text{in}} = \boldsymbol{0}, \quad \boldsymbol{w}_{\text{in}} = \boldsymbol{0}. \tag{11}$$

Simply said, every input of the WDF which is not a function of a particular n-th wave variable is set to zero, otherwise one, which is done for every $\boldsymbol{e}_n$ from (10), therefore requiring N evaluations of the WDF. Static Jacobian matrix is then composed as

$$\boldsymbol{S}_{\boldsymbol{r}} = \begin{bmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \cdots & \boldsymbol{r}_N \end{bmatrix}^T \tag{12}$$

of consecutive outputs $\boldsymbol{r}_n$, $n \in 1, 2, \ldots, N$. Determination of the Jacobian matrix for a given WD structure utilizing the structure itself by appropriately setting inputs according to (10) and (11) without altering or extending the structure is whats defining the term ADWDF.

B. Diode Model

Diodes are to be implemented according to the Shockley diode model [17] as

$$I_D = oI_s \left[\exp\left(\frac{oU_D}{U_t n_e} \right) - 1 \right], \tag{13}$$

whereby $o \in \{-1, 1\}$ describes the orientation, reverse or forward, respectively, with saturation current I_s, thermal voltage U_t, and ideality factor n_e. To obtain a closed-form WD model of (13), we apply the so-called Lambert $\mathcal{W}$ function which has already been successfully used to model diodes as an explicit WD equation [3], [7]. Hereby, a single parameterizable model can be used for various exponential circuit models, such as diode configurations in backward and forward orientation, utilizing the same lookup table.

The Lambert $\mathcal{W}$ function is defined as a multivalued function [18] that satisfies $z = \mathcal{W}(z) \exp(\mathcal{W}(z))$, $z \in \mathbb{C}$, whereby only two real branches $\mathcal{W}_0, \mathcal{W}_{-1} \in \mathbb{R}$ are considered. A mapping between Kirchhoff (voltages U and currents I) and WD-domain (incident a and reflective b waves) utilizes

$$a = U + IR_p\,, \quad b = U - IR_p\,, \tag{14}$$

where $R_p > 0$ is the freely selectable port resistance [1]. Inserting (14) in (13) leads to

$$\frac{a-b}{2R_p} = oI_S \cdot \left[\exp\left(\frac{o(a+b)}{2U_t n_e} \right) - 1 \right], \tag{15}$$

resulting in

$$b = a + 2oI_s R_p - 2oU_t n_e \mathcal{W}\left[\frac{I_s R_p}{U_t n_e} \exp\left(\frac{a + oI_s R_p}{oU_t n_e} \right) \right], \tag{16}$$

as an explicit diode model wave mapping [18], [19]. Derivative of (16) is likewise available as a closed-form expression [18] according to

$$\frac{\partial b}{\partial a} = 1 - \frac{2\mathcal{W}(\hat{b})}{\mathcal{W}(\hat{b}) + 1}, \quad \hat{b} = \frac{I_s R_p}{U_t n_e} \exp\left(\frac{a + oI_s R_p}{oU_t n_e} \right), \tag{17}$$

hence utilizing the same lookup table.

Regarding an AD procedure of a nonlinear function sub-evaluation step v_i of the diode model for an independent wave variable a_l and its partial derivative w.r.t. a_m are given by

$$v_i = b_l(a_l) = \text{ according to (16)}\,, \tag{18a}$$

$$\frac{\partial v_i}{\partial a_m} = \begin{cases} \text{according to (17)} & , m = l \\ 0 & , m \neq l \end{cases}. \tag{18b}$$

C. Seed Initialization Procedure

In order to obtain the (combined, nonlinear and linear) Jacobian matrix, a SIP is proposed as depicted in Figure 2a whereby N independent variables for incident or reflected waves are assumed. As an extension to the SIP of linear ADWDFs (see Section II-A), N connected nonlinearities are assumed given by $\boldsymbol{f}_{\text{NL}}(\boldsymbol{r}) \in \mathbb{R}^N$ and their (assumed to exist) associated Jacobian matrix $\boldsymbol{F}_{\text{NL}}(\boldsymbol{r}) \in \mathbb{R}^{N \times N}$.

In contrast to purely linear ADWDFs, the resulting Jacobian matrix is not static and depends on $\boldsymbol{r}$, thus has to be computed

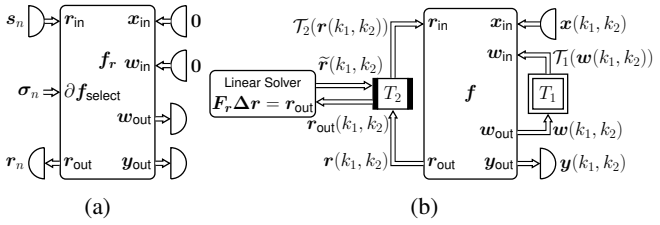


Fig. 2: (a) Proposed SIP of the ADWDF containing multiple nonlinearities. (b) Proposed enhanced MDWDF approach.

prior to every iteration step k_2, requiring N initial evaluations of the ADWDF. Furthermore, partial derivatives of nonlinear functions must be available during SIPs for the respective wave variable as output of all connected nonlinearities in order to perform an appropriate sub-evaluation step v_i, as e.g. for diodes (18). Since seed initializations are independent of each other, the procedure is highly parallelizable for $M \leq N$ simultaneous computational tasks evaluating M instances of the ADWDF at once. The very first seed values are set to $\boldsymbol{s}_n = \boldsymbol{e}_n$, $k_1 = 0$, $k_2 = 0$ for every $n = 1, \ldots, N$ according to (10). Simultaneously, a seed initialization selection vector

$$\boldsymbol{\sigma}_n = \begin{bmatrix} \frac{\partial f_{\text{NL},1}}{\partial a_n} & \frac{\partial f_{\text{NL},2}}{\partial a_n} & \cdots & \frac{\partial f_{\text{NL},N}}{\partial a_n} \end{bmatrix}^T, \; k_1 > 0, \; k_2 \geq 0 \quad (19)$$

selects the output of all nonlinearities as partial derivatives w.r.t the n-th wave variable being evaluated. Subsequent seed values are set to solutions $\widetilde{\boldsymbol{r}}$ of the previous Newton step $k_2 - 1$, $k_2 > 0$, in particular, $\boldsymbol{s}_n$ contains the n-th component of $\widetilde{\boldsymbol{r}}$ as the n-th component of $\boldsymbol{s}_n$, otherwise 0, hence

$$\boldsymbol{s}_n = \begin{bmatrix} 0 & \cdots & \widetilde{r}_{n,n} & \cdots & 0 \end{bmatrix}^T. \quad (20)$$

The (inseparable) Jacobian matrix is then composed out of outputs $\boldsymbol{r}_n$ after N evaluations as

$$\boldsymbol{S}_{\boldsymbol{r}} + \boldsymbol{F}_{\text{NL}}(\boldsymbol{r}) = \begin{bmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \cdots & \boldsymbol{r}_N \end{bmatrix}^T. \quad (21)$$

The resulting nonlinear ADWDF is identical to the underlying foundation WDF, whereby partial derivatives of the nonlinear functions are implemented to be selectable by (19) besides the actual function evaluation (cf. (18) for the diode), thereby forming an entirely new nonlinear WDF building block. The chain rule always remains intact as long as the input of a selected partial derivative nonlinearity e.g. $\partial/\partial a_n f_{\text{NL},1}(r_n)$ is directly initialized with the appropriate seed $\widetilde{r}_{n,n}$ during the SIP, hence $\partial/\partial a_n f_{\text{NL},n}(\widetilde{r}_{n,n})$, and no arithmetic blocks (e.g. adaptors) are located in between as a function e.g. $g(\widetilde{r}_{n,n})$, which would cause $\partial/\partial a_n f_{\text{NL},n}(g(\widetilde{r}_{n,n}))$ as opposed to the correct $\partial/\partial a_n f_{\text{NL},n}(g(\widetilde{r}_{n,n})) \cdot \partial/\partial a_n g(\widetilde{r}_{n,n})$, hence, violating the chain rule. This requirement is equivalent to the necessity of delays T_2 to be directly connected to the input of the respective nonlinearity, since the input/output relation of the system to be solved (3) is considered from the point of view of T_2.

III. Enhanced Multi-Dimensional Approach

An enhanced MDWDF approach is presented as depicted in Figure 2b, which is mainly based on the methods described in section II-C. The method of the original approach is adopted (see I-A) whereby fixed point iteration over k_2 is replaced by Newton's method iterating over k_2, while retaining the original delays T_1 and T_2 according to (1) and (2), respectively. A linear solver is connected to T_2 which determines the optimal convergence matrices (21) via ADWDFs prior to each iteration cycle by means of the proposed SIP and updates inputs as $\widetilde{\boldsymbol{r}}(k_1, k_2) = \boldsymbol{r}(k_1, k_2 - 1) + \boldsymbol{\Delta r}(k_1, k_2)$, for $k_2 = 1, 2, \ldots, K$ with a Newton step $\boldsymbol{\Delta r}(k_1, k_2)$ as the solution of the linear system. As already mentioned, our algorithm comprises two consecutive steps for every k_2 :

1) Determine $\boldsymbol{F_r}(\boldsymbol{r})$ by means of (21) using proposed ADWDF for every $\boldsymbol{r}_n$, $n = 1, \ldots, N$.
2) Evaluate WDF to obtain $\boldsymbol{r}_{\text{out}}$ and solve (6) for $\boldsymbol{\Delta r}$.

Although the linear solver is a global process, the WD structure remains at least semi-modular, since any modifications of a given structure are immediately taken into account during SIPs of the respective ADWDF in an automatic fashion.

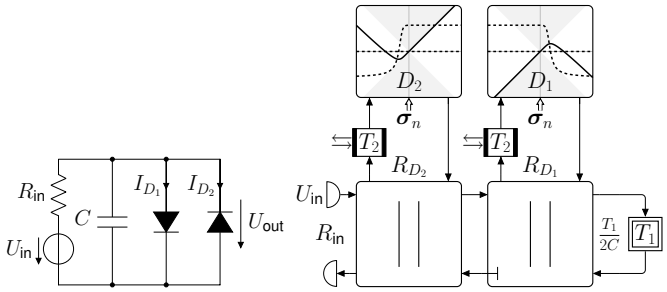
A. Reduced Convergence Matrices

Computational effort of the linear solver increases w.r.t. N connected nonlinearities, depending on the used method and matrix sparsity, which generally counteracts real-time capabilities. Therefore, reductions of $\boldsymbol{F_r}$, namely only triangular or diagonal components are considered w.r.t. convergence speed influences exemplarily in the subsequent Section. Thus, for a triangular (bottom left $\boldsymbol{L_r}$, upper right $\boldsymbol{R_r}$) or diagonal $\boldsymbol{D_r}$ reduction of $\boldsymbol{F_r}$, generally denoted as $\widetilde{\boldsymbol{F}}_{\boldsymbol{r}}$, remaining components of $\boldsymbol{F_r}$ are set to 0. Such reductions are feasible, assuming that $\rho(\boldsymbol{F_r}) < 1$, since the spectral radius consistently amounts to $\rho(\widetilde{\boldsymbol{F}}_{\boldsymbol{r}}) < \rho(\boldsymbol{F_r})$ for any of the mentioned reductions [20], hence convergence is assured, though, expected to be slower. However, a diagonal reduction enables a strictly modular approach similar to [13], [14], which does not require a global linear solver.

IV. Example: Diode Clipper

To give an example of the presented approach, the well-known diode clipper circuit is analyzed focusing on the influence of the choice of port resistance values on convergence speed. As depicted in Figure 3a, it consists of the parallel connection of a resistive voltage source with voltage U_{in} and internal resistance $R_{\text{in}} = 2.2\,\text{k}\Omega$, a capacitor $C = 470\,\text{nF}$ and a pair of anti-parallel diodes D_1 and D_2 that are, apart from their orientation, identical. Diodes are modeled according to (18), whereby $I_s = 2.52\,\text{nA}$, $U_t = 25.9\,\text{mV}$, and $n_e = 1.752$. Derivation of a WD structure has been performed using the proposed method and is depicted in Figure 3b. Delays T_2 are connected to the linear solver (indicated by double arrows) with port resistances $R_D = R_{D_1} = R_{D_2}$, whereby $R_D/2 = R_Z = R_{\text{in}} \parallel R_C$ (impedance matching). Diode outputs are implemented to be selectable w.r.t partial derivatives according to (19) as (18) (indicated by dashed lines) in addition to the actual wave mapping (solid line).

Simulations are compared for an amplified Heaviside step function input with a step size of 10 V, which easily drives the prototype circuit into a strongly nonlinear operating range.



(a) (b)

Fig. 3: (a) Prototype circuit of a diode clipper. (b) Proposed enhanced WD model of the diode clipper.

Choice of R_D	Maximum Number of Iterations					
	Reference [14]		Proposed			
	PR	PSM I	$\boldsymbol{F_r}$	$\boldsymbol{R_r}$	$\boldsymbol{L_r}$	$\boldsymbol{D_r}$
$100 \cdot R_Z$	397	10	7	7	7	8
$10 \cdot R_Z$	87	11	5	5	5	9
$1 \cdot R_Z$	11	19	2	8	8	15
$0.1 \cdot R_Z$	109	200	4	47	47	85
$0.01 \cdot R_Z$	1085	564	5	331	331	591
$10^{-10} \cdot R_Z$	-	-	5	-	-	-

TABLE I: Maximum number of iterations needed for the proposed approach compared to [14].

For different values of R_D, the number of maximum iterations required to make the respective simulations fall below a maximum absolute error smaller than 10^{-5} is presented in Table I for the proposed approach utilizing a fully populated convergence matrix $\boldsymbol{F_r}$ and reduced ones (bottom left triangular $\boldsymbol{L_r}$, upper right triangular $\boldsymbol{R_r}$, and diagonal $\boldsymbol{D_r}$) compared to results from [14] (Pure Relaxation (PR) and PSM I). Simulations have been performed using sampling rates $F_s = 40\,\text{kHz}$ with 10^4 samples in Matlab. The proposed approach achieves very fast convergence speeds with 2-7 maximum iterations utilizing a fully populated $\boldsymbol{F_r}$ and outperforms PR and PSM I on the whole range of different R_D, even for very remote $R_D = 10^{-10} \cdot R_Z$, for which no feasible results are available of the remaining methods due to very long simulation times. $\boldsymbol{R_r}$ and $\boldsymbol{L_r}$ achieve same results which perform slightly better than PSM I, whereas $\boldsymbol{D_r}$ performs slightly better than PSM I for $R_D \geq 0.1 \cdot R_Z$. Note that the proposed approach utilizes initially $N = 2$ evaluations of the ADWDF for the given example prior to the actual iteration cycle to determine convergence matrices. However, as already mentioned, due to a potential parallelization of the procedure the $N = 2$ initial steps should not be considered as an equivalent addition to the overall iteration steps required, hence neglected for the presented results.

V. Conclusion

In the present paper, an enhanced semi-modular approach based on MDWDFs is presented for the realization of multiple nonlinearities. Therefore, the concept of ADWDFs has been extended for nonlinearities to overcome convergence speed limitations induced by port resistances, making the proposed approach virtually independent of the same. An appropriate diode model has been studied as a novel application to nonlinear ADWDFs, which is verified in a concrete example.

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