A Multi-Stage Parallel LMS Structure and its Stability Analysis Using Transfer Function Approximation

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***Abstract*—Generally, the least mean square (LMS) adaptive algorithm is widely used in antenna array beamforming given its target tracking capability and its low computational requirements. However, the classical LMS implementation still suffers from a trade-off between convergence speed and residual error floor. Numerous variants to the classical LMS have been suggested as a solution for the previous problem at the cost of a considerable increase in the computational complexity and degraded performance in low signal to noise ratio (SNR). Thus, in this paper, we propose a multi-stage parallel LMS structure with an error feedback for accelerating the LMS convergence while maintaining a minimal steady state error and a computational complexity of order $O(N)$, where N represents the number of antenna elements. In parallel LMS (pLMS), the second LMS stage (LMS_2) error is delayed by one sample and fed-back to combine with that of the first LMS stage (LMS_1) to form the total pLMS error. A transfer function approximation to the pLMS is derived in order to numerically assess the pLMS stability and to determine the approximate maximum parametric value of the step size for which the pLMS remains stable. Simulation result highlight the superior performance of the pLMS in demonstrating accelerated convergence and low steady state error compared to previous variants and for different SNR environment.**

***Index Terms*—LMS, Parallel LMS, Adaptive Beamforming, Transfer Function, Farrow Filter.**

I. Introduction

Adaptive Beamforming (ABF) is a spatial multiplexing technique and an essential feature of any smart antenna array [1]. ABF is employed in various applications such as: wireless communication, sonar and radar tracking to increase spectral capacity and efficiency [2]. ABF implements an adaptive algorithm whose input is modeled as a linear combination of the observed noisy signal and the varying filter weights. The filter weights are concurrently computed by the adaptive algorithm attenuating interfering signals through directional signal transmission or reception [2].

In modern wireless communications, a requirement for adaptive beamforming is the ability to continuously adapt, in real-time, to the ever changing signal conditions and users mobility. Such requirement is reflected by the system convergence rate, computational complexity and beam pointing accuracy [3]. Popular, non-blind, adaptive algorithms such as the least mean square (LMS) and recursive least square (RLS) iteratively minimize the mean square error between the filtered output signal and a known reference signal [3]. In contrast to the LMS with a linear computational complexity of order $O(N)$, where N represents the number of antenna elements, the RLS presents an undesirable quadratic complexity of order $O(N^2)$. However simple and effective, the LMS still suffers from a trade-off between its convergence speed and its residual error floor [3]. Several variants of the classical LMS have been proposed to accelerate the convergence rate while trying to maintain an acceptably low error floor. These techniques include the cascaded least mean square-least mean square (LLMS) [3] and the recursive least mean square (RLMS) [4].

The LLMS and RLMS structures presented a multi stage LMS-LMS and RLS-LMS structures cascaded by an estimate of the array steering vector [2], [3]. This technique shows superior performance over previous LMS variants at the cost of doubling the computational requirement of the classical LMS algorithm and presenting additional $N+1$ complex divisions to compute the steering vector estimate block. As such, the cascade RLMS was simplified in [4] by eliminating the need to compute the array image cascading block hence obtaining a parallel input RLMS structure. However, the complexity of the system remained of order $O(N^2)$ and the RLS still requires a division operation.

Our contribution in this paper is summarized as follows. First, we propose a multi-stage parallel input LMS (pLMS) with an error feedback for accelerating convergence, maintaining minimal error floor and a low computational complexity. To derive pLMS total error, the second LMS stage (LMS_2) error is multiplied by the imaginary number $j = \sqrt{-1}$ and is subject to a one sample delay to combine with that of the first stage LMS (LMS_1). The multiplication by j allows robustness against recurring samples as a result of symbol re-transmission or high speed data acquisition. Second, we presented a transfer function approximation for the pLMS to numerically assess its stability and to determine the upper bound value of the step size.

II. Mathematical Review

Here we present a brief background review of the LMS beamformer for narrow-band complex signals and a uniform linear array (ULA) of N equally spaced antenna elements [3]. Let the input vector $\mathbf{x}(k) = [x_1(k), x_2(k), ..., x_N(k)]^T$ impinging the array from the far field [5], at the discrete time instant k defined by

$$\mathbf{x}(k) = \mathbf{a}_{\mathrm{d}} s_{\mathrm{d}}(k) + \sum_{\mathrm{l}=0}^{\mathrm{N}-1} \mathbf{i}_{\mathrm{l}}(k) + \mathbf{n}(k) \tag{1}$$

with $[.]^T$ is the matrix transpose, $s_d(k)$ and $\mathbf{i}_{\mathrm{l}}(k)$ are the desired and interfering signals, with $l < N$, $\mathbf{a}_{\mathrm{d}}$ is the $N \times 1$ complex array steering vector for the desired signal, and $\mathbf{n}(k)$ is the complex additive white Gaussian noise (CAWGN) vector. A general form of $\mathbf{a}_d$ is given by

$$\mathbf{a}_d = [1, e^{-j\psi}, e^{-j2\psi}, ...e^{-j(N-1)\psi}]^T \tag{2}$$

and

$$\psi = 2\pi \frac{D\sin(\theta)}{\lambda} \tag{3}$$

where, the first antenna element acting as a reference, θ is the angle of arrival, D is the distance between consecutive antenna elements, and λ is the signal wavelength. Hence, the delay between consecutive antenna elements is defined as

$$t = \frac{D\sin(\theta)}{c} \tag{4}$$

where c is the velocity of light. The output of the beamformer subject to a linear combiner is given by

$$y(k) = \mathbf{w}^{\mathbf{H}}(k)\mathbf{x}(k) \tag{5}$$

where $[.]^H$ represents the matrix Hermitian transpose and $\mathbf{w}(k)$ is the array weight vector.

A. Least Mean Square Algorithm

The LMS is a non-blind adaptive algorithm which iteratively minimizes the mean square error (MSE) between a desired signal$\mathbf{d}(k)$ and the filter output $y(k)$ [6]. The LMS is presented by its error signal, $e(k)$ its weight update equation as follows

$$e(k) = d(k) - y(k) \tag{6}$$

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu_{LMS} e^*(k)\mathbf{x}(k) \tag{7}$$

where $*$ is the complex conjugate operator and μ_{LMS} is the learning rate, i.e. step size [6]. The optimal weight, $\mathbf{w}_{oplms}$, [7] is given as

$$\mathbf{w}_{oplms} = \mathbf{R}^{-1}\mathbf{p} \tag{8}$$

Where, $\mathbf{R} = \mathbf{R}(0)$ and $\mathbf{R}^{-1} = \mathbf{R}^{-1}(0)$ are the input signal auto-correlation matrix and its inverse, respectively, and $\mathbf{p} = \mathbf{p}(0)$ is the cross correlation vector of the input $\mathbf{x}(k)$ and desired signal $d(k)$. $\mathbf{R}(0)$ and $\mathbf{p}(0)$ are defined at lag $\tau = 0$ as

$$\mathbf{R}(\tau) = E[\mathbf{x}(k-\tau)\mathbf{x}^{\mathbf{H}}(k)] \tag{9}$$

$$\mathbf{p}(\tau) = E[d^*(k-\tau)\mathbf{x}(k)] \tag{10}$$

where, $E[.]$ is the expectation operator, the lag $\tau = k_1 - k_2$ and k_1, k_2 are different time instances from which an observation of the random process is taken [7].

III. Parallel LMS (pLMS)

As previously defined, pLMS is a multi stage LMS with parallel input where the overall error signal, $e_{pLMS}(k)$, is derived as a combination of individual stage errors as shown in Fig. 1. As described in Fig. 1, the error signal of the LMS_2

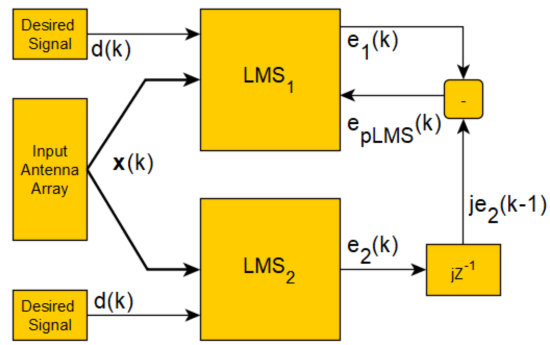

Fig. 1. pLMS Diagram

stage, $e_2(k)$, is multiplied by j and subject to a one sample delay, represented by the block jZ^{-1}. The resulting error, $je_2(k-1)$ is then combined with that of LMS_1, consequently allowing parallel operation of both stages, hence the notation pLMS. We assume the process to be wide sense stationary (WSS) and all signals are Gaussian with zero mean. The overall mean square error, $\xi_{pLMS}(k)$, is defined as

$$\begin{aligned}\xi_{pLMS}(k) =& E[|e_{pLMS}(k)|^2] \\ =& E[|e_1(k)|^2 + je_1(k)e_2^*(k-1) \\ & - je_1^*(k)e_2(k-1) + |e_2(k-1)|^2]\end{aligned} \tag{11}$$

where, $e_i(k) = d(k) - \mathbf{w}_i^{\mathbf{H}}(k)\mathbf{x}(k)$, i represents the stage identifier, $|.|$ signifies complex modulus and $*$ is the complex conjugate operator.

Proposition – I: The multiplication by j, allows robustness against error nulls as a result of recurring samples in the input and/or, desired signal. Recurring samples are a result of a high sampling frequency and/or symbol re-transmission. Thus preserving accelerated convergence.

Proof: let $d(k) = a + jb$ and $d(k-1) = c + jd$. For recurring samples, $D = d(k) - d(k-1)$ with $d(k) \approx d(k-1)$, $a \approx c$, and $b \approx d$ thus $S(k) \approx 0 + j0$. However, for $S(k) = d(k) - jd(k-1)$, i.e. left side term of (11), becomes $S(k) = a + d + j(b - c) \neq 0$.

■

where $S(k)$ is the new pLMS reference signal, such as $S(k) = d(k) - jd(k-1)$. The pLMS algorithm is now defined by Algorithm 1 where $y_1(k)$ and $y_2(k)$ are the first and second stage output, μ_1 and μ_2 are the LMS_1 and LMS_2 step sizes, respectively and $\mathbf{w}_2(k-1)$ is the weight vector of the LMS_2 stage delayed by one sample.

Moreover, the first term of (11) can be expressed as

$$E[|e_1(k)|^2] = E[|d(k)|^2] - \mathbf{p^H}\mathbf{w}_1(k) - \mathbf{w}_1^{\mathbf{H}}(k)\mathbf{p} + \mathbf{w}_1^{\mathbf{H}}(k)\mathbf{R}\mathbf{w}_1(k) \quad (12)$$

Furthermore, the last term of (11) can be developed as follows

$$\begin{aligned} E[|e_2(k-1)|^2] =& E[|d(k-1)|^2] \\ &- E[d_1(k)\mathbf{x}_1^{\mathbf{H}}(k-1)]\mathbf{w}_2(k-1) \\ &- \mathbf{w}_2^{\mathbf{H}}(k-1)E[d_1^*(k)\mathbf{x}_1(k-1)] \\ &+ \mathbf{w}_2^{\mathbf{H}}(k-1)E[\mathbf{x}_1(k)\mathbf{x}_1^{\mathbf{H}}(k-1)] \\ &\times \mathbf{w}_2(k-1) \end{aligned} \quad (13)$$

Using (12), (13), and expanding the terms in (11) the MSE ξ_{pLMS} becomes

$$\begin{aligned} \xi_{pLMS}(k) =& E[|d(k)|^2] - \mathbf{p^H}\mathbf{w}_1(k) - \mathbf{w}_1^{\mathbf{H}}(k)\mathbf{p} \\ &+ \mathbf{w}_1^{\mathbf{H}}(k)\mathbf{R}\mathbf{w}_1(k) + E[|d(k-1)|^2] \\ &- E[d_1(k)\mathbf{x}_1^{\mathbf{H}}(k-1)]\mathbf{w}_2(k-1) \\ &- \mathbf{w}_2^{\mathbf{H}}(k-1)E[d_1^*(k)\mathbf{x}_1(k-1)] \\ &+ \mathbf{w}_2^{\mathbf{H}}(k-1)E[\mathbf{x}_1(k)\mathbf{x}_1^{\mathbf{H}}(k-1)]\mathbf{w}_2(k-1) \\ &+ jE[d(k)d^*(k-1)] - jE[d^*(k)d(k-1)] \\ &- jE[d(k)\mathbf{x}^{\mathbf{H}}(k-1)]\mathbf{w}_2(k-1) \\ &- j\mathbf{w}_1^{\mathbf{H}}(k)E[d^*(k-1)\mathbf{x}(k)] \\ &+ j\mathbf{w}_1^{\mathbf{H}}(k)E[\mathbf{x}(k)\mathbf{x}^{\mathbf{H}}(k-1)]\mathbf{w}_2(k-1) \\ &+ jE[d(k-1)\mathbf{x}^{\mathbf{H}}(k)]\mathbf{w}_1(k) \\ &+ j\mathbf{w}_2^{\mathbf{H}}(k-1)E[d^*(k)\mathbf{x}(k-1)] \\ &- jE[\mathbf{x}^{\mathbf{H}}(k)\mathbf{w}_1(k)\mathbf{w}_2^{\mathbf{H}}(k-1)\mathbf{x}(k-1)] \end{aligned} \quad (14)$$

The pLMS gradient, ∇_{pLMS}, can be obtained by differentiating (14) with respect to $\mathbf{w^H}_1(k)$.

$$\begin{aligned} \frac{\partial \xi_{pLMS}(k)}{\partial \mathbf{w^H}_1(k)} =& \nabla_{pLMS} \\ =& -\mathbf{p} + \mathbf{R}\mathbf{w}_1(k) - jE[d^*(k-1)\mathbf{x}(k)] \\ &+ jE[\mathbf{x}(k)\mathbf{x}^{\mathbf{H}}(k-1)]\mathbf{w}_2(k-1) \end{aligned} \quad (15)$$

Thus, the optimal weight vector, $\mathbf{w}_{op}$, assuming both stages converges, is obtained by setting the gradient, $\nabla_{pLMS} = 0$ such as

$$\begin{aligned} \mathbf{w}_{op} =& \mathbf{w}_{oplms} + j\mathbf{R}^{-1}E[d^*(k-1)\mathbf{x}(k)] \\ &- j\mathbf{R}^{-1}E[\mathbf{x}(k)\mathbf{x}^{\mathbf{H}}(k-1)]\mathbf{w}_{oplms} \end{aligned} \quad (16)$$

Table I, presents a comparison of resource complexity where cMultiply, cAdd, cDivide and RLMSp denotes complex multiplication, complex addition, complex division and parallel RLMS, respectively. From Table I, it is clear that the RLMS, RLMSp and RLS require an undesirable complexity of order $O(N^2)$. Additionally, the RLMS and RLMSp require $N+1$ and 1 complex division, respectively. On the other hand, compared to the LLMS, the proposed pLMS provides a considerable reduction in resource usage, i.e. $2N$ complex multiplications, N complex additions and N complex divisions.

Algorithm 1 Parallel LMS (pLMS)

Conditions:

$$e_2(-1) = e_2(0) = 0$$
$$d(-1) = d(0)$$
$$\mathbf{x}(-1) = \mathbf{x}(0)$$

pLMS:

LMS_1 :

$$y_1(k) = \mathbf{w}_1^{\mathbf{H}}(k)\mathbf{x}(k)$$
$$e_1(k) = d(k) - y_1(k)$$
$$e_{pLMS}(k) = e_1(k) - je_2(k-1)$$
$$\mathbf{w}_1(k+1) = \mathbf{w}_1(k) + \mu_1 e_{pLMS}^*(k)\mathbf{x}(k)$$

LMS_2 :

$$y_2(k) = \mathbf{w}_2^{\mathbf{H}}(k)\mathbf{x}(k)$$
$$e_2(k) = d(k) - y_2(k)$$
$$\mathbf{w}_2(k+1) = \mathbf{w}_2(k) + \mu_2 e_2^*(k)\mathbf{x}(k)$$

Algorithm	cMultiply	cAdd	cDivide
RLMS [8]	$3N^2+11N+2$	$2N^2+9N+6$	$N+1$
RLMSp [4]	$3N^2+7N+1$	$2N^2+6N+3$	1
RLS	$3N^2+5N$	$2N^2+4N+2$	1
LLMS [3]	$6N+2$	$5N+4$	N
pLMS	$4N+2$	$4N+4$	0
LMS	$2N+1$	$2N+1$	0

TABLE I
THEORETICAL COMPLEXITY AND RESOURCE USAGE

IV. Transfer Function Approximation

In order to numerically assess the stability and performance of the proposed system, this section presents a discrete time transfer function approximation of the pLMS. Moreover, the input system described by the N equally spaced, identical antenna elements is modeled as a N^{th} order fractional delay filter employing a Farrow structure and Lagrange interpolation [9]. Using (4) the new pLMS input signal $\mathbf{x}_f(k)$ can now be defined as:

$$\mathbf{x}_f(k) = \mathbf{y}_{\mathrm{d}}(k) + \sum_{\mathrm{j}=0}^{\mathrm{N}-1} \mathbf{y}_{\mathrm{i,j}}(k) + \mathbf{n}(k) \quad (17)$$

where $\mathbf{y}_d(k)$, $\mathbf{y}_{\mathrm{i,j}}(k)$ are the message and interfering signals subject to a fractional delay filter and $\mathbf{n}(k)$ is a complex additive white Gaussian noise (CAWGN). The pLMS transfer function approximation can now be derived from (6) and (11) such that

$$e_{pLMS}(k) = d(k) - jd(k-1) - y_1(k) + jy_2(k-1) \quad (18)$$

Applying the Z transform for both sides of (18), we obtain

$$\begin{aligned} E[D(Z)] - jZ^{-1}E[D(Z)] =& E[J_{pLMS}(Z)] + E[Y_1(Z)] \\ &- jZ^{-1}E[Y_2(Z)] \end{aligned} \quad (19)$$

with

$$E[J_{pLMS}(Z)] = E[e_{pLMS}(0)] + E[e_{pLMS}(1)]Z^{-1} + ... \tag{20}$$

$$E[D(Z)] = E[d(0)] + E[d(1)]Z^{-1} + ... \tag{21}$$

and

$$y_i(k) = \mathbf{w}_i^{\mathbf{H}}(k)\mathbf{x}_f(k) \tag{22}$$

putting $\mathbf{w}_1(0) = \mathbf{w}_2(0) = 0$, we obtain

$$y_1(k) = \mu_1 \sum_{i=0}^{k-1} \beta_i(k) e_{pLMS}(i) \tag{23}$$

$$y_2(k) = \mu_2 \sum_{i=0}^{k-1} \beta_i(k) e_2(i) \tag{24}$$

where $\beta_i(k) = \mathbf{x}_f^{\mathbf{H}}(i)\mathbf{x}_f(k)$. Moreover, since $\beta_i(k)$ is time varying [10], it is difficult to achieve a solvable difference equation. However, we know that

$$\beta_i(k) = \sum_{j=0}^{N-1} x_{f(i-j)} x_{f(k-j)} = N r_{ki} \tag{25}$$

$$r_{ki} = \frac{1}{N} \sum_{j=0}^{N-1} x_{f(i-j)} x_{f(k-j)} \tag{26}$$

and r_{ki} is the input signals auto-correlation estimate [10]. Considering the input signal is WSS and its properties can be estimated by a time average we get $r_{ki} \approx r_{k-i}$. Hence, (23) and (24) can be approximated as having constant coefficients

$$y_1(k) = N\mu_1 \sum_{i=0}^{k-1} r_{k-i} e_{pLMS}(i) \tag{27}$$

$$y_2(k) = N\mu_2 \sum_{i=0}^{k-1} r_{k-i} e_2(i) \tag{28}$$

Furthermore, applying the Z transform to both sides of (27) and (28), we get

$$E[Y_1(Z)] = \mu_1 N E[J_{pLMS}(Z)]R(Z) \tag{29}$$

$$E[Y_2(Z)] = \mu_2 N E[J_2(Z)]R(Z) \tag{30}$$

where

$$E[J_2(Z)] = E[e_2(0)] + E[e_2(1)]Z^{-1} + ... \tag{31}$$

Additionally, by taking the expectation, it is assumed that r_{k-i} is the auto-correlation coefficient instead of its estimate [10] and $R(z) = r_1 Z^{-1} + r_2 Z^{-2} + ...$ is a polynomial in the Z field. Furthermore, from (19), (29) and (30), we get

$$\begin{aligned} E[D(Z)] - jZ^{-1}E[D(Z)] = & E[J_{pLMS}(Z)] \\ & + \mu_1 N E[J_{pLMS}(Z)]R(Z) \\ & - jZ^{-1}\mu_2 N E[J_2(Z)]R(Z) \end{aligned} \tag{32}$$

Using (32), and the LMS transfer function approximation in [10], we can write

$$E[J_2(Z)] = \frac{E[D(Z)]}{1 + \mu_2 N R(Z)} \tag{33}$$

The pLMS transfer function approximate becomes

$$H_{pLMS}(Z) = \frac{1 + \mu_2 N R(Z) - jZ^{-1}}{(1 + \mu_1 N R(Z))(1 + \mu_2 N R(Z))} \tag{34}$$

While (34) presented a simple approximation for the behavior of the pLMS adaptive filters, it is not and doesn't represent the optimal least-square solutions for the steady-state behavior of the filters. Additionally the presented relationship, starting at $k = 0$, includes both convergence and steady-state results [10], [11].

V. Simulation Results and Discussion

Simulations are conducted for a linear antenna array of $N = 8$ elements with 500 realizations of 500 samples. The input signals are one message signal and two interferes with an angle of arrival (AOA) of 45^o, 20^o and 70^o, respectively. The generated inputs are independent random complex Gaussian sequences u of the form $u = \mathcal{N}(0, \sigma_r^2) + j\mathcal{N}(0, \sigma_c^2)$ where $\mathcal{N}(m, \sigma^2)$ denotes normal (Gaussian) distribution with mean m and variances σ^2. The resulting input is corrupted by CAWGN noise with a $SNR = 10dB$. The parameters and initial conditions at $k = 0$ are given as, $\mu_1 = \mu_2 = \mu = 0.01$, $d(-1) = d(0)$, $\mathbf{x}(-1) = \mathbf{x}(0)$, $\sigma_r^2 = 0.02$ and $\sigma_c^2 = 0.05$. As for the RLS the initial parameters are the forgetting factor $\alpha = 0.97$ and initial matrix $\mathbf{P}(0) = 0.6I$, where I is the $N \times N$ identity matrix. The simulation was re-made with 10 realizations of 10 samples each for the transfer function pole-zero plot.

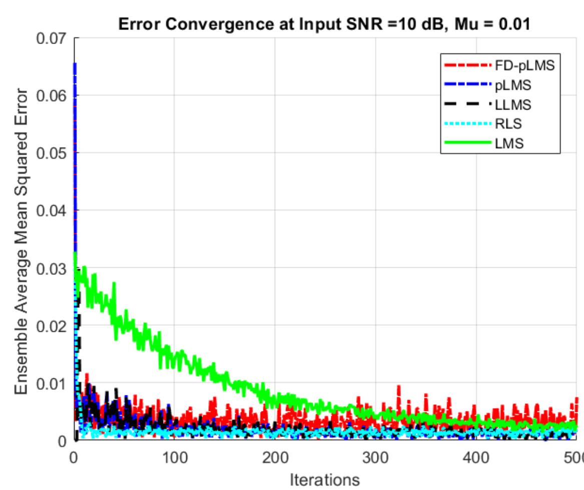

Fig. 2. pLMS MSE Convergence Behavior

Simulation results describing the mean square error (MSE) convergence behavior is presented in Fig. 2 for the pLMS, pLMS transfer function subject to fractional delay (FD-pLMS), LLMS, LMS and the RLS. As shown in Fig. 2 compared to the LLMS and RLS the pLMS achieved accelerated

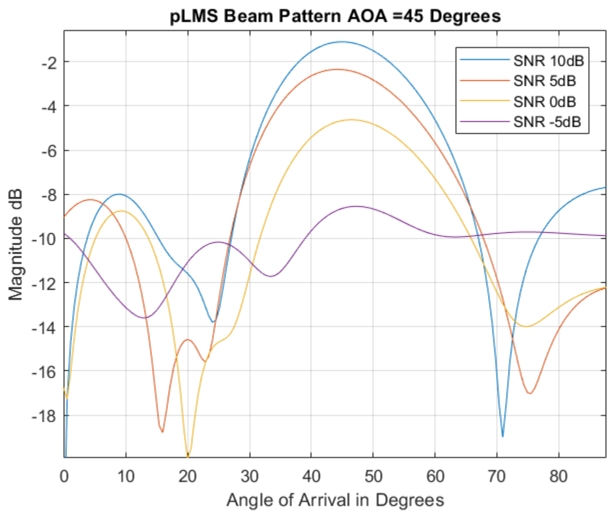

Fig. 3. pLMS MSE Convergence Behavior for Different SNR

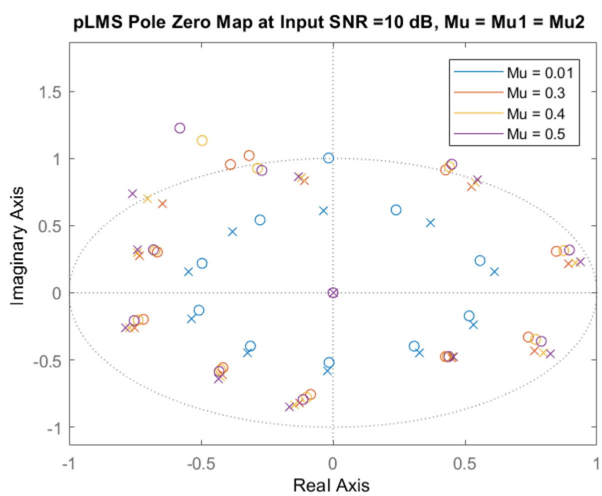

Fig. 4. pLMS Pole Zero Map for Different μ

convergence behavior while maintaining an acceptably low residual error at steady state for a lower complexity and a division free structure, i.e. computational complexity of order $O(N)$. Moreover, the pLMS-FD presented the same accelerated convergence behavior as the pLMS and RLS however at the cost of a larger residual error floor. The increase in the steady state error is a result of the fractional delay filter representing spatially sampled signals in temporal form.

To further assess the performance of the pLMS, the beam pattern is simulated for SNRs ranging from $-5dB$ to $10dB$ with a step of $5dB$ and shown in Fig. 3. From Fig. 3, it is clear that the pLMS achieved its convergence and maintained its beam pointing accuracy for the desired direction of 45^o, up to a $SNR = -5dB$. Furthermore, stability analysis for the pLMS is studied through the pole zero plot for different μ, to numerically determine the maximum step size for which the pLMS remains stable and is shown in Fig. 4. From Fig. 4, it is clear that as μ increases the poles and zeros move closer to the outside of the unit circle, where the maximum step size for stability falls in the range of $0.3 < \mu < 0.4$.

VI. Conclusion and Future Work

In this paper, we proposed a multi stage parallel LMS (pLMS) structure for adaptive beamforming and its transfer function approximation to numerically assess its stability. pLMS is achieved by two least mean square (LMS) stages with parallel inputs and connected by an error feedback. Experimental results have shown superior performance for the pLMS demonstrated by its accelerated convergence, low residual error and low computational complexity for different signal to noise ratio environments. Additionally, a transfer function approximation of the pLMS is derived to numerically assess the systems stability and determine the maximum value of the step size for which the system remains stable. Future work includes performing additional complexity reduction for implementing the pLMS structure in a pipelined manner on re-configurable hardware and embedded systems.

Acknowledgment

The authors are grateful to AID - DGA (l'Agence de l'Innovation de Défense à la Direction Générale de l'Armement – Minitère des Armées) & ANR (Agence Nationale de le Recherche en France) for supporting our ANR-ASTRID – Project (ANR-19-ASTR-0005-03).

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