Persymmetric Detection of Subspace Signals Embedded in Subspace Interference and Gaussian Noise

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***Abstract*—We consider the problem of detecting subspace signals embedded in subspace interference and Gaussian noise with unknown covariance matrix. According to the criterion of generalized likelihood ratio test, we exploit persymmetry to propose an adaptive detector. Moreover, the statistical characterization of the proposed detector in the absence of target signals is obtained, which exhibits a constant false alarm rate property against the noise covariance matrix. Numerical examples illustrate that the proposed detector outperforms its counterparts, especially when the number of training data is small.**

***Index Terms*—Adaptive detection, persymmetry, generalized likelihood ratio test, subspace signal, interference, constant false alarm rate.**

I. Introduction

Adaptive detection of a target embedded in Gaussian noise with an unknown covariance matrix has received much attention [1]–[4]. Kelly did pioneering work in [5], where an adaptive detector was designed according to the criterion of generalized likelihood ratio test (GLRT). Interestingly, Kelly's GLRT exhibits a constant false alarm rate (CFAR) property against the noise covariance matrix. In [6], Robey et al. designed an adaptive matched filter (AMF) which has less complexity than the GLRT to implement. In [7], De Maio devised an adaptive detector according to the Rao criterion.

The problem addressed in [5]–[7] is to detect a rank-one signal. In many practical scenarios, however, the signal to be detected satisfies a subspace model where the signal belongs to a subspace spanned by the column vectors of a known matrix. This subspace model has been widely used in the adaptive detection problem. For example, target signals from multiple polarization channels of a polarimetric radar can be formulated by a subspace model [8]–[11]. In addition, the subspace model can be adopted for target detection to account for the uncertainties in the target pointing direction and/or Doppler frequency [12]–[15].

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In practice, multiple observations (or test data) are available when 1) target echoes occupy several range cells [16], [17]; 2) the signals collected from multiple radar bands are combined for target detection [18], [19]; 3) the signals collected from multiple dwells or coherent processing intervals (CPIs) are used for target detection [20], [21]. Based on multiple observations, the target detection problem in the presence of interference and noise has been extensively studied. In [22], [23], several detectors were devised on the assumption that the noise covariance matrix is known. In more practical situations where the noise covariance matrix is unknown, the authors in [24] proposed several adaptive detectors according to the GLRT criteria, by using multiple observations in the presence of subspace interference and noise. The false alarm rate of one of the GLRT detectors was derived in [25]. In [26], the Rao test was designed for the same detection problem as that in [24]. A unifying framework was established for adaptive detection with multiple observations in the presence of interference and noise in [27], [28], where a maximal invariant statistic and several adaptive detectors were proposed.

It is worth noting that no prior information on the covariance matrix is exploited for the detection problem with multiple observations in the presence of interference and noise. In practice, some special structures may exist in the covariance matrix, e.g., persymmetry [29]. It is well-known that persymmetry exists in the covariance matrix, when a radar system is equipped with a symmetrically spaced linear array and/or symmetrically spaced pulse trains [30]–[32].

In this paper, we exploit persymmetry for target detection in the presence of interference and noise. Based on multiple observations, we propose an GLRT detector by using persymmetry. The statistical distribution of the proposed detector in the absence of target signals is derived, which reveals its CFAR property against the noise covariance matrix. Numerical examples show that the proposed detector outperforms its competitors, especially when the training data size is small.

Notation: Vectors (matrices) are denoted by boldface lower (upper) case letters. Superscripts $(\cdot)^T$, $(\cdot)^*$ and $(\cdot)^\dagger$ denote transpose, complex conjugate and conjugate transpose, respectively. $\mathbb{C}^{m\times n}$ is a complex-valued matrix space of dimension

$m \times n$. $\mathbf{I}_n$ stands for an identity matrix of $n \times n$, and $\mathbf{0}_{m\times n}$ represents a null matrix of dimension $m \times n$. For notational simplicity, we sometimes omit the subscripts in $\mathbf{I}_n$ and $\mathbf{0}_{m\times n}$ if no confusion occurs. $\mathfrak{Re}$ and $\mathfrak{Im}$ represent the real and imaginary parts of a complex quantity, respectively. The notation $\sim$ means "be distributed as". $\mathcal{CN}(\boldsymbol{\mu}_c, \mathbf{R}_c)$ (or $\mathcal{N}(\boldsymbol{\mu}_c, \mathbf{R}_c)$) denotes the circularly symmetric, complex-valued (or real-valued) Gaussian distribution with mean $\boldsymbol{\mu}_c$ and covariance matrix $\mathbf{R}_c$. $\mathcal{W}_n(m, \mathbf{R})$ denotes the n-dimensional real-valued central Wishart distribution with m degrees of freedom and scale matrix $\mathbf{R}$. The $n \times m$ matrix $\mathbf{C} = [\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_m] \sim \mathcal{N}(\mathbf{0}_{n\times m}, \mathbf{R} \otimes \mathbf{I}_m)$ means that the column vectors $\mathbf{c}_j$ are independent identically distributed (IID) as $\mathbf{c}_j \sim \mathcal{N}(\mathbf{0}_{n\times 1}, \mathbf{R})$ for $j = 1, 2, \cdots, m$, where $\otimes$ is the Kronecker product of matrices. $|\cdot|$ represents the modulus of a scalar and the determinant of a matrix, when the argument is a scalar and matrix, respectively. $\beta(a, b)$ is the real-valued Beta distribution of parameters a and b. tr$(\cdot)$ denotes the trace of a square matrix and $\jmath = \sqrt{-1}$. $\lceil\cdot\rceil$ denotes the smallest integer greater than or equal to a given number.

II. Problem Formulation

Assume that multiple test data denoted by a set of $N \times 1$ vectors $\{\dot{\mathbf{x}}_h\}_{h=1}^{H}$ are available, where H is the number of observations, and N is the number of (temporal, spatial, or spatial-temporal) channels. In radar applications, these observations can be collected from multiple range cells, bands, and/or CPIs. When a target is present, $\dot{\mathbf{x}}_h$ can be expressed by

$$\dot{\mathbf{x}}_h = \dot{\mathbf{H}}\boldsymbol{\theta}_h + \dot{\mathbf{J}}\boldsymbol{\phi}_h + \dot{\mathbf{n}}_h, \quad h = 1, 2, \dots, H, \tag{1}$$

where $\dot{\mathbf{H}} \in \mathbb{C}^{N\times p}$ is a known full-column-rank target subspace matrix; $\boldsymbol{\theta}_h \in \mathbb{C}^{p\times 1}$ is a deterministic but unknown coordinate vector; $\dot{\mathbf{J}} \in \mathbb{C}^{N\times q}$ is a known full-column-rank interference subspace matrix, and $\boldsymbol{\phi}_h \in \mathbb{C}^{q\times 1}$ is a deterministic but unknown interference coordinate vector; $\dot{\mathbf{n}}_h$ denotes the noise, and $\dot{\mathbf{n}}_h \sim \mathcal{CN}(\mathbf{0}, \dot{\mathbf{R}})$ with $\dot{\mathbf{R}}$ being an unknown covariance matrix.

As in [22], [24], [26]–[28], [33]–[35], a standard assumption is that the columns of $\dot{\mathbf{H}}$ and $\dot{\mathbf{J}}$ are linearly independent, and the augmented matrix $\dot{\mathbf{M}} \triangleq [\dot{\mathbf{H}}, \dot{\mathbf{J}}] \in \mathbb{C}^{N\times(p+q)}$ is of full rank. This implies $p + q \leq N$. In order to estimate the noise covariance matrix, a set of homogeneous training (secondary) data $\{\dot{\mathbf{x}}_k\}_{k=H+1}^{H+K}$ only containing noise is assumed available, i.e., $\dot{\mathbf{x}}_k \sim \mathcal{CN}(\mathbf{0}, \dot{\mathbf{R}}), k = H+1, H+2, \dots, H+K$. Suppose further that the noise in the test and training data is IID.

We aim to declare the presence or absence of target signals in the test data. Let the null hypothesis H_0 be such that the target signals are absent, and the alternative hypothesis H_1 be that the target signals are present. The target detection problem can be formulated as the following binary hypotheses test:

$$\begin{cases} H_0: \begin{cases} \dot{\mathbf{x}}_h \sim \mathcal{CN}(\dot{\mathbf{J}}\boldsymbol{\phi}_h, \dot{\mathbf{R}}), & h = 1, 2, \dots, H, \\ \dot{\mathbf{x}}_k \sim \mathcal{CN}(\mathbf{0}, \dot{\mathbf{R}}), & k = H+1, \dots, H+K, \end{cases} \\ H_1: \begin{cases} \dot{\mathbf{x}}_h \sim \mathcal{CN}(\dot{\mathbf{H}}\boldsymbol{\theta}_h + \dot{\mathbf{J}}\boldsymbol{\phi}_h, \dot{\mathbf{R}}), & h = 1, 2, \dots, H, \\ \dot{\mathbf{x}}_k \sim \mathcal{CN}(\mathbf{0}, \dot{\mathbf{R}}), & k = H+1, \dots, H+K. \end{cases} \end{cases} \tag{2}$$

For the detection problem above, several adaptive detectors, such as the GLRT [24], Rao test [26] and Wald test [36], have been developed. However, no prior structure of the noise covariance matrix was exploited therein.

In practice, $\dot{\mathbf{H}}$, $\dot{\mathbf{J}}$ and $\dot{\mathbf{R}}$ have persymmetric structures when a symmetrically spaced linear array with its center at the origin and/or symmetrically spaced pulse trains are used. When the persymmetry exists, we have

$$\begin{cases} \dot{\mathbf{R}} = \mathbf{D}_0 \dot{\mathbf{R}}^* \mathbf{D}_0 \in \mathbb{C}^{N\times N}, \\ \dot{\mathbf{H}} = \mathbf{D}_0 \dot{\mathbf{H}}^* \in \mathbb{C}^{N\times p}, \\ \dot{\mathbf{J}} = \mathbf{D}_0 \dot{\mathbf{J}}^* \in \mathbb{C}^{N\times q}, \end{cases} \tag{3}$$

where $\mathbf{D}_0$ is a permutation matrix with unit antidiagonal elements and zeros elsewhere [32].

In the following, we exploit persymmetry to design a GLRT detector for the detection problem (2). Now the constraint $\lceil \frac{N}{2} \rceil \leq K$ has to be satisfied, which is much less restrictive than that in [24], [26], [36].

III. GLRT Detection

The GLRT for the detection problem in (2) is given by

$$\frac{\max_{\{\boldsymbol{\theta}_h, \boldsymbol{\phi}_h, \dot{\mathbf{R}}\}_{h=1}^{H}} f\left(\dot{\mathbf{X}}_t, \dot{\mathbf{X}}_s | H_1\right)}{\max_{\{\boldsymbol{\phi}_h, \dot{\mathbf{R}}\}} f\left(\dot{\mathbf{X}}_t, \dot{\mathbf{X}}_s | H_0\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_0, \tag{4}$$

where λ_0 is a detection threshold, $f\left(\dot{\mathbf{X}}_t, \dot{\mathbf{X}}_s | H_0\right)$ and $f\left(\dot{\mathbf{X}}_t, \dot{\mathbf{X}}_s | H_1\right)$ are the probability density functions (PDFs) of the test and training data under H_0 and H_1, respectively,

$$\dot{\mathbf{X}}_t = [\dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2, \dots, \dot{\mathbf{x}}_H] \in \mathbb{C}^{N\times H}, \tag{5}$$

and

$$\dot{\mathbf{X}}_s = [\dot{\mathbf{x}}_{H+1}, \dot{\mathbf{x}}_{H+2}, \dots, \dot{\mathbf{x}}_{H+K}] \in \mathbb{C}^{N\times K}. \tag{6}$$

Due to the independence among the test and training data, $f\left(\dot{\mathbf{X}}_t, \dot{\mathbf{X}}_s | H_0\right)$ can be written as

$$f\left(\dot{\mathbf{X}}_t, \dot{\mathbf{X}}_s | H_0\right) = \left\{ \frac{1}{\pi^N |\dot{\mathbf{R}}|} \exp\left[-\text{tr}(\dot{\mathbf{R}}^{-1}\dot{\mathbf{T}}_0)\right] \right\}^{K+H}, \tag{7}$$

where

$$\dot{\mathbf{T}}_0 = \frac{1}{K+H}\left[\dot{\mathbf{X}}_s\dot{\mathbf{X}}_s^\dagger + \sum_{h=1}^{H}\left(\dot{\mathbf{x}}_h - \dot{\mathbf{J}}\boldsymbol{\phi}_h\right)\left(\dot{\mathbf{x}}_h - \dot{\mathbf{J}}\boldsymbol{\phi}_h\right)^\dagger\right] \in \mathbb{C}^{N\times N}. \tag{8}$$

In addition, $f\left(\dot{\mathbf{X}}_t, \dot{\mathbf{X}}_s | H_1\right)$ can be expressed by

$$f\left(\dot{\mathbf{X}}_t, \dot{\mathbf{X}}_s | H_1\right) = \left\{ \frac{1}{\pi^N |\dot{\mathbf{R}}|} \exp\left[-\text{tr}(\dot{\mathbf{R}}^{-1}\dot{\mathbf{T}}_1)\right] \right\}^{K+H}, \tag{9}$$

where

$$\dot{\mathbf{T}}_1 = \frac{1}{K+H}\left[\dot{\mathbf{X}}_s\dot{\mathbf{X}}_s^\dagger + \sum_{h=1}^{H}\left(\dot{\mathbf{x}}_h - \dot{\mathbf{M}}\boldsymbol{\psi}_h\right)\left(\dot{\mathbf{x}}_h - \dot{\mathbf{M}}\boldsymbol{\psi}_h\right)^\dagger\right] \in \mathbb{C}^{N\times N}, \tag{10}$$

with

$$\boldsymbol{\psi}_h = [\boldsymbol{\theta}_h^T, \boldsymbol{\phi}_h^T]^T \in \mathbb{C}^{(p+q)\times 1}, \quad h = 1, 2, \ldots, H. \tag{11}$$

Using the persymmetric structure in the covariance matrix $\dot{\mathbf{R}}$, we have

$$\begin{aligned}\mathrm{tr}(\dot{\mathbf{R}}^{-1}\dot{\mathbf{T}}_j) =& \mathrm{tr}[\mathbf{D}_0(\dot{\mathbf{R}}^*)^{-1}\mathbf{D}_0\dot{\mathbf{T}}_j] \\ =& \mathrm{tr}[\dot{\mathbf{R}}^{-1}\mathbf{D}_0\dot{\mathbf{T}}_j^*\mathbf{D}_0], \qquad j = 0, 1.\end{aligned} \tag{12}$$

As a result, (7) can be rewritten as

$$f\left(\dot{\mathbf{X}}_t, \dot{\mathbf{X}}_s | H_0\right) = \left\{\frac{1}{\pi^N|\dot{\mathbf{R}}|}\exp\left[-\mathrm{tr}(\dot{\mathbf{R}}^{-1}\mathbf{T}_0)\right]\right\}^{K+H}, \tag{13}$$

where

$$\begin{aligned}\mathbf{T}_0 =& \frac{1}{2}\left(\dot{\mathbf{T}}_0 + \mathbf{D}_0\dot{\mathbf{T}}_0^*\mathbf{D}_0\right) \\ =& \frac{1}{K+H}\left[\hat{\mathbf{R}} + \left(\dot{\mathbf{X}} - \dot{\mathbf{J}}\boldsymbol{\Phi}\right)\left(\dot{\mathbf{X}} - \dot{\mathbf{J}}\boldsymbol{\Phi}\right)^\dagger\right] \in \mathbb{C}^{N\times N},\end{aligned} \tag{14}$$

with

$$\hat{\mathbf{R}} = \frac{1}{2}\left[\dot{\mathbf{X}}_s\dot{\mathbf{X}}_s^\dagger + \mathbf{D}_0\left(\dot{\mathbf{X}}_s\dot{\mathbf{X}}_s^\dagger\right)^*\mathbf{D}_0\right] \in \mathbb{C}^{N\times N}, \tag{15}$$

$$\begin{aligned}\boldsymbol{\Phi} =& [\mathfrak{Re}(\boldsymbol{\phi}_1), \ldots, \mathfrak{Re}(\boldsymbol{\phi}_H), \jmath\mathfrak{Im}(\boldsymbol{\phi}_1), \ldots, \jmath\mathfrak{Im}(\boldsymbol{\phi}_H)] \\ &\in \mathbb{C}^{q\times 2H},\end{aligned} \tag{16}$$

$$\dot{\mathbf{X}} = [\dot{\mathbf{x}}_{e1}, \ldots, \dot{\mathbf{x}}_{eH}, \dot{\mathbf{x}}_{o1}, \ldots, \dot{\mathbf{x}}_{oH}] \in \mathbb{C}^{N\times 2H}, \tag{17}$$

and

$$\begin{cases}\dot{\mathbf{x}}_{eh} = \frac{1}{2}\left(\dot{\mathbf{x}}_h + \mathbf{D}_0\dot{\mathbf{x}}_h^*\right) \in \mathbb{C}^{N\times 1}, \\ \dot{\mathbf{x}}_{oh} = \frac{1}{2}\left(\dot{\mathbf{x}}_h - \mathbf{D}_0\dot{\mathbf{x}}_h^*\right) \in \mathbb{C}^{N\times 1},\end{cases} \tag{18}$$

for $h = 1, 2, \ldots, H$. Similarly, we obtain

$$f\left(\dot{\mathbf{X}}_t, \dot{\mathbf{X}}_s | H_1\right) = \left\{\frac{1}{\pi^N|\dot{\mathbf{R}}|}\exp\left[-\mathrm{tr}(\dot{\mathbf{R}}^{-1}\mathbf{T}_1)\right]\right\}^{K+H}, \tag{19}$$

where

$$\begin{aligned}\mathbf{T}_1 =& \frac{1}{2}\left(\dot{\mathbf{T}}_1 + \mathbf{D}_0\dot{\mathbf{T}}_1^*\mathbf{D}_0\right) \\ =& \frac{1}{K+H}\left[\hat{\mathbf{R}} + \left(\dot{\mathbf{X}} - \dot{\mathbf{M}}\boldsymbol{\Psi}\right)\left(\dot{\mathbf{X}} - \dot{\mathbf{M}}\boldsymbol{\Psi}\right)^\dagger\right] \in \mathbb{C}^{N\times N},\end{aligned} \tag{20}$$

with

$$\begin{aligned}\boldsymbol{\Psi} =& [\mathfrak{Re}(\boldsymbol{\psi}_1), \ldots, \mathfrak{Re}(\boldsymbol{\psi}_H), \jmath\mathfrak{Im}(\boldsymbol{\psi}_1), \ldots, \jmath\mathfrak{Im}(\boldsymbol{\psi}_H)] \\ &\in \mathbb{C}^{(q+p)\times 2H}.\end{aligned} \tag{21}$$

Similar to [24], we can derive the persymmetric GLRT (per-GLRT) detector for the detection problem (2) as

$$\Lambda = \frac{\left|\mathbf{I}_{2H} + \dot{\mathbf{X}}^\dagger\dot{\mathbf{P}}_1\dot{\mathbf{X}}\right|}{\left|\mathbf{I}_{2H} + \dot{\mathbf{X}}^\dagger\dot{\mathbf{P}}_2\dot{\mathbf{X}}\right|} \underset{H_1}{\overset{H_0}{\gtrless}} \lambda, \tag{22}$$

where λ is a detection threshold,

$$\dot{\mathbf{P}}_1 = \hat{\mathbf{R}}^{-1} - \hat{\mathbf{R}}^{-1}\dot{\mathbf{J}}\left(\dot{\mathbf{J}}^\dagger\hat{\mathbf{R}}^{-1}\dot{\mathbf{J}}\right)^{-1}\dot{\mathbf{J}}^\dagger\hat{\mathbf{R}}^{-1} \in \mathbb{C}^{N\times N}, \tag{23}$$

and

$$\dot{\mathbf{P}}_2 = \hat{\mathbf{R}}^{-1} - \hat{\mathbf{R}}^{-1}\dot{\mathbf{M}}\left(\dot{\mathbf{M}}^\dagger\hat{\mathbf{R}}^{-1}\dot{\mathbf{M}}\right)^{-1}\dot{\mathbf{M}}^\dagger\hat{\mathbf{R}}^{-1} \in \mathbb{C}^{N\times N}. \tag{24}$$

The proposed per-GLRT detector in (22) is different from that in [24], since the former exploits the persymmetry, whereas the latter does not.

IV. Statistical Distribution Under H_0

Now we examine the distribution of Λ under H_0. It is easy to check that under H_0

$$\dot{\mathbf{X}} \sim \mathcal{CN}\left(\mathbf{0}, \frac{\dot{\mathbf{R}}}{2} \otimes \mathbf{I}_{2H}\right). \tag{25}$$

Define two unitary matrices as follows

$$\mathbf{D}_1 = \frac{1}{2}\left[(\mathbf{I}_N + \mathbf{D}_0) + \jmath(\mathbf{I}_N - \mathbf{D}_0)\right] \in \mathbb{C}^{N\times N}, \tag{26}$$

and

$$\mathbf{V}_{2H} = \begin{bmatrix}\mathbf{I}_H & \mathbf{0} \\ \mathbf{0} & -\jmath\mathbf{I}_H\end{bmatrix} \in \mathbb{C}^{2H\times 2H}. \tag{27}$$

Using $\mathbf{D}_1$ and $\mathbf{V}_{2H}$, we define

$$\begin{aligned}\bar{\mathbf{X}}_t =& \mathbf{D}_1\dot{\mathbf{X}}\mathbf{V}_{2H} \\ =& [\dot{\mathbf{x}}_{er1}, \ldots, \dot{\mathbf{x}}_{erH}, \dot{\mathbf{x}}_{or1}, \ldots, \dot{\mathbf{x}}_{orH}] \in \mathbb{R}^{N\times 2H},\end{aligned} \tag{28}$$

where

$$\begin{aligned}\dot{\mathbf{x}}_{erh} =& \mathbf{D}_1\dot{\mathbf{x}}_{eh} \\ =& \frac{1}{2}\left[(\mathbf{I}_N + \mathbf{D}_0)\mathfrak{Re}(\dot{\mathbf{x}}_h) - (\mathbf{I}_N - \mathbf{D}_0)\mathfrak{Im}(\dot{\mathbf{x}}_h)\right] \in \mathbb{R}^{N\times 1},\end{aligned} \tag{29}$$

and

$$\begin{aligned}\dot{\mathbf{x}}_{orh} =& -\jmath\mathbf{D}_1\dot{\mathbf{x}}_{oh} \\ =& \frac{1}{2}\left[(\mathbf{I}_N - \mathbf{D}_0)\mathfrak{Re}(\dot{\mathbf{x}}_h) + (\mathbf{I}_N + \mathbf{D}_0)\mathfrak{Im}(\dot{\mathbf{x}}_h)\right] \in \mathbb{R}^{N\times 1},\end{aligned} \tag{30}$$

for $h = 1, 2, \ldots, H$. It follows from (25) that

$$\bar{\mathbf{X}}_t \sim \mathcal{N}\left(\mathbf{0}, \mathbf{R} \otimes \mathbf{I}_{2H}\right), \tag{31}$$

where

$$\mathbf{R} = \frac{1}{2}\mathbf{D}_1\dot{\mathbf{R}}\mathbf{D}_1^\dagger = \frac{1}{2}\left[\mathfrak{Re}(\dot{\mathbf{R}}) + \mathbf{D}_0\mathfrak{Im}(\dot{\mathbf{R}})\right] \in \mathbb{R}^{N\times N}. \tag{32}$$

Let

$$\begin{cases}\bar{\mathbf{R}} = \mathbf{D}_1\hat{\mathbf{R}}\mathbf{D}_1^\dagger = \mathfrak{Re}(\hat{\mathbf{R}}) + \mathbf{D}_0\mathfrak{Im}(\hat{\mathbf{R}}) \in \mathbb{R}^{N\times N}, \\ \bar{\mathbf{J}} = \mathbf{D}_1\dot{\mathbf{J}} = \mathfrak{Re}(\dot{\mathbf{J}}) - \mathfrak{Im}(\dot{\mathbf{J}}) \in \mathbb{R}^{N\times q}, \\ \bar{\mathbf{M}} = \mathbf{D}_1\dot{\mathbf{M}} = \mathfrak{Re}(\dot{\mathbf{M}}) - \mathfrak{Im}(\dot{\mathbf{M}}) \in \mathbb{R}^{N\times(p+q)},\end{cases} \tag{33}$$

and then, the proposed per-GLRT in (22) can be recast to

$$\Lambda = \frac{\left|\mathbf{I}_{2H} + \bar{\mathbf{X}}_t^T\mathbf{P}_1\bar{\mathbf{X}}_t\right|}{\left|\mathbf{I}_{2H} + \bar{\mathbf{X}}_t^T\mathbf{P}_2\bar{\mathbf{X}}_t\right|} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda, \tag{34}$$

where

$$\mathbf{P}_1 = \bar{\mathbf{R}}^{-1} - \bar{\mathbf{R}}^{-1}\bar{\mathbf{J}}\left(\bar{\mathbf{J}}^T\bar{\mathbf{R}}^{-1}\bar{\mathbf{J}}\right)^{-1}\bar{\mathbf{J}}^T\bar{\mathbf{R}}^{-1} \in \mathbb{R}^{N\times N}, \tag{35}$$

and

$$\mathbf{P}_2 = \bar{\mathbf{R}}^{-1} - \bar{\mathbf{R}}^{-1}\bar{\mathbf{M}}\left(\bar{\mathbf{M}}^T\bar{\mathbf{R}}^{-1}\bar{\mathbf{M}}\right)^{-1}\bar{\mathbf{M}}^T\bar{\mathbf{R}}^{-1} \in \mathbb{R}^{N\times N}. \tag{36}$$

Note that all the quantities in (34) are real-valued.

Using the similar approach in [25], we can obtain that the test statistic Λ in (34) under H_0 is distributed as

$$\begin{aligned}\Lambda^{-1} &\sim U_{p,2H,2K+p-M} \\ &\overset{\mathrm{d}}{=} \prod_{j=1}^{p} \beta\left(\frac{2K+p-M-j+1}{2}, H\right).\end{aligned} \tag{37}$$

It is obvious that the distribution of Λ is irrelevant to $\dot{\mathbf{R}}$. Therefore, the proposed per-GLRT detector exhibits the CFAR property against the noise covariance matrix.

Note that we cannot derive the distribution of the proposed per-GLRT detector under H_1, because the noncentral Wilks' distribution is difficult to handle analytically [37].

V. Numerical Examples

Assume that a pulsed Doppler radar is used, where the number of pulses in a CPI is N. The (i, j)th element of the noise covariance matrix is chosen as $[\dot{\mathbf{R}}]_{i,j} = \sigma^2 0.9^{|i-j|}$, where σ^2 is the noise power. Without loss of generality, we set $\boldsymbol{\theta} = \sigma^2_{\boldsymbol{\theta}}[1, \cdots, 1]^T$ and $\boldsymbol{\phi} = \sigma^2_{\boldsymbol{\phi}}[1, \cdots, 1]^T$, where $\sigma^2_{\boldsymbol{\theta}}$ and $\sigma^2_{\boldsymbol{\phi}}$ are the signal and interference powers, respectively. The signal-to-noise ratio (SNR) and interference-to-noise ratio (INR) are defined as $\text{SNR} = 10\log_{10}\frac{\sigma^2_{\boldsymbol{\theta}}}{\sigma^2}$, and $\text{INR} = 10\log_{10}\frac{\sigma^2_{\boldsymbol{\phi}}}{\sigma^2}$, respectively. The INR is set to be 30 dB.

In Fig. 1, the detection probability as a function of SNR is presented. Here, we select $N = 12$, $p = 2$, $q = 3$, $H = 3$ and the probability of false alarm $P_{\text{FA}} = 10^{-4}$. For comparison purposes, we consider the GLRT and AMF developed in Eq. (10) and Section IV.A of [24], respectively. Note that the GLRT and AMF in [24] do not exploit persymmetry. It can be seen that the proposed detector outperforms its counterparts. When the training data size increases, the performance gains of the proposed detector over its counterparts become small. This is because the estimation accuracy of the noise covariance matrix is high, even when the persymmetry is not exploited.

The detection probability versus K is plotted in Fig. 2, where $N = 12$, $p = 2$, $q = 3$, $H = 3$, $\text{SNR} = 5$ dB and $P_{\text{FA}} = 10^{-4}$. These results highlight that the proposed detector clearly outperforms its counterparts in the cases of small training data size. The proposed detector can work in the sample-starved case (for example, $K = 8$), whereas its counterparts cannot work for $K < N$.

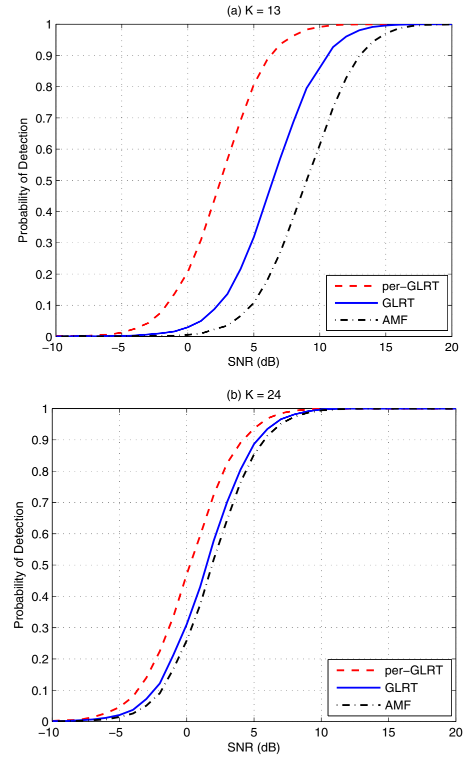

Fig. 1. Detection probability versus SNR for $N = 12$, $p = 2$, $q = 3$, $H = 3$ and $P_{\text{FA}} = 10^{-4}$.

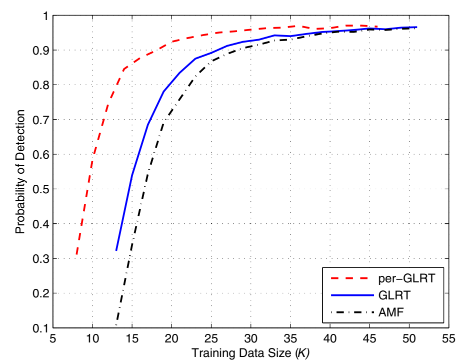

Fig. 2. Detection probability versus K for $N = 12$, $p = 2$, $q = 3$, $H = 3$, $\text{SNR} = 5$ dB and $P_{\text{FA}} = 10^{-4}$.

VI. Conclusions

The problem of detecting subspace signals was considered in the presence of subspace interference and Gaussian noise with unknown covariance matrix. We have exploited persymmetry to devise an adaptive CFAR detector, e.g., per-GLRT. Compared to the conventional detectors, the proposed detector can work in sample-starved cases where $\lceil \frac{N}{2} \rceil \leq K < N$. The statistical property of the proposed per-GLRT detector under H_0 was derived. Numerical examples show that the proposed detector has better performance than its counterparts, especially when the training data size is small.

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