Rao Test With Improved Robustness for Range-Spread Target Detection

Shengyin Sun, Jun Liu
Department of Electronic Engineering and Information Science
University of Science and Technology of China
230027, Hefei, China
email: junliu@ustc.edu.cn

Weijian Liu
Wuhan Electronic Information Institute
430019, Wuhan, China

***Abstract*—This paper deals with the problem of detecting range-spread targets in Gaussian noise with unknown covariance matrix. We model the received signal under the signal-plus-noise hypothesis as the sum of noise, useful target echoes and fictitious signals, which makes the signal-plus-noise hypothesis more plausible in the mismatched case. An adaptive detector is designed according to the Rao test. We prove the proposed Rao test exhibits constant false alarm rate property against the covariance matrix. Numerical examples show that the proposed Rao outperforms its counterparts in the mismatched case.**

***Index Terms*—Adaptive detection, distributed targets, constant false alarm rate, steering vector mismatch, Rao test.**

I. Introduction

Adaptive detection under the background of Gaussian noise with unknown covariance matrix has received an increasing attention in the past decades [1]–[10]. The adaptation here means all unknown parameters can be estimated from the received data during the design procedure of detectors. Based on the criterion of generalized likelihood ratio test (GLRT), Kelly first proposed an adaptive GLRT detector to detect point-like targets in [11], which is deemed to be the pioneering work in this field. Robey *et al.* proposed an adaptive matched filter (AMF) in [12], which is more computationally efficient than the GLRT detector. According to the criterion of Rao test, De Maio proposed an adaptive detector in [13], which has strong rejection capabilities of mismatched signals. In addition, it is shown in [14] that the Wald test for the detection problem in [11]–[13] is the same as the AMF. It should be pointed out that the GLRT, AMF and Rao test exhibit the constant false alarm rate (CFAR) properties against the noise covariance matrix.

All the studies in [11]–[13] are aimed at solving the problem of detecting point-like targets. In practice, when the size of the target to be detected is large or the radar resolution is high, the echoes reflected from the target would occupy several range cells. This type of target is called range-spread or distributed target. To solve the problem of range-spread target detection, many studies have been carried out [15]–[22]. In [23], the range-spread target is assumed to occupy entire range cells. In this case, a detector was proposed to detect range-spread targets in the background of white noise with known power. When the scattering centers of the target occupy only a part of the range cells, the detector proposed in [23] suffers huge performance loss. Faced with this problem, researchers considered using prior knowledge of spatial scattering intensity to design detectors [24]. Detectors based on the GLRT, Rao test and Wald test were proposed to address the range-spread target detection in [25], [26].

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For most of the above-mentioned studies, the steering vector mismatches are not taken into consideration. However, mismatches in steering vectors often occur due to uncalibrated arrays, waveform distortions, etc [27]–[29]. Therefore, in practical applications, we prefer to get a detector that is robust to mismatches. In order to address the problem of detecting point-like targets, a framework for designing robust detectors has been proposed in [30]. Under this framework, the received signal under signal-plus-noise hypothesis is modeled as the sum of noise, useful target echoes and fictitious signals, which makes the detector more robust to mismatches. Inspired by this framework, a robust detector based on the GLRT criterion was proposed to solve the problem of range-spread target detection in [31]. Since there exists no uniformly most powerful (UMP) test for the range-spread target detection (noise covariance matrix is unknown), it is feasible to design the detector by using some criteria other than the GLRT.

In this paper, we consider the range-spread target detection problem in the case of signal mismatch in Gaussian noise with unknown covariance matrix. We model the received signal under signal-plus-noise hypothesis as the sum of noise, useful target echoes and fictitious signals. The fictitious signals are added to make the signal-plus-noise hypothesis more plausible, and hence renders the detector more robust to the steering vector mismatches. Based on the criterion of Rao test, an adaptive detector is derived, which exhibits the CFAR property against the covariance matrix. Simulation results show that the proposed Rao test exhibits the strongest robustness in the mismatched case.

The remainder of this paper is organized as follows.[1] In Section II, we formulate the problem of distributed target detection. Based on the design criteria of Rao test, a detector is proposed in Section III. Section IV proves the CFAR properties of the proposed detector against the noise covariance matrix. Numerical examples are provided in Section V. Finally, the paper is summarized in Section VI.

II. Problem Formulation

Suppose that data are collected from a pulsed Doppler radar system which sends N pulse trains in a coherent processing interval (CPI). We examine the problem of detecting the presence of a range-spread target across K range cells. Denote the data in the kth range cell by an $N \times 1$ column vector $\mathbf{x}_k, k = 1, 2, \ldots, K$. These data $\mathbf{x}_k$ are usually called primary data or test data. The detection problem is equivalent to making a decision between hypothesis H_0 and hypothesis H_1, where H_0 means $\mathbf{x}_k$ only contains noise $\mathbf{n}_k$ and H_1 means $\mathbf{x}_k$ contains noise $\mathbf{n}_k$ and a signal of interest $\mathbf{s}_k$. To make H_1 more plausible, a fictitious component $\boldsymbol{\xi}_k$ is added under H_1 [31], which hopefully improves the robustness of the detector in the case of target steering vector mismatch[2]. Precisely, $\mathbf{x}_k$ under H_1 can be written as

$$\mathbf{x}_k = \alpha_k \mathbf{v} + \mathbf{n}_k + \boldsymbol{\xi}_k, \quad k = 1, 2, \ldots, K, \tag{1}$$

where α_k is an unknown complex scalar accounting for the target reflectivity and channel propagation effects, $\mathbf{v} \in \mathbb{C}^{N\times 1}$ is a known target steering vector, $\mathbf{n}_k \in \mathbb{C}^{N\times 1}$ is the noise in the kth range cell.

Assume that $\mathbf{n}_k$ has independent and identically distributed (IID), complex, circularly symmetric, Gaussian distribution with zero mean and unknown (Hermitian) positive definite matrix $\mathbf{R}$, i.e., $\mathbf{n}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{R})$ for $k = 1, 2, \ldots, K$. The fictitious signals $\boldsymbol{\xi}_k$, $k = 1, 2, \ldots, K$, are assumed to be IID, complex, circularly symmetric, Gaussian vector with zero mean and a covariance matrix $\gamma\boldsymbol{\Sigma}$, i.e., $\boldsymbol{\xi}_k \sim \mathcal{CN}(\mathbf{0}, \gamma\boldsymbol{\Sigma})$. The parameter γ is used to control the robustness of the detector and the matrix $\boldsymbol{\Sigma}$ is designed to capture some leakage of the signal caused by mismatches, which makes the detector more inclined to decide for H_1 in the mismatched case [31]. In the following discussion, we chose $\boldsymbol{\Sigma} = \mathbf{R}$. It is reasonable when prior knowledge about the mismatch is not available [31]. Suppose further that the fictitious term $\boldsymbol{\xi}_k$ is independent of the noise term $\mathbf{n}_k$. It is easy to check $\mathbf{x}_k$ has the following distribution

$$\mathbf{x}_k \sim \mathcal{CN}(\alpha_k \mathbf{v}, (1+\gamma)\mathbf{R}), \quad k = 1, 2, \ldots, K. \tag{2}$$

In addition to primary data $\mathbf{x}_k$, a set of secondary data $\{\dot{\mathbf{x}}_l\}_{l=1}^{L}$ is often assumed to be available, which can be used to estimate $\mathbf{R}$. These secondary data free of the target echoes are usually collected in the vicinity of the primary data.

Under the assumptions above, the range-spread target detection problem at hand turns out to be the following binary hypothesis test

$$H_0 : \begin{cases} \mathbf{x}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}), \ k = 1, 2, \cdots, K, \\ \dot{\mathbf{x}}_l \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}), \ l = 1, 2, \cdots, L, \end{cases} \tag{3a}$$

$$H_1 : \begin{cases} \mathbf{x}_k \sim \mathcal{CN}(\alpha_k \mathbf{v}, (1+\gamma)\mathbf{R}), \ k = 1, 2, \cdots, K, \\ \dot{\mathbf{x}}_l \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}), \ l = 1, 2, \cdots, L, \end{cases} \tag{3b}$$

where $\alpha_k, \mathbf{R}$ and $\gamma \geq 0$ are unknown, $\mathbf{v}$ is a known vector. It is needed to emphasize that the hypothesis H_1 in the detection problem (3) is different from that in [25], [26], since the term $\gamma\mathbf{R}$ is added to the noise covariance matrix $\mathbf{R}$.

Note that the GLRT-based detectors for the detection problem in (3) were derived in [31]. However, there is no UMP test for the detection problem in (3). Hence, we design a new detector according to the criterion of Rao test in the following.

III. Robust Rao Detection

To solve the detection problem in (3), we resort to the design principle of Rao test. For future reference, we denote by

- $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_K] \in \mathbb{C}^{N\times K}$ and $\mathbf{X_L} = [\dot{\mathbf{x}}_1, \ldots, \dot{\mathbf{x}}_L] \in \mathbb{C}^{N\times L}$ the primary and secondary data matrices, respectively;
- $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_K]^T \in \mathbb{C}^{K\times 1}$ a K-dimensional column vector;
- $\mathbf{X}_\alpha = [\mathbf{x}_1 - \alpha_1\mathbf{v}, \ldots, \mathbf{x}_K - \alpha_K\mathbf{v}] \in \mathbb{C}^{N\times K}$ the matrix with useful signal components removed under H_1 hypothesis;
- $\boldsymbol{\Theta}_\mathrm{r} = [\mathfrak{Re}(\alpha_1), \mathfrak{Im}(\alpha_1), \ldots, \mathfrak{Re}(\alpha_K), \mathfrak{Im}(\alpha_K), \gamma]^T \in \mathbb{C}^{(2K+1)\times 1}$ the relative parameters. In the detection problem (3), we can see that parameter γ is related to the target signal, so it makes sense to put parameter γ in the relative parameters;
- $\boldsymbol{\Theta}_\mathrm{s} = \mathrm{vec}(\mathbf{R}) \in \mathbb{C}^{\mathrm{N}^2\times 1}$ the nuisance parameters;
- $\boldsymbol{\Theta} = [\boldsymbol{\Theta}_\mathrm{r}^T, \boldsymbol{\Theta}_\mathrm{s}^T]^T \in \mathbb{C}^{(2K+N^2+1)\times 1}$ the parameter column vector.

Here, we derive a detector based on the Rao test. First, we assume that the noise covariance matrix $\mathbf{R}$ is known. The PDF

[1] *Notation*: Vectors (matrices) are denoted by boldface lower (upper) case letters. Superscripts $(\cdot)^*$, $(\cdot)^T$ and $(\cdot)^H$ denote complex conjugate, transpose and complex conjugate transpose, respectively. $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$ are real and complex matrix spaces of dimension $m\times n$. $\mathbf{I}_n$ stands for an identity matrix of $n\times n$, $\mathbf{0}_{m\times n}$ represents a null matrix of dimension $m\times n$ and $\mathbf{U}_n$ represents a unitary matrix of dimension $n \times n$. The notation $\sim$ means " be distributed as ", $\mathcal{CN}(\boldsymbol{\mu}_c, \mathbf{R}_c)$ denotes a circularly symmetric, complex-valued Gaussian distribution with mean $\boldsymbol{\mu}_c$ and covariance matrix $\mathbf{R}_c$, $\mathcal{N}(\boldsymbol{\mu}_r, \mathbf{R}_r)$ represents a real-valued Gaussian distribution with mean $\boldsymbol{\mu}_r$ and covariance matrix $\mathbf{R}_r$, $\mathcal{CW}(N, M, \mathbf{R})$ denotes the N-dimensional real-valued Wishart distribution with M degrees of freedom and scale matrix $\mathbf{R}$, and $\mathrm{E}(\cdot)$ is the statistical expectation. $\otimes$ is the Kronecker product. The matrix $\mathbf{P_B} = \mathbf{B}(\mathbf{B}^H\mathbf{B})^{-1}\mathbf{B}^H$ is the projection matrix onto the subspace spanned by the columns of the matrix $\mathbf{B}$, and $\mathbf{P}_\mathbf{B}^\perp = \mathbf{I} - \mathbf{P_B}$. $\mathrm{vec}(\mathbf{B})$ vectorizes $\mathbf{B}$ by stacking its columns. $|\cdot|$ represents the determinant of a matrix and the modulus of a scalar, when the argument is a matrix and a scalar, respectively. $\|\cdot\|$ is the Euclidean norm of a vector. $\mathfrak{Re}(\cdot)$ and $\mathfrak{Im}(\cdot)$ represent the real and imaginary parts of a complex quantity, respectively. $\mathrm{Tr}(\cdot)$ denotes the trace of a square matrix, and $\jmath = \sqrt{-1}$.

[2] The idea of adding a fictitious signal under some hypothesis can also be found in [32], where the fictitious signal is added under H_0 to improve the mismatched discrimination performance (i.e., selectivity).

of $\mathbf{X}$ under hypothesis H_1 can be expressed as

$$f_1(\mathbf{X}|\mathbf{R},\gamma,\boldsymbol{\alpha})=\frac{\pi^{-NK}}{(1+\gamma)^{NK}|\mathbf{R}|^K}e^{-\mathrm{tr}\left(\frac{1}{1+\gamma}\mathbf{R}^{-1}\mathbf{X}_\alpha\mathbf{X}_\alpha{}^{\mathrm{H}}\right)}. \tag{4}$$

The design criterion of Rao test for the problem at hand is [33]

$$\left.\frac{\partial \ln f_1(\mathbf{X}|\boldsymbol{\Theta})}{\partial\boldsymbol{\Theta}_{\mathrm{r}}}\right|^T_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}_0}\left[\mathbf{I}^{-1}(\hat{\boldsymbol{\Theta}}_0)\right]_{\boldsymbol{\Theta}_{\mathrm{r}}\boldsymbol{\Theta}_{\mathrm{r}}}\left.\frac{\partial \ln f_1(\mathbf{X}|\boldsymbol{\Theta})}{\partial\boldsymbol{\Theta}_{\mathrm{r}}}\right|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}_0}, \tag{5}$$

where $\hat{\boldsymbol{\Theta}}_0$ is the MLE of $\boldsymbol{\Theta}$ under H_0, $\mathbf{I}(\boldsymbol{\Theta})$ is the Fisher information matrix (FIM) with respect to (w.r.t.) $\boldsymbol{\Theta}$, and $[\mathbf{I}^{-1}(\boldsymbol{\Theta})]_{\boldsymbol{\Theta}_{\mathrm{r}}\boldsymbol{\Theta}_{\mathrm{r}}}$ is part of the inversion of the $\mathbf{I}(\boldsymbol{\Theta})$. The $\mathbf{I}(\boldsymbol{\Theta})$ can be partitioned as

$$\mathbf{I}(\boldsymbol{\Theta})=\begin{bmatrix}\mathbf{I}_{\boldsymbol{\Theta}_{\mathrm{r}},\boldsymbol{\Theta}_{\mathrm{r}}}(\boldsymbol{\Theta}) & \mathbf{I}_{\boldsymbol{\Theta}_{\mathrm{r}},\boldsymbol{\Theta}_{\mathrm{s}}}(\boldsymbol{\Theta})\\ \mathbf{I}_{\boldsymbol{\Theta}_{\mathrm{s}},\boldsymbol{\Theta}_{\mathrm{r}}}(\boldsymbol{\Theta}) & \mathbf{I}_{\boldsymbol{\Theta}_{\mathrm{s}},\boldsymbol{\Theta}_{\mathrm{s}}}(\boldsymbol{\Theta})\end{bmatrix}, \tag{6}$$

and

$$\begin{aligned}\left[\mathbf{I}^{-1}(\boldsymbol{\Theta})\right]_{\boldsymbol{\Theta}_{\mathrm{r}},\boldsymbol{\Theta}_{\mathrm{r}}}=&\big[\mathbf{I}_{\boldsymbol{\Theta}_{\mathrm{r}},\boldsymbol{\Theta}_{\mathrm{r}}}(\boldsymbol{\Theta})-\mathbf{I}_{\boldsymbol{\Theta}_{\mathrm{r}},\boldsymbol{\Theta}_{\mathrm{s}}}(\boldsymbol{\Theta})\\ &\times\mathbf{I}^{-1}_{\boldsymbol{\Theta}_{\mathrm{s}},\boldsymbol{\Theta}_{\mathrm{s}}}(\boldsymbol{\Theta})\mathbf{I}_{\boldsymbol{\Theta}_{\mathrm{s}},\boldsymbol{\Theta}_{\mathrm{r}}}(\boldsymbol{\Theta})\big]^{-1}.\end{aligned} \tag{7}$$

In order to get the FIM, we need to take the derivative of the logarithm of (4) w.r.t. $\boldsymbol{\Theta}_{\mathrm{r}}$. For our problem it can be easily shown that

$$\frac{\partial \ln f_1(\mathbf{X}|\mathbf{R},\gamma,\boldsymbol{\alpha})}{\partial\mathfrak{Re}(\alpha_k)}=\frac{2\mathfrak{Re}\{\mathbf{v}^H\mathbf{R}^{-1}(\mathbf{x}_k-\alpha_k\mathbf{v})\}}{1+\gamma}, \tag{8}$$

$$\frac{\partial \ln f_1(\mathbf{X}|\mathbf{R},\gamma,\boldsymbol{\alpha})}{\partial\mathfrak{Im}(\alpha_k)}=\frac{2\mathfrak{Im}\{\mathbf{v}^H\mathbf{R}^{-1}(\mathbf{x}_k-\alpha_k\mathbf{v})\}}{1+\gamma}, \tag{9}$$

$$\frac{\partial \ln f_1(\mathbf{X}|\mathbf{R},\gamma,\boldsymbol{\alpha})}{\partial\gamma}=\frac{\mathrm{tr}(\mathbf{R}^{-1}\mathbf{X}_\alpha\mathbf{X}_\alpha^{\mathrm{H}})}{(1+\gamma)^2}-\frac{NK}{1+\gamma}. \tag{10}$$

According to (8)–(10), we can obtain the derivative of the logarithm of (4) w.r.t. $\boldsymbol{\Theta}_{\mathrm{r}}$ evaluated at $\hat{\boldsymbol{\Theta}}_0$ as

$$\begin{aligned}\left.\frac{\partial \ln f_1(\mathbf{X}|\boldsymbol{\Theta})}{\partial\boldsymbol{\Theta}_{\mathrm{r}}}\right|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}_0}=2\Big[&\mathfrak{Re}(\mathbf{v}^H\mathbf{R}^{-1}\mathbf{x}_1),\mathfrak{Im}(\mathbf{v}^H\mathbf{R}^{-1}\mathbf{x}_1),\ldots,\\ &\mathfrak{Re}(\mathbf{v}^H\mathbf{R}^{-1}\mathbf{x}_K),\mathfrak{Im}(\mathbf{v}^H\mathbf{R}^{-1}\mathbf{x}_K),\frac{\mathrm{tr}(\mathbf{R}^{-1}\mathbf{X}\mathbf{X}^{\mathrm{H}})-NK}{2}\Big]^T.\end{aligned} \tag{11}$$

The blocks of the FIM can be obtained as follows

$$\begin{aligned}\mathbf{I}_{\boldsymbol{\Theta}_{\mathrm{r}},\boldsymbol{\Theta}_{\mathrm{r}}}(\boldsymbol{\Theta})&=-\mathrm{E}\left[\frac{\partial^2\ln f_1(\mathbf{X}|\mathbf{R},\gamma,\boldsymbol{\alpha})}{\partial\boldsymbol{\Theta}_{\mathrm{r}}^*\partial\boldsymbol{\Theta}_{\mathrm{r}}^T}\right]\\ &=\begin{bmatrix}\frac{2}{1+\gamma}\mathbf{v}^H\mathbf{R}^{-1}\mathbf{v}\mathbf{I}_{2K} & \mathbf{0}_{2K\times1}\\ \mathbf{0}_{1\times2K} & \frac{NK}{(1+\gamma)^2}\end{bmatrix},\end{aligned} \tag{12}$$

$$\begin{aligned}\mathbf{I}_{\boldsymbol{\Theta}_{\mathrm{s}},\boldsymbol{\Theta}_{\mathrm{r}}}(\boldsymbol{\Theta})&=-\mathrm{E}\left[\frac{\partial^2\ln f_1(\mathbf{X}|\mathbf{R},\gamma,\boldsymbol{\alpha})}{\partial\boldsymbol{\Theta}_{\mathrm{s}}^*\partial\boldsymbol{\Theta}_{\mathrm{r}}^T}\right]\\ &=\begin{bmatrix}\mathbf{0}_{N^2\times2K} & \mathrm{vec}\left(\frac{K}{1+\gamma}\mathbf{R}^{-1}\right)\end{bmatrix},\end{aligned} \tag{13}$$

$$\begin{aligned}\mathbf{I}_{\boldsymbol{\Theta}_{\mathrm{r}},\boldsymbol{\Theta}_{\mathrm{s}}}(\boldsymbol{\Theta})&=-\mathrm{E}\left[\frac{\partial^2\ln f_1(\mathbf{X}|\mathbf{R},\gamma,\boldsymbol{\alpha})}{\partial\boldsymbol{\Theta}_{\mathrm{r}}^*\partial\boldsymbol{\Theta}_{\mathrm{s}}^T}\right]\\ &=\begin{bmatrix}\mathbf{0}_{2K\times N^2}\\ \mathrm{vec}^T\left(\frac{K}{1+\gamma}\mathbf{R}^{-T}\right)\end{bmatrix},\end{aligned} \tag{14}$$

and

$$\begin{aligned}\mathbf{I}_{\boldsymbol{\Theta}_{\mathrm{s}},\boldsymbol{\Theta}_{\mathrm{s}}}(\boldsymbol{\Theta})&=-\mathrm{E}\left[\frac{\partial^2\ln f_1(\mathbf{X}|\mathbf{R},\gamma,\boldsymbol{\alpha})}{\partial\boldsymbol{\Theta}_{\mathrm{s}}^*\partial\boldsymbol{\Theta}_{\mathrm{s}}^T}\right]\\ &=(K+L)\mathbf{R}^{-T}\otimes\mathbf{R}^{-1}.\end{aligned} \tag{15}$$

Plugging (12), (14), (13) and (15) into (7) leads to

$$\left[\mathbf{I}^{-1}(\boldsymbol{\Theta})\right]_{\boldsymbol{\Theta}_{\mathrm{r}},\boldsymbol{\Theta}_{\mathrm{r}}}=\begin{bmatrix}\frac{2}{1+\gamma}\mathbf{v}^H\mathbf{R}^{-1}\mathbf{v}\mathbf{I}_{2K} & \mathbf{0}_{2K\times1}\\ \mathbf{0}_{1\times2K} & c_0\end{bmatrix}^{-1}, \tag{16}$$

where

$$c_0=\frac{NKL}{(K+L)(1+\gamma)^2}. \tag{17}$$

According to (16), the $\left[\mathbf{I}^{-1}(\boldsymbol{\Theta})\right]_{\boldsymbol{\Theta}_{\mathrm{r}}\boldsymbol{\Theta}_{\mathrm{r}}}$ evaluated at $\hat{\boldsymbol{\Theta}}_0$ is

$$\left[\mathbf{I}^{-1}(\hat{\boldsymbol{\Theta}}_0)\right]_{\boldsymbol{\Theta}_{\mathrm{r}}\boldsymbol{\Theta}_{\mathrm{r}}}=\begin{bmatrix}(2\mathbf{v}^H\mathbf{R}^{-1}\mathbf{v}\mathbf{I}_{2K})^{-1} & \mathbf{0}_{2K\times1}\\ \mathbf{0}_{1\times2K} & c_1{}^{-1}\end{bmatrix}, \tag{18}$$

where

$$c_1=\frac{NKL}{K+L}. \tag{19}$$

Substituting (11) and (18) into (5) leads to the Rao test with known $\mathbf{R}$

$$t_{\mathrm{Rao|}\mathbf{R}}=\mathbf{b}^T(2\mathbf{v}^H\mathbf{R}^{-1}\mathbf{v}\mathbf{I}_{2K})^{-1}\mathbf{b}+c_1{}^{-1}c_2{}^2, \tag{20}$$

where $c_2=\mathrm{tr}(\mathbf{R}^{-1}\mathbf{X}\mathbf{X}^H)-NK$, and $\mathbf{b}=\left[b_{R1},b_{I1},\ldots,b_{RK},b_{IK}\right]^T$, with $b_{Rk}=2\mathfrak{Re}\left(\mathbf{v}^H\mathbf{R}^{-1}\mathbf{x}_k\right)$, and $b_{Ik}=2\mathfrak{Im}\left(\mathbf{v}^H\mathbf{R}^{-1}\mathbf{x}_k\right)$.

Define $\mathbf{S}$ as the sample covariance matrix (SCM), i.e.,

$$\mathbf{S}=\frac{1}{L}\mathbf{X}_{\mathbf{L}}\mathbf{X}_{\mathbf{L}}{}^H. \tag{21}$$

Replacing $\mathbf{R}$ in (20) with $\mathbf{S}$, after some algebra, yields a more compact representation of the Rao test

$$t_{\mathrm{Rao}}=2Lt_{\mathrm{GAMF}}+\frac{K+L}{NKL}\hat{c}_2^2, \tag{22}$$

where t_{GAMF} is the generalized adaptive matched filter, i.e.,

$$t_{\mathrm{GAMF}}=\frac{\mathbf{v}^H\mathbf{S}_0^{-1}\mathbf{X}\mathbf{X}^H\mathbf{S}_0^{-1}\mathbf{v}}{\mathbf{v}^H\mathbf{S}_0^{-1}\mathbf{v}}, \tag{23}$$

and

$$\hat{c}_2=\mathrm{tr}(\mathbf{S}^{-1}\mathbf{X}\mathbf{X}^H)-NK. \tag{24}$$

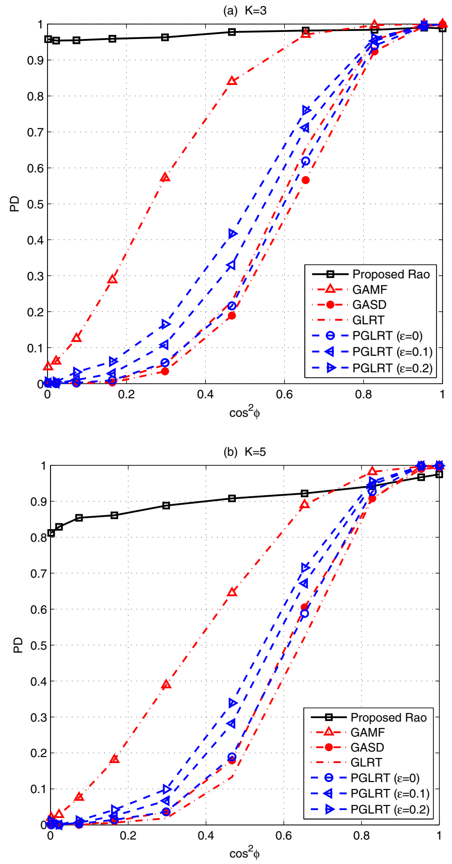

Fig. 1. PD versus $\cos^2\phi$, $N = 16$, SNR $= 21$ dB, $L = 32$.

IV. CFAR Analysis

Note that the Rao test in (22) is related to two quantities t_{GAMF} and $\hat{c}_2$. It has already been proven in [25] that t_{GAMF} is independent of the covariance matrix $\mathbf{R}$ under H_0. Now, we only need to prove $\hat{c}_2$ under H_0 has a distribution independent of $\mathbf{R}$, if we want to prove the CFAR property of the proposed Rao test.

Let $\bar{\mathbf{S}} = \mathbf{R}^{-1/2}\mathbf{S}\mathbf{R}^{-1/2}$, and $\bar{\mathbf{X}} = \mathbf{R}^{-1/2}\mathbf{X}$. We observe that

$$\mathrm{tr}(\mathbf{S}^{-1}\mathbf{X}\mathbf{X}^{\mathrm{H}}) = \mathrm{tr}(\mathbf{X}^H\mathbf{S}^{-1}\mathbf{X}) = \mathrm{tr}(\bar{\mathbf{X}}_u^H\bar{\mathbf{S}}_u^{-1}\bar{\mathbf{X}}_u), \quad (25)$$

where

$$\bar{\mathbf{S}}_u = \mathbf{U}_N^H\bar{\mathbf{S}}\mathbf{U}_N \sim \mathcal{CW}(N, K, \mathbf{I}_N \otimes \mathbf{I}_N), \quad (26)$$

and

$$\bar{\mathbf{X}}_u = \mathbf{U}_N^H\bar{\mathbf{X}} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_N \otimes \mathbf{I}_K). \quad (27)$$

Obviously both $\bar{\mathbf{X}}_u$ and $\bar{\mathbf{S}}_u$ are independent of $\mathbf{R}$. As a result, $\hat{c}_2$ is independent of $\mathbf{R}$.

Note taht the CFAR property of the GAMF has been described in [25], hence the proposed Rao test exhibits the CFAR property against the noise covariance matrix $\mathbf{R}$.

V. Numerical Examples

In this section, we assess the performance of the proposed detectors by Monte Carlo simulation. Assume that a pulsed Doppler radar is used and the number of pulses in a CPI is $N = 16$. The probability of false (PFA) is set to be 10^{-4}. In order to obtain the detection threshold and the probability of detection (PD), $100/\mathrm{PFA}$ and 10^4 independent trials are used, respectively.

Base on the simplified model for the spectral shape in [34], we construct $\mathbf{R}$ as the sum of a colored noise covariance $\mathbf{R}_1$ plus a thermal noise matrix 10 dB weaker, i.e., $\mathbf{R} = \mathbf{R}_1 + \sigma_n^2\mathbf{I}_N$. The (i,j)th element of $\mathbf{R}_1$ is chosen as $[\mathbf{R}_1]_{i,j} = \exp[-2\pi^2\sigma^2(i-j)^2]$. We assume $\sigma = 0.05$, which corresponds to a one-lag correlation coefficient of $\rho = \exp(-2\pi^2\sigma^2) = 0.95$. The target steering vector $\mathbf{v}$ is selected as $\mathbf{v} = [1, e^{\jmath 2\pi f_d}, \ldots, e^{\jmath 2\pi(N-1)f_d}]^T$, where $f_d = 0.08$ is the normalized Doppler frequency. The signal-to-noise ratio is defined as

$$\mathrm{SNR} = \sum_{k=1}^{K} |\alpha_k|^2 \mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}. \quad (28)$$

To analyze the robustness of the detectors, we define the mismatch between $\mathbf{v}$ and true target steering vector (denoted by $\mathbf{p}$) is

$$\cos^2\phi = \frac{|\mathbf{v}^H\mathbf{R}^{-1}\mathbf{p}|^2}{(\mathbf{v}^H\mathbf{R}^{-1}\mathbf{v})(\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p})}, \quad (29)$$

where ϕ is the angle between assumed steering vector $\mathbf{v}$ and actual steering vector $\mathbf{p}$. In the case of mismatch, the Doppler frequency of $\mathbf{p}$ is $f_d + \delta/N$. It is worth noting that $\delta = 0$ ($\cos^2\phi = 1$) corresponds to the matched case, while $\delta = 0.5$ implies $\cos^2\phi = 0.30$.

For comparison purposes, the parametric GLRT (PGLRT) [31, eq. (14)], the GLRT [25, eq. (12)], the GAMF [25, eq. (25)] and the generalized adaptive subspace detector (GASD) [25, eq. (26)] are considered.

Fig. 1 shows the detection probability versus $\cos^2\phi$ for the case of SNR $= 21$ dB. It can be seen that the proposed Rao test processes the strongest robustness against the mismatches. More specifically, when $\cos^2\phi$ changes from 1 to 0, the performance of the proposed Rao test is basically unchanged, while its counterparts suffer huge performance losses with the increases in the mismatches. The strong robustness of the proposed Rao test in the mismatched case is obtained at the price of performance losses in the matched case.

VI. Conclusions

We have considered the range-spread target detection problem in Gaussian noise with unknown covariance matrix. To make H_1 more plausible in the mismatched case, we model the received signal under H_1 as the sum of noise, useful target echoes and fictitious signals. According to the criterion of Rao test, we have designed an adaptive detector. The proposed Rao test shows the CFAR property against the covariance matrix. Simulation results demonstrate that the proposed Rao test has the strongest robustness in the mismatched case, at the price of sacrificing performance in the matched case.

References

[1] A. De Maio and M. S. Greco, Eds., *Modern Radar Detection Theory*. ser. Radar, Sonar and Navigation. Stevenage: U.K.: Inst. Eng. Technol., 2015.

[2] W. Liu, W. Xie, J. Liu, and Y. Wang, "Adaptive double subspace signal detection in Gaussian background–Part I: Homogeneous environments," *IEEE Transactions on Signal Processing*, vol. 62, no. 9, pp. 2345–2357, May 2014.

[3] A. De Maio, D. Orlando, C. Hao, and G. Foglia, "Adaptive detection of point-like targets in spectrally symmetric interference," *IEEE Transactions on Signal Processing*, vol. 64, no. 12, pp. 3207–3220, December 2016.

[4] A. Coluccia and G. Ricci, "Adaptive radar detectors for point-like gaussian targets in Gaussian noise," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 53, no. 3, pp. 1284–1294, June 2017.

[5] P. Wang, H. Li, and B. Himed, "Knowledge-aided parametric tests for multichannel adaptive signal detection," *IEEE Transactions on Signal Processing*, vol. 59, no. 12, pp. 5970–5982, December 2011.

[6] Y. Gao, H. Li, and B. Himed, "Adaptive subspace tests for multichannel signal detection in auto-regressive disturbance," *IEEE Transactions on Signal Processing*, vol. 66, no. 21, pp. 5577–5587, November 1 2018.

[7] D. Orlando and G. Ricci, "Adaptive radar detection and localization of a point-like target," *IEEE Transactions on Signal Processing*, vol. 59, no. 9, pp. 4086–4096, September 2011.

[8] A. De Maio and D. Orlando, "Adaptive radar detection of a subspace signal embedded in subspace structured plus Gaussian interference via invariance," *IEEE Transactions on Signal Processing*, vol. 64, no. 8, pp. 2156–2167, April 2016.

[9] D. Ciuonzo, A. De Maio, and D. Orlando, "A unifying framework for adaptive radar detection in homogeneous plus structured interference–Part II: Detectors design," *IEEE Transaction on Signal Processing*, vol. 11, no. 64, pp. 2907–2919, June 1 2016.

[10] D. Ciuonzo, V. Carotenuto, and A. De Maio, "On multiple covariance equality testing with application to sar change detection," *IEEE Transactions on Signal Processing*, vol. 65, no. 19, pp. 5078–5091, May 2017.

[11] E. J. Kelly, "An adaptive detection algorithm," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 22, no. 1, pp. 115–127, March 1986.

[12] F. C. Robey, D. R. Fuhrmann, E. J. Kelly, and R. Nitzberg, "A CFAR adaptive matched filter detector," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 28, no. 1, pp. 208–216, January 1992.

[13] A. De Maio, "Rao test for adaptive detection in Gaussian interference with unknown covariance matrix," *IEEE Transactions on Signal Processing*, vol. 55, no. 7, pp. 3577–3584, July 2007.

[14] ——, "A new derivation of the adaptive matched filter," *IEEE Signal Processing Letters*, vol. 11, no. 10, pp. 792–793, October 2004.

[15] F. Bandiera, A. De Maio, A. S. Greco, and G. Ricci, "Adaptive radar detection of distributed targets in homogeneous and partially homogeneous noise plus subspace interference," *IEEE Transactions on Signal Processing*, vol. 55, no. 4, pp. 1223–1237, April 2007.

[16] Y. Gao, H. Li, and B. Himed, "Knowledge-aided range-spread target detection for distributed MIMO radar in nonhomogeneous environments," *IEEE Transactions on Signal Processing*, vol. 65, no. 3, pp. 617–627, February 2017.

[17] C. Hao, X. Ma, X. Shang, and L. Cai, "Adaptive detection of distributed targets in partially homogeneous environment with Rao and Wald tests," *Signal Processing*, vol. 92, pp. 926–930, 2012.

[18] Z. Wang, M. Li, H. Chen, Y. Lu, R. Cao, P. Zhang, L. Zuo, and Y. Wu, "Persymmetric detectors of distributed targets in partially homogeneous disturbance," *Signal Processing*, vol. 128, pp. 382–388, 2016.

[19] C. Hao, J. Yang, X. Ma, C. Hou, and D. Orlando, "Adaptive detection of distributed targets with orthogonal rejection," *IET Radar, Sonar and Navigation*, vol. 6, no. 6, pp. 483–493, 2012.

[20] M. Tang, Y. Rong, J. Zhou, and X. R. Li, "Invariant adaptive detection of range-spread targets under structured noise covariance," *IEEE Transactions on Signal Processing*, vol. 65, no. 12, pp. 3048–3061, June 15 2017.

[21] L. Xiao, Y. Liu, T. Huang, X. Liu, and X. Wang, "Distributed target detection with partial observation," *IEEE Transactions on Signal Processing*, vol. 66, no. 6, pp. 1551–1565, March 2018.

[22] L. Xiao, Y. Liu, T. Huang, L. Wang, and X. Wang, "Detection of subspace distributed target in partial observation scenario with Rao test," *Signal Processing*, vol. 166, 2020.

[23] G. Van der Spek, "Detection of a distributed target," *IEEE Transactions on Aerospace and Electronic Systems*, no. 5, pp. 922–931, September 1971.

[24] K. Gerlach, M. Steiner, and F. C. Lin, "Detection of a spatially distributed target in white noise," *IEEE Signal Processing Letters*, vol. 4, no. 7, pp. 198–200, July 1997.

[25] E. Conte, A. De Maio, and G. Ricci, "GLRT-based adaptive detection algorithms for range-spread targets," *IEEE Transactions on Signal Processing*, vol. 49, no. 7, pp. 1336–1348, July 2001.

[26] X. Shuai, L. Kong, and J. Yang, "Adaptive detection for distributed targets in Gaussian noise with Rao and Wald tests," *Science China Information Sciences*, vol. 55, no. 6, pp. 1290–1300, 2012.

[27] F. Bandiera, D. Orlando, and G. Ricci, *Advanced radar detection schemes under mismatched signal models*. Morgan & Claypool, 2009.

[28] W. Liu, J. Liu, Y. Gao, G. Wang, and Y.-L. Wang, "Multichannel signal detection in interference and noise when signal mismatch happens," *Signal Processing*, vol. 166, p. 107268, 2020.

[29] Y. Gu, N. A. Goodman, S. Hong, and Y. Li, "Robust adaptive beamforming based on interference covariance matrix sparse reconstruction," *Signal Processing*, vol. 96, 2014.

[30] A. Coluccia, G. Ricci, and O. Besson, "Design of robust radar detectors through random perturbation of the target signature," *IEEE Transactions on Signal Processing*, vol. 67, no. 19, pp. 5118–5129, August 2019.

[31] A. Coluccia, A. Fascista, and G. Ricci, "A novel approach to robust radar detection of range-spread targets," *Signal Processing*, vol. 166, pp. 1–8, January 2019.

[32] N. B. Pulsone and C. M. Rader, "Adaptive beamformer orthogonal rejection test," *IEEE Transactions on Signal Processing*, vol. 49, no. 3, pp. 521–529, March 2001.

[33] W. Liu, Y. Wang, and W. Xie, "Fisher information matrix, Rao test, and Wald test for complex-valued signals and their applications," *Signal Processing*, vol. 94, 2014.

[34] A. Farina, F. Gini, M. V. Greco, and P. H. Y. Lee, "Improvement factor for real sea-clutter doppler frequency spectra," *IEE Proceedings on radar, sonar and navigation*, vol. 143, no. 5, pp. 341–344, October 1996.