

# A Lie-group based modelling for centroid and shape estimation of a cluster of space debris

Samy Labsir, Audrey Giremus  
Laboratory IMS, Bordeaux University  
Talence, France  
{samy.labsir, audrey.giremus}@u-bordeaux.fr

Brice Yver, Thomas Benoudiba–Campanini  
CEA-CESTA  
Le Barp, France  
{brice.yver, thomas.benoudiba-campanini}@cea.fr

**Abstract**—In the context of spatial surveillance, we are interested in estimating the outline and centroid position of a cluster of debris from a set of noisy sensor observations. The motion of the pieces of debris is completely driven by Kepler’s law, therefore they scatter taking a specific curvature. This spreading resembles that of samples drawn on the Lie group  $SE(3)$ . For this reason, we propose a reformulation of the cluster observation model on Lie groups to intrinsically capture its shape. Then, we derive an optimization algorithm on Lie group to solve the estimation problem. The presented approach is validated on simulated data and compared to a state-of-the-art method based on a Gaussian process modelling.

**Index Terms**—Space debris, extended target, Bayesian estimation, optimization on Lie group.

## I. INTRODUCTION

Spatial surveillance has become a crucial challenge over the last decade [1]–[4] due to the exponentially increasing number of space debris. They can be natural or artificial orbiting objects, which are no longer functional but can cause critical damages when colliding with operational spacecrafts [5][6]. The objective is to determine the position of the pieces of debris from imperfect sensor measurements. For this purpose, Bayesian inference methods [7][8] can be applied. In addition to the measurements, they rely on a prior probabilistic model of the parameters of interest. The pointwise estimates of the latter are based on their posterior distribution conditionally upon the observations.

This study addresses the early stages of the evolution of a cluster right after its formation consecutive, for instance, to the disintegration of a satellite or a rocket fragment. The different pieces of debris are then very close to each other and form a compact set. In this case, they can be considered as a single extended object and dedicated methods can be applied [9][10]. A target is assumed extended if it results in several sensor measurements spatially distributed within its volume. Different Bayesian approaches, generally in a dynamic framework, have been proposed to estimate the position of the object centroid and its shape. A seminal method, presented in [11], assumes that the object is ellipsoidal and parameterized by a symmetric positive definite (SPD) extent matrix. The measurements are then considered to be normally distributed around the centroid position and with a covariance directly related to that matrix. However, the ellipsoidal assumption is too restrictive in many

applications. Another more generic category of methods relies on a star-convex parametric representation of the object contour such as the random hypersurface model [12][14] or the Gaussian process model [15]. However, these approaches would not fully take advantage of the physics of the problem at hand. Indeed, as the pieces of debris are only subject to the gravitational force, their trajectories are constrained and they move away from each other taking a specific curvature [3][5][16]. This spreading resembles that of objects evolving on a manifold of the type Lie group (LG) and distributed according to a concentrated Gaussian distribution (CGD) [18][19].

In this study, we build upon this observation to adapt the random-matrix approach in [11] so as to capture the specific shape of the cluster of debris. For that purpose, the first contribution is to propose a new parameterization of the cluster on LG. It consists of its centroid position and a rotation matrix indicating its orientation with respect to a reference frame. Both of them can be concatenated in a matrix belonging to the *Special Euclidean Group*  $SE(3)$ . The second contribution is to define a new observation model which ensures the desired spatial distribution of the measurements. Following [11], it is characterized by an extent matrix, subsequently referred to as the shape matrix. The originality resides in the probability density function (pdf) of the measurements which is chosen to be a CGD. Preliminary results were presented in [17] but only the centroid position was estimated. In this communication, we also determine the surface of the cluster by estimating the shape matrix jointly with the remainder of the unknown variables. The latter is re-written as a product of elements defined on LGs so as to tackle jointly the estimation of all the parameters in the maximum *a posteriori* sense using a single optimisation algorithm on LG. A specific Newton method on LG is proposed that takes advantage of the structure of the error criterion to be minimized.

After this introduction, the necessary material on LGs for the subsequent theoretical developments is provided in section II. The proposed model and algorithm to estimate the shape and centroid position of a cluster of space debris are detailed in section III. The section IV illustrates the relevance of the developed approach by simulation results and includes a comparison with a state-of-the-art method. Finally, conclusions and perspectives are drawn in section V.

## II. BACKGROUND ON LIE GROUPS

In this section, we provide the necessary background on LG theory. Firstly, we introduce general mathematical definitions. Then, we focus on two topics: defining uncertainty on LGs through the formalism of CGDs and solving optimization problems on LGs.

### A. LG definition and properties

A LG  $G$  is a group equipped with a structure of smooth manifold. In our work, we deal with matrix LGs so that  $G \subset \mathbb{R}^{n \times n}$ . As a group, it is endowed with an internal law, an inverse and a neutral element. Furthermore, the smooth manifold property means that the group operations are differentiable [21][22]. Also,  $\forall \mathbf{X} \in G$  a tangent vector space to  $G$  at  $\mathbf{X}$ , denoted  $T_{\mathbf{X}}G$ , can be defined. The tangent space to the identity  $\mathbf{I}_n$  is paid a special attention and called the Lie algebra  $\mathfrak{g}$ . Indeed, due to the group structure,  $\forall \mathbf{Y} \in \mathfrak{g}$ , the tangent application  $L_{\mathbf{X}} : \mathbf{Y} \rightarrow \mathbf{Y} \mathbf{X}$  transports every element of  $\mathfrak{g}$  to  $T_{\mathbf{X}}G$ . The elements of  $\mathfrak{g}$  and  $G$  are related through the group exponential mapping  $\text{Exp}_G : \mathfrak{g} \rightarrow G$ , and its local inverse, the group logarithm mapping,  $\text{Log}_G : G \rightarrow \mathfrak{g}$ . For matrix groups,  $\text{Exp}_G$  and  $\text{Log}_G$  are merely the matrix exponential and logarithm. The dimension  $m$  of the manifold  $G$  is defined as the dimension of the vector space  $\mathfrak{g}$ . It represents the intrinsic number of degrees of freedom of the elements in  $G$ . Finally, we can define an isomorphism linking  $\mathfrak{g}$  and the Euclidean space  $\mathbb{R}^m$ . If  $\mathbf{a}$  is an element of  $\mathbb{R}^m$  and  $\mathbf{a}$  its image in  $\mathfrak{g}$ , we have  $\mathbf{a} = [\mathbf{a}]_G^\wedge$  and  $\mathbf{a} = [\mathbf{a}]_G^\vee$ , respectively. This isomorphism enables to manipulate  $m$ -size vectors instead of  $n \times n$  matrices, which is computationally more efficient.

For the sake of brevity, we use the summarized notations  $\mathbf{a} = \text{Log}_G^\vee(\mathbf{X}) = [\text{Log}_G(\mathbf{X})]^\vee$  and  $\mathbf{X} = \text{Exp}_G^\wedge(\mathbf{a}) = \text{Exp}_G([\mathbf{a}]_G^\wedge)$ , with  $\mathbf{X} \in G$  and  $\mathbf{a} \in \mathbb{R}^m$ .

### B. Defining uncertainty on LGs

Bayesian approaches are based on a probabilistic formulation of estimation problems. When the parameters of interest are defined on LGs, it is thus necessary to consider pdfs that intrinsically account for this constraint. Different manifold-defined pdfs have been proposed in the literature such as the Von Mises-Fisher, the Fisher-Bingham [23] and the Riemannian Gaussian pdfs [24]. The formalism of CGDs has the advantage of applying to any LG. These distributions exhibit nice properties and can be interpreted as generalizations on LGs of the multivariate Gaussian pdf in the sense that they maximize the entropy under constraints on the mean and covariance [25][26][27]. Concentrated Gaussian distributed vectors can be obtained by mapping Euclidean vectors. If  $\epsilon$  is a centered and  $m$ -dimensional Gaussian-distributed vector with covariance  $\mathbf{P}$  and  $\mu$  is an element of the LG  $G$ , then  $\mathbf{X} = \mu \text{Exp}_G^\wedge(\epsilon)$  is distributed according to a left<sup>1</sup> CGD on a LG  $G$ , with parameters  $\mu$  and  $\mathbf{P} \in \mathbb{R}^{m \times m}$ . We denote  $\mathbf{X} \sim \mathcal{N}_G^L(\mu, \mathbf{P})$ . The generation process of CGD variables, as

<sup>1</sup>Note that a right CGD could be defined in a similar manner by switching  $\mu$  and  $\text{Exp}_G^\wedge(\epsilon)$  in the definition of  $\mathbf{X}$ .

well as the mappings between a LG and its locally isomorph vector space are illustrated in figure 1. In the neighborhood of  $\mu$ , this pdf can be approximated as follows:

$$p(\mathbf{X}) \simeq \frac{1}{\sqrt{(2\pi)^p |\mathbf{P}|}} \exp^{-\frac{1}{2} \|\text{Log}_G^\vee(\mu^{-1} \mathbf{X})\|_{\mathbf{P}}^2}. \quad (1)$$

In the sequel,  $p(\mathbf{X})$  is denoted  $\mathcal{N}_G^L(\mathbf{X}; \mu, \mathbf{P})$ .

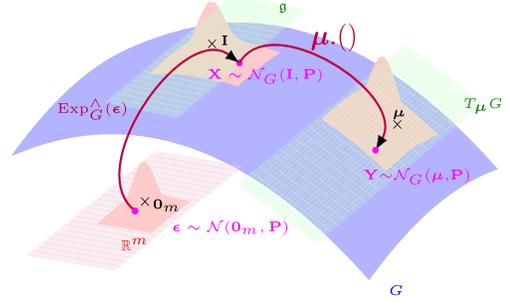


Fig. 1. Link between  $G$ ,  $\mathfrak{g}$  and  $\mathbb{R}^m$  and associated uncertainties.

### C. Optimization on LGs

In a Bayesian framework, the pointwise estimates of the unknown parameters are computed as the modes of their posterior distribution conditionally upon the observations. It often amounts to solving a minimization issue. Consequently, various dedicated optimization algorithms on LGs were developed in the literature. In [28][29], a generalization of the Newton algorithm on LGs was proposed and a Gauss-Newton on LGs (LG-GN) was introduced in [30]. These approaches require to define the notion of first and second derivative of a function on LGs. Let  $f : G \rightarrow \mathbb{R}$  be a smooth function. The right Lie derivative of  $f$  is defined by [27]:

$$\nabla f_X = \left. \frac{\partial (f(\mathbf{X} \text{Exp}_G^\wedge(\mathbf{a})))}{\partial \mathbf{a}} \right|_{\mathbf{a}=\mathbf{0}} \quad (2)$$

Note that for a multidimensional function, we can define its Jacobian matrix in the same way.

The second derivative makes sense by introducing the notion of affine connection which is required to differentiate elements of vector fields. If the symmetric Cartan-Schouten connection is considered, then the second derivative can be written as [29]:

$$\mathbf{H}_X = \left. \frac{\partial^2 (f(\mathbf{X} \text{Exp}_G^\wedge(\mathbf{a}_1) \text{Exp}_G^\wedge(\mathbf{a}_2)))}{\partial \mathbf{a}_1 \partial \mathbf{a}_2} \right|_{\mathbf{a}_1=\mathbf{0}, \mathbf{a}_2=\mathbf{0}} \quad (3)$$

For mathematical details on affine connections and the Riemannian geometry, the reader can refer to [31][32].

## III. PROPOSED CLUSTER FITTING APPROACH

### A. Cluster modelling on LG

As already mentioned in [5][16], we could observe through simulations that a set of spatial objects whose motion is only constrained by the gravitational force evolves taking a specific banana-shape illustrated in [17]. This spreading mimics that of a set of samples distributed on the LG *special Euclidean*

$SE(3)$  according to a CGD [18][19]. Thus, to capture the shape of the cluster, we propose an original formulation of the measurement model on LGs. It is based on a parameterization of the cluster characterized, on the one hand by the position  $\mathbf{p}$  of its centroid and, on the other hand, by a rotation matrix  $\mathbf{R}$ . The latter represents the transition matrix between a local coordinate system attached to the centroid and a reference global coordinate system.  $\mathbf{R}$  and  $\mathbf{p}$  can be gathered in a matrix defined on  $SE(3)$  as follows:

$$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix}. \quad (4)$$

The elements of  $SE(3)$  are classically used to represent rigid transformations and  $\mathbf{p}$  is referred to as the translation component.

At a given instant, only  $n$  reflectors are detected by the radar sensor. To properly represent their dispersion, we assume that they are distributed around the centroid  $\mathbf{M}$  according to a CGD:

$$\mathbf{Z}_i = \mathbf{M} \text{Exp}_{SE(3)}^{\wedge}(\boldsymbol{\epsilon}_i), \text{ for } i = 1, \dots, n, \quad (5)$$

where  $\boldsymbol{\epsilon}_i$  is a centered Gaussian vector with covariance  $\mathbf{S}$ . In this way, the "banana"-like scattering is ensured and the cluster properties (scale, orientation, volume) are completely characterized by the unknown SPD matrix  $\mathbf{S}$ . Since the radar only provides the position of the reflectors, the rotation components of the matrices  $\mathbf{Z}_i$  are not observable. The actual observation model finally takes the form:

$$\mathbf{z}_i = \Pi(\mathbf{Z}_i) + \mathbf{u}_i, \quad (6)$$

where for  $i = 1, \dots, n$ ,  $\mathbf{u}_i$  is the sensor noise that is assumed centered and with a covariance  $\mathbf{U}$ .  $\Pi : SE(3) \rightarrow \mathbb{R}^3$ , is a mapping which picks out the translation component of an element of  $SE(3)$ . The proposed cluster modelling as well as the measurements are depicted in figure 2. In the rest of the paper, we use the condensed notations  $\mathbf{Z} = \{\mathbf{Z}_i\}_{i=1}^n \in SE(3)^n$  ( $n$  direct products of  $SE(3)$ ) and  $\mathbf{z} = \{\mathbf{z}_i\}_{i=1}^n$ .

The cluster shape is fitted by estimating the matrix  $\mathbf{S}$ . In order to develop a unified formalism to jointly estimate it with  $\mathbf{M}$ , we parameterize it as a set of elements also defined on LGs. Due to its properties, it can be diagonalized in an orthogonal basis with strictly positive eigenvalues as follows:

$$\mathbf{S} = \mathbf{P}^{\top} \mathbf{D} \mathbf{P} \quad (7)$$

where  $\mathbf{P}$  belongs to the LG  $SO(6)$  and  $\mathbf{D}$  belongs to the LG  $D_6(\mathbb{R}^{+*})$  with:

- $SO(6) = \{\mathbf{P} \in \mathbb{R}^{6 \times 6} | \mathbf{P}^{\top} \mathbf{P} = \mathbf{I}_6, |\mathbf{P}| = 1\}$
- $D_6(\mathbb{R}^{+*}) = \{\text{diag}([d_1, \dots, d_6]) | \forall i \in \llbracket 1, 6 \rrbracket, d_i > 0\}$ .

Thus, estimating  $\mathbf{S}$  is equivalent to estimating  $(\mathbf{P}, \mathbf{D})$ . Hereafter, all the unknown variables are concatenated in  $\mathbf{X} = (\mathbf{M}, \mathbf{P}, \mathbf{D}) \in SE(3) \times SO(6) \times D_6(\mathbb{R}^{+*})$ .

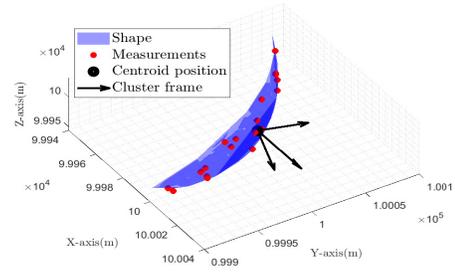


Fig. 2. Cluster modelling

## B. Posterior estimator

We consider a Bayesian approach, well-suited to integrate prior constraints on the parameters to be estimated. A difficulty is that we have to address the presence of the hidden variables  $\mathbf{Z}$  in the hierarchical measurement model (5) and (6). They are intricate to marginalize, therefore we choose to estimate them jointly with our parameters of interest. We perform the estimation in the maximum *a posteriori* sense:

$$\hat{\mathbf{X}}^a = \arg \max_{\mathbf{X}^a \in G} p(\mathbf{X}^a | \mathbf{z}), \quad (8)$$

where  $\mathbf{X}^a = (\mathbf{M}, \mathbf{P}, \mathbf{D}, \mathbf{Z})$  and  $G$  is the LG  $SE(3) \times SO(6) \times D_6(\mathbb{R}^{+*}) \times SE(3)^n$  with dimension  $m_a$ .

The posterior distribution to maximize can be decomposed by using the Bayes' rule. Indeed, by exploiting the conditional independencies, we can write:

$$p(\mathbf{X}^a | \mathbf{z}) \propto \left( \prod_{i=1}^n p(\mathbf{z}_i | \mathbf{Z}_i) p(\mathbf{Z}_i | \mathbf{M}, \mathbf{P}, \mathbf{D}) \right) p(\mathbf{M}) p(\mathbf{P}) p(\mathbf{D}). \quad (9)$$

According to the models (5) and (6), we obtain:

$$\begin{cases} p(\mathbf{z}_i | \mathbf{Z}_i) = \mathcal{N}(\mathbf{z}_i; \Pi(\mathbf{Z}_i), \mathbf{U}), \\ p(\mathbf{Z}_i | \mathbf{M}, \mathbf{P}, \mathbf{D}) = \mathcal{N}_{SE(3)}^L(\mathbf{Z}_i; \mathbf{M}, \mathbf{P}^{\top} \mathbf{D} \mathbf{P}). \end{cases} \quad (10) \quad (11)$$

where  $\mathcal{N}(\mathbf{x}; \mathbf{m}, \boldsymbol{\Sigma})$  is the Euclidean Gaussian pdf of 1<sup>st</sup> and 2<sup>nd</sup> order moments  $\mathbf{m}$  and  $\boldsymbol{\Sigma}$ , respectively, evaluated at  $\mathbf{x}$ .

To model the prior information on the unknown matrices  $\mathbf{M}$ ,  $\mathbf{P}$  and  $\mathbf{D}$ , we use three left CGDs respectively on  $SE(3)$ ,  $SO(6)$  and  $D_6(\mathbb{R}^{+*})$  with hyperparameters  $(\boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M)$ ,  $(\boldsymbol{\mu}_P, \boldsymbol{\Sigma}_P)$  and  $(\boldsymbol{\mu}_D, \boldsymbol{\Sigma}_D)$ . It can be noted that  $p(\mathbf{D})$  is well-adapted because it ensures the positivity of its realizations.

## C. Optimization algorithm

Maximizing  $p(\mathbf{X}^a | \mathbf{z})$  is equivalent to minimizing  $J(\mathbf{X}^a) = -2 \log p(\mathbf{X}^a | \mathbf{z})$ . Thus, we aim at solving the following optimization problem:

$$\hat{\mathbf{X}}^a = \arg \min_{\mathbf{X}^a \in G} J(\mathbf{X}^a). \quad (12)$$

After some rearrangements, this criterion can be decomposed in two terms:

$$J(\mathbf{X}^a) = \underbrace{J_{nq}(\mathbf{D})}_{\text{Non-quadratic part}} + \underbrace{J_q(\mathbf{M}, \mathbf{P}, \mathbf{D}, \mathbf{Z})}_{\text{Quadratic part}} + C, \quad (13)$$

with  $C$  a constant and  $J_{nq}(\mathbf{D}) = n \sum_{j=1}^6 \log(d_j)$ . Finally,  $J_q$  is expressed as:

$$\begin{aligned}
J_q(\mathbf{M}, \mathbf{P}, \mathbf{D}, \mathbf{Z}) = & \\
& \sum_{i=1}^n \left( \left\| \mathbf{z}_i - \Pi(\mathbf{Z}_i) \right\|_{\mathbf{U}}^2 + \left\| \mathbf{D}^{-1/2} \mathbf{P} \text{Log}_{SE(3)}^{\vee}(\mathbf{M}^{-1} \mathbf{Z}_i) \right\|_2^2 \right) \\
& + \left\| \text{Log}_{SO(6)}^{\vee}(\boldsymbol{\mu}_{\mathbf{P}}^{-1} \mathbf{P}) \right\|_{\Sigma_{\mathbf{P}}}^2 + \left\| \text{Log}_{SE(3)}^{\vee}(\boldsymbol{\mu}_{\mathbf{M}}^{-1} \mathbf{M}) \right\|_{\Sigma_{\mathbf{M}}}^2 \\
& + \left\| \text{Log}_{D_6(\mathbb{R}^{+*})}^{\vee}(\boldsymbol{\mu}_{\mathbf{D}}^{-1} \mathbf{D}) \right\|_{\Sigma_{\mathbf{D}}}^2. \tag{14}
\end{aligned}$$

Owing to its quadratic structure,  $J_q$  can be written compactly as:

$$J_q(\mathbf{X}^a) = \|\phi(\mathbf{X}^a)\|_{\Sigma_q}^2 \tag{15}$$

where  $\phi: G \rightarrow \mathbb{R}^{m_a}$  is a smooth function and  $\Sigma_q = \text{blkdiag}(\tilde{\mathbf{U}}, \tilde{\mathbf{I}}, \Sigma_M, \Sigma_P, \Sigma_D)$  with  $\tilde{\mathbf{U}} = \mathbf{I}_n \odot \mathbf{U}$  and  $\tilde{\mathbf{I}} = \mathbf{I}_n \odot \mathbf{I}_3$  ( $\odot$  defines the Kronecker product and  $\text{blkdiag}$  the block diagonal operator). The criterion (13) is smooth but it is difficult to obtain an analytical expression of its minimum. To overcome this problem, a specific iterative Newton-based algorithm is proposed in this work. The Newton method on LG consists in finding, at each iteration  $l$ , an optimal descent direction  $\delta_{l+1}^* \in \mathbb{R}^{m_a}$  by minimizing a second order approximation of  $J(\mathbf{X}_l^a \text{Exp}_G^{\wedge}(\delta))$  where  $\mathbf{X}_l^a$  stands for the current iterate. It is then updated as [29]:

$$\mathbf{X}_{l+1}^a = \mathbf{X}_l^a \exp_G^{\wedge}(\delta_{l+1}^*). \tag{16}$$

such that the set  $\{\mathbf{X}_l^a\}_{l \in \mathbb{N}}$  converges toward a critical point of the criterion. In our case, the particular structure of  $J$  motivates us to find  $\delta_{l+1}^*$  minimizing the sum of:

- 1/a second-order approximation of  $J_q$  based on a first-order Taylor expansion of  $\phi(\mathbf{X}_l^a \text{Exp}_G^{\wedge}(\delta))$ ,
- 2/a second-order expansion of  $J_{nq}(\mathbf{X}_l^a \text{Exp}_G^{\wedge}(\delta))$ . In these formulas, the first and second order derivatives are calculated according to the equations (2) and (3).

After some computations, we obtain:

$$\delta_{l+1}^* = -(\mathbf{H}_l + 2\mathbf{J}_l^{\top} \Sigma_q^{-1} \mathbf{J}_l)^{-1} (\nabla J_l + 2\mathbf{J}_l^{\top} \Sigma_q^{-1} \phi(\mathbf{X}_l^a)) \tag{17}$$

where  $\mathbf{H}_l$  and  $\nabla J_l$  correspond to the second derivative and the Lie derivative of  $J_{nq}$ , while  $\mathbf{J}_l$  is the Jacobian of  $\phi$  computed at  $\mathbf{X}_l^a$ .

#### IV. SIMULATION RESULTS

##### A. Performance evaluation

We evaluate the proposed approach by considering a cluster of debris located in low orbit, such that the altitude of each piece is around  $10^5$  m relative to the ground. For validation purposes, the observations are directly generated from the models (5) and (6) by considering a variance of the measurement noise equal to  $500\text{m}^2$ . Only 10 reflectors are considered and the matrix  $\mathbf{S}$  is set so that the extent of the cluster is about  $10^3$  m according to the three axes.

To apply the proposed methodology, the hyperparameters of the prior distributions have to be adjusted. The prior ‘‘mean’’

and covariance matrix of the position component of  $\mathbf{M}$  are merely taken as the empirical statistics of the measurements. As for the moments associated to the rotation component, they are chosen in a non informative manner. Regarding the hyperparameters of the prior pdfs  $p(\mathbf{P})$  and  $p(\mathbf{D})$ , they are set so as to generate a great variety of ‘‘banana’’-like shapes. Finally, if we denote  $\mathbf{X}_0 = (\mathbf{M}_0, \mathbf{P}_0, \mathbf{D}_0)$  the initial guess of the optimization algorithm, its value is determined from the available sensor information.  $\mathbf{P}_0$  and  $\mathbf{D}_0$  are obtained by diagonalizing the empirical covariance matrix of the measurements. As for  $\mathbf{M}_0$ , it could be taken as their centroid position. However, to make the problem more challenging, we introduce a systematic error of order  $10^3$  m.

To begin with, the figure 3 illustrates the quality of the cluster reconstruction, in terms of shape and centroid position, at different iterations of the optimization algorithm. The true and estimated cluster boundaries are obtained by three-dimensional triangulation from 2000 samples simulated according to model (5) by taking the actual and estimated value of the matrix  $\mathbf{S}$ , respectively. By comparing the figures 3-(a) and 3-(b), we observe a very good coincidence between the actual and the estimated shape at the last iteration. Furthermore, the centroid position is well-recovered in spite of a high initialization error. To quantify the estimation performance, a classical Euclidean metric does not make sense, hence an intrinsic error on LG is considered. Let  $\mathbf{V}$  be a matrix belonging to a LG  $G_{\mathbf{V}}$  and  $\hat{\mathbf{V}}$  be its estimate, then the LG-error is defined as  $\left\| \text{Log}_{G_{\mathbf{V}}}^{\vee}(\mathbf{V}^{-1} \hat{\mathbf{V}}) \right\|_2$ . Average values are computed from  $N_r = 50$  Monte-Carlo runs corresponding to different realizations of the model noises: they are plotted in figures 4-(a) and 4-(b). It can be noted that the errors converge and stabilize to fixed and small values after only a few iterations.

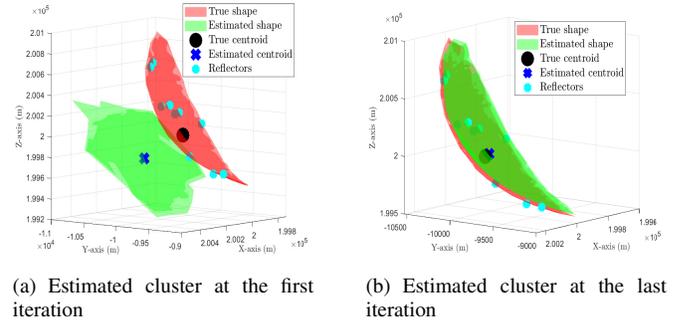


Fig. 3. Shape and centroid reconstruction

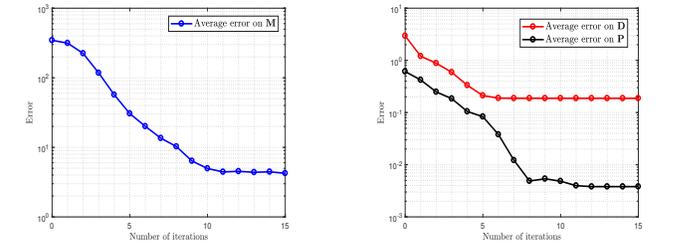
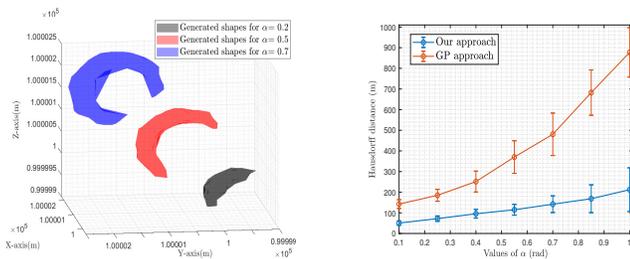


Fig. 4. Influence of the distortion parameter

## B. Comparison with a state-of-the-art technique

We compare our approach with a recent one proposed in the literature that is based on Gaussian process (GP) modelling [33]. As opposed to [11], it has the advantage of applying whatever the shape of the extended object is. We study the robustness of both algorithms by testing them on shapes with different curvatures. For that purpose, we introduce an empirical angular parameter  $\alpha$  (rad) which makes it possible to control the degree of distortion of the cluster by twisting the shape according to one of the three axes as illustrated in figure 5-(a). We evaluate the cluster reconstruction for different values of  $\alpha$  for both the proposed approach and the GP-based algorithm. However, since they rely on different parameterizations, their estimation results are difficult to compare. To overcome this issue, we generate a significant number of samples from (5) using the actual and the estimated values of  $\mathbf{M}$  and  $\mathbf{S}$ . We also simulate in the same way the GP-based cluster probabilistic model. Then, we can compare the clouds of points thus generated using Hausdorff distances [34]. Figure 5-(b) plots the average Hausdorff distance, obtained from 200 MC runs, between the actual cluster and the one estimated by our approach and by the GP method as a function of  $\alpha$ . The error bars represent the standard deviations. It appears that, with the LG modelling, the distance remains relatively stable when the shape becomes very distorted. Contrariwise, the distance of the GP approach significantly increases. Thus, our method appears to be more adapted to perform the reconstruction of shapes with high curvatures.



(a) Evolution of the shapes for different distortions

(b) Evolution of the Hausdorff distance

Fig. 5. Estimations error on unknown parameters

## V. CONCLUSIONS

In this communication, a new model and a new algorithm are introduced to estimate the centroid position and the shape of a cluster of space debris. The originality lies on a LG-modelling of the cluster which is well-suited to represent the spreading of the debris. The resulting estimation problem is solved by a Newton optimization technique on LG. The proposed approach is shown to be efficient and robust on different simulated scenarios. The perspective is to derive a filtering algorithm on LGs wherein the temporal evolution of the cluster is considered.

## REFERENCES

[1] J.A. Kennewell and B.-N. Vo, *An overview of space situational awareness*, International Conference on Information Fusion, Istanbul, Turkey, July 2013.

[2] K.J. DeMars, I.I. Hussein, C. Frueh, M.K. Jah and R. S. Erwin, *Multiple-object space surveillance tracking using finite-set statistics*, Journal of Guidance, Control, and Dynamics, vol. 38 (9), pp. 1741-1756, March 2015.

[3] C. Yanez, M. Gupta, V. Morand and J. C. Dolado, *On the Gaussianity Validity Time for Orbital Uncertainty Propagation*, ESA NEO and Debris Detection Conference, Darmstadt, January 2019.

[4] D.Mehrholtz and al., *Detecting, tracking and imaging space debris*, ESA bulletin. Bulletin ASE, European Space Agency, vol. 109 (109), February 2002

[5] H. Klinkrad, *Space debris*, Wiley Online Library, 2010.

[6] *The Threat of Orbital Debris and Protecting NASA Space Assets from Satellite Collisions*, NASA, April 2009.

[7] S. Arulampalam, S. Maskell, N. Gordon and T. Clapp, *A Tutorial on Particle Filter for Online Nonlinear/Non-Gaussian Bayesian Tracking*, IEEE Transactions on Signal Processing, vol. 50 (2), pp. 174-188, February 2002.

[8] S.Särkkä, *Bayesian Filtering and Smoothing*, Cambridge, 2013.

[9] K. Granstrom, M. Baum and S. Reuter, *Extended Object Tracking: Introduction, Overview and Applications*, Journal of Advances in Information Fusion, vol. 12 (2), pp. 139-174, December 2016.

[10] K. Gilholm and D. Salmond, *Spatial distribution model for tracking extended objects*, IEEE Proceedings of Radar, Sonar and Navigation, vol. 152(5), pp. 364-371, October 2005.

[11] J.W. Koch, *Bayesian Approach to Extended Object and Cluster Tracking Using Random Matrices*, IEEE Transactions on Aerospace and Electronic Systems, vol. 4 (3), pp. 1042-1059, July 2008.

[12] M. Baum and U.D. Hanebeck, *Extended Object Tracking with Random Hypersurface Models*, IEEE Transactions on Aerospace and Electronic Systems, vol. 50 (1), pp. 149-159, January 2014.

[13] M.Baum and U.D. Hanebeck, *Shape tracking of extended objects and group targets with star-convex RHMs*, International Conference on Information Fusion, Chicago, USA, 2011.

[14] F. Faion, M. Baum, and U.D. Hanebeck, *Tracking 3D shapes in noisypoint clouds with random hypersurface models*, International Conference on Information Fusion, Singapore, August 2012.

[15] N. Wahlström and E. Özkan, *Extended Target Tracking Using Gaussian Processes*, IEEE Transactions on Signal Processing, vol. 63 (16), pp. 4165-4178, August 2015.

[16] J.T. Horwood and A.B. Poore, *Orbital State Uncertainty Realism*, Proceedings of the Advanced Maui Optical and Space Surveillance Technologies Conference, Maui, Hawaii, USA, September 2011.

[17] S. Labsir, A. Giremus, B. Yver and T. Benoudiba-Campanini, *Tracking a cluster of space debris by filtering on Lie groups*, International Conference on Acoustic, Speech and Signal Processing, Brighton, UK, 2019.

[18] A. W. Long and K.C. Wolfe and M. Mashner and G.S. Chirikjian, *The Banana Distribution is Gaussian: A Localization Study with Exponential Coordinates*, Robotics: Science and Systems, 2012.

[19] R. A. Srivatsan and M. Travers and H. Choset, *Using Lie algebra for shape estimation of medical snake robots*, International Conference on Intelligent Robots and Systems, Chicago, USA, 2012.

[20] J.E. Prussing and B.A. Conway, *Orbital Mechanics*, Oxford University Press, 1993.

[21] A. Baker, *Matrix Groups: an Introduction to Lie Group Theory*, Springer, 2003.

[22] J.Faraut, *Analysis on Lie group : an Introduction*, Cambridge University Press, 2008.

[23] C.J.Bordin, C.G. de Figueredo, M. G. S. Bruno, *Nonlinear state estimation on unit spheres using manifold particle filtering*, Digital Signal Processing, vol. 81, pp. 50-56, 2018.

[24] S. Saïd, L. Bombrun, Y. Berthoumiou, J. H. Manton, *Riemannian Gaussian Distributions on the Space of Symmetric Positive Definite Matrices*, IEEE Transactions on Information Theory, vol. 63 (4), pp. 2153-2170, April 2017.

[25] T. Barfoot and P. T. Furgale, *Associating Uncertainty With Three-Dimensional Poses for Use in Estimation Problem*, IEEE Transactions on Robotics, vol. 30 (3), pp. 679-693, June 2014.

[26] T. Barfoot, *State Estimation for robotics*, Cambridge, 2014.

[27] G.-S. Chirikjian, *Stochastic Models, Information Theory, and Lie Groups, Volume 2: Classical Results and Geometric Methods*, Applied and Numerical Harmonic Analysis, 2009.

[28] B. Owren and B. Welfert, *The Newton iteration on Lie groups*, BIT Numerical Mathematics, vol. 40(1), pp. 121-145, 2000.

[29] J.H.Manton and R.Mahony, *The Geometry of the Newton Method on Non-Compact Lie Group*, Journal of Global Optimization, vol. 23, pp. 309-327, 2002.

[30] C.J.Taylor and D.J.Kriegman, *Minimization on the Lie group SO(3) and related manifolds*, 1994.

[31] X.Pennec, *Intrinsic statistics on Riemannian manifolds : basic tools for geometric measurements*, Journal of Mathematical Imaging and Vision, vol. 25, pp. 127-155, 2006.

[32] P.A. Absil, R.Mahony and R. Sepulchre, *Optimization algorithm on matrix manifolds*, Princeton, 2008.

[33] M.Kurut and E.Özkan *3D Extended Object Tracking Using Recursive Gaussian Processes*, International Conference on Information Fusion, Cambridge, UK, 2018.

[34] W. Rucklidge *Efficient Visual Recognition Using the Hausdorff Distance*, Springer-Verlag, 1996.