# BLIND EQUALIZATION VIA POLYNOMIAL OPTIMIZATION

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## ABSTRACT

A polynomial optimization based blind equalizer (POBE) is proposed. Different from the popular constant modulus algorithm and its variants, the POBE adopts an eighth-order multivariate polynomial as the loss function. Since the loss function is sensitive to phase rotation, the POBE can achieve automatic carrier phase recovery. A gradient descent method with optimal step size is developed for solving the optimization problem. We reveal that this optimal step size is one root of a seventh-order univariate polynomial and hence, can be computed easily. Compared with the blind equalizers based on stochastic gradient descent with empirical step size, which suffers from slow convergence or even divergence, the POBE significantly accelerates the convergence rate. Moreover, it attains a much lower inter-symbol interference (ISI), resulting in a noticeable improvement of equalization performance. Simulation results demonstrate the superiority of POBE over several representative blind equalizers.

***Index Terms***— Blind equalization, carrier phase recovery, constant modulus algorithm, polynomial optimization, root finding.

## 1. INTRODUCTION

A fundamental problem in digital communications is to mitigate the inter-symbol interference (ISI) induced by the propagation channel, which is called channel equalization [1]. Blind equalization refers to directly recovering the transmitted data without the use of training symbols. One attractiveness of blind equalization is that it improves spectrum efficiency.

One widely-used technique for blind equalization of two-dimensional modulation schemes [2] is the constant modulus algorithm (CMA) [2]–[6]. Since the constant modulus (CM) loss function is invariant to phase rotation, the CMA lacks the capability of recovering the carrier phase. As a result, an additional phase-locked loop is required at the output of the CMA equalizer, which increases the hardware complexity at the steady-state. By using the phase sensitive $\ell_p$-modulus ($p \neq 2$) of the complex numbers [7]–[9], the generalized CMA (GCMA) was designed for joint equalization and carrier phase recovery [7, 8]. The GCMA of [8] was applied to optical fiber communication systems [10]. Another blind equalizer that achieves phase recovery is the multi-modulus algorithm (MMA) [11]–[13]. The MMA is only applicable to squared constellation such as 16-QAM [11], which constitutes one limitation of it.

The stochastic gradient descent method is often employed to minimize the CM, GCM and MM loss functions. In spite of the low computational complexity, its convergence rate is quite slow. When the number of data samples is not large enough, the stochastic gradient method even diverges. Several Newton-based algorithms have been developed to accelerate the convergence speed for the CMA [6], GCMA [8], and MMA [12]. However, the per-iteration complexity of the blind equalization technique is greatly increased since it needs to invert the Hessian matrix. One problem of stochastic gradient/Newton methods is that it is not easy to determine the step size. In most cases, the step size is selected by a heuristic manner. It needs "tuning" to choose an appropriate step size according to the practical settings.

In this paper, we propose a polynomial optimization based blind equalizer (POBE). Unlike the CMA, GCMA, and MMA, the POBE uses an eighth-order multivariate polynomial as the loss function, which is sensitive to phase rotation and thus, can automatically recover the carrier phase. Note that the GCMA with $p = 4$ [8] is different from the proposed POBE. We devise a gradient descent method with optimal step size for solving the POBE using a "mini-batch" of samples. When the batch size is one, it reduces to the stochastic gradient method. The optimal step size can be easily computed by finding the roots of a seventh-order univariate polynomial. Compared with existing algorithms using empirical step size, the POBE significantly speeds up the convergence rate. More attractively, the POBE can attain a lower ISI than the prior approaches, which demonstrate it has a superior performance for interference suppression.

The superscripts $(\cdot)^\top$, $(\cdot)^*$ and $(\cdot)^{\mathrm{H}}$ denote the transpose, complex conjugate and Hermitian transpose, respectively. The imaginary unit is $\mathrm{j} = \sqrt{-1}$ while $\mathbb{E}[\cdot]$ is expectation

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operator. The subscripts $(\cdot)_{\text{R}}$ and $(\cdot)_{\text{I}}$ stand for the real and imaginary parts of a complex-valued quantity. The $|\cdot|$ represents the absolute value of a real number or the modulus of a complex number.

## 2. SIGNAL MODEL AND PRIOR ARTS

Consider a communication system with discrete-time complex baseband signal model

$$x(n) = s(n) * h(n) + \nu(n) \tag{1}$$

where $x(n)$ is the received signal, $s(n)$ is the transmitted data symbol, $h(n)$ is the channel impulse response, $\nu(n)$ is the additive white noise, and $*$ denotes convolution. It is assumed that $s(n)$, and its real and imaginary parts are independent and identically distributed (i.i.d.), which is easily to satisfy in practice. The received signal is distorted due to the ISI induced by the propagation channel. Herein, blind deconvolution is employed to mitigate the ISI, i.e., recover the transmitted symbols without estimating the channel response. To be more specific, we aim to design a equalizer with impulsive response being $w(n)$ such that the output of equalizer $y(n) = w(n) * x(n)$ recovers $s(n)$. Considering the finite impulse response (FIR) equalizer with $L$ coefficients $\boldsymbol{w} = [w_0, \cdots, w_{L-1}]^\top$, the equalizer output is written as

$$y(n) = \sum_{i=0}^{L-1} w_i^* x(n-i) = \boldsymbol{w}^{\text{H}} \boldsymbol{x}_n \tag{2}$$

where $\boldsymbol{x}_n \triangleq [x(n), \cdots, x(n-L+1)]^\top$.

The prior arts of blind equalization are mainly based on the CM property. The celebrated CMA [3] solves

$$\min_{\boldsymbol{w}} f_{\text{CM}}(\boldsymbol{w}) := \sum_n \left(|\boldsymbol{w}^{\text{H}} \boldsymbol{x}_n|^2 - r_{\text{CM}}\right)^2 \tag{3}$$

to obtain the equalizer, where $|y| = \sqrt{y_{\text{R}}^2 + y_{\text{I}}^2}$ is the modulus of the complex number $y$, $r_{\text{CM}} > 0$ is the dispersion constant of the CMA defined as [3]:

$$r_{\text{CM}} = \frac{\mathbb{E}\left[|s(n)|^4\right]}{\mathbb{E}\left[|s(n)|^2\right]}. \tag{4}$$

Based on the $\ell_p$-modulus $|y|_p = (|y_{\text{R}}|^p + |y_{\text{I}}|^p)^{1/p}$ with $p \geq 1$, the GCMA [7, 8] solves

$$\min_{\boldsymbol{w}} f_{\text{GCM}}(\boldsymbol{w}) := \sum_n \left(|\boldsymbol{w}^{\text{H}} \boldsymbol{x}_n|_p^2 - r_{\text{GCM}}\right)^2 \tag{5}$$

with the dispersion constant being

$$r_{\text{GCM}} = \frac{\mathbb{E}\left[|s(n)|_p^4\right]}{\mathbb{E}\left[|s(n)|_p^2\right]}. \tag{6}$$

When $p = 2$, the GCMA reduces to CMA. The widely-used algorithm for solving the minimization problems in (3) and (5) is the stochastic gradient descent. Also, the stochastic Newton methods are applied to accelerate the convergence speed [8].

Another important approach for blind equalization is the MMA:

$$\min_{\boldsymbol{w}} f_{\text{MM}}(\boldsymbol{w}) := \sum_n \left( \left(y_{\text{R}}^2(n) - r_{\text{R}}\right)^2 + \left(y_{\text{I}}^2(n) - r_{\text{I}}\right)^2 \right) \tag{7}$$

with $y(n) = \boldsymbol{w}^{\text{H}} \boldsymbol{x}_n$, and the dispersion constants

$$r_{\text{R}} = \frac{\mathbb{E}\left[s_{\text{R}}^4(n)\right]}{\mathbb{E}\left[s_{\text{R}}^2(n)\right]},\ r_{\text{I}} = \frac{\mathbb{E}\left[s_{\text{I}}^4(n)\right]}{\mathbb{E}\left[s_{\text{I}}^2(n)\right]}. \tag{8}$$

One limitation of the MMA is that it is only applicable to symbols with squared constellation, such as 16-QAM.

## 3. POLYNOMIAL OPTIMIZATION BASED BLIND EQUALIZER

### 3.1. Polynomial Loss Function

In this paper, we propose to minimize the following loss function for blind equalization

$$\min_{\boldsymbol{w}} f(\boldsymbol{w}) := \sum_n \left(|y_{\text{R}}(n)|^p + |y_{\text{I}}(n)|^p - r_p\right)^2. \tag{9}$$

where $y(n) = \boldsymbol{w}^{\text{H}} \boldsymbol{x}_n$ and $r_p$ is the dispersion constant, whose optimal value is given by

$$r_p = \frac{\mathbb{E}\left[|s(n)|_p^{2p}\right]}{\mathbb{E}\left[|s(n)|_p^p\right]}. \tag{10}$$

In general, $p$ can take arbitrary value from $[1, \infty]$. In this paper, it is of particular interest to consider the case when $p$ is a positive even number. In such a case, the loss function of (9) is a multivariate polynomial with degree of $2p$. Therefore, we refer to the equalizer given by (9) as POBE. For $p = 2$, the POBE of (9) reduces to the most common CMA of (3). However, it is clear that the POBE is different from the MMA and GCMA. The GCMA fits the square of the $\ell_p$-modulus but the POBE fits the $p$th power of the $\ell_p$-modulus. There are two guidelines to select an appropriate $p$. On one hand, $p > 2$ is preferred since it achieves phase recovery and improves the performance. On the other hand, it is not advisable to select too large $p$. First, large $p$ leads to a high order polynomial and makes optimization more difficult. Second, it could result in overflow when calculating the $p$th power of a number. For example, $p = 4$, which results in minimization of an eighth-order polynomial, is an appropriate choice. When $p \neq 2$, $f(\boldsymbol{w})$ of (9) is sensitive to phase rotation, i.e., $f(\boldsymbol{w}) \not\equiv f(\mathrm{e}^{\mathrm{j}\phi}\boldsymbol{w})$ with $\phi$ being a phase angle. As a result, the POBE can achieve automatic carrier phase recovery.

We compute the gradient of $f(\boldsymbol{w})$ as

$$\nabla f(\boldsymbol{w}) = p\sum_n (|y_{\text{R}}(n)|^p + |y_{\text{I}}(n)|^p - r_p) \times \\ (|y_{\text{R}}(n)|^{p-2}y_{\text{R}}(n) - \text{j}|y_{\text{I}}(n)|^{p-2}y_{\text{I}}(n))\boldsymbol{x}_n. \quad (11)$$

For the case of $p=4$, the gradient reduces to

$$\nabla f(\boldsymbol{w}) = 4\sum_n \left(y_{\text{R}}^4(n) + y_{\text{I}}^4(n) - r_4\right)(y_{\text{R}}^3(n) - \text{j}y_{\text{I}}^3(n))\boldsymbol{x}_n. \quad (12)$$

The gradient descent is employed to solve (9) with update formation being

$$\boldsymbol{w} \leftarrow \boldsymbol{w} - t\nabla f(\boldsymbol{w}) \quad (13)$$

where $t > 0$ is the step size. Most existing blind equalization algorithms use stochastic gradient and their step sizes depend on "tuning" in an empirical way. The step size is usually set small enough to guarantee convergence. Our algorithm is based on batch processing, where $P$ samples $\{x_n\}_{n=0}^{P-1}$ are available. Using the $N$ samples, we can form $N = P - L + 1$ sample blocks $\{\boldsymbol{x}_n\}_{n=L-1}^{P-1}$. Denoting the index set $\mathcal{N} = \{L-1, L, \cdots, P-1\}$, the sample blocks are expressed as $\{\boldsymbol{x}_n\}_{n\in\mathcal{N}}$. When $N = 1$, it reduces to stochastic gradient method.

### 3.2. Optimal Step Size

Given $\boldsymbol{w}$ and $\nabla f(\boldsymbol{w})$ at the current iteration, the optimal step size can be obtained by using the line search

$$t^\star = \arg\min_{t>0} \left\{\tilde{f}(t) \triangleq f(\boldsymbol{w} - t\nabla f(\boldsymbol{w}))\right\} \quad (14)$$

which is a one-dimensional optimization problem with respect to $t$. Furthermore, $\tilde{f}(t)$ is a univariate $2p$th-order polynomial. The optimal step size is the root of

$$\frac{\mathrm{d}\tilde{f}(t)}{\mathrm{d}t} = 0$$

where the derivative $\mathrm{d}\tilde{f}(t)/\mathrm{d}t$ is a $(2p-1)$th-order polynomial. When $p = 4$, exploiting $y(n) = \boldsymbol{w}^{\text{H}}\boldsymbol{x}_n$ and denoting $u(n) = -\nabla f(\boldsymbol{w})^{\text{H}}\boldsymbol{x}_n$, the eighth-order polynomial $\tilde{f}(t) = \sum_{i=0}^{8}\beta_i t^i$ equals

$$\sum_{n\in\mathcal{N}}\left(\left(y_{\text{R}}(n) + tu_{\text{R}}(n)\right)^4 + \left(y_{\text{I}}(n) + tu_{\text{I}}(n)\right)^4 - r_4\right)^2. \quad (15)$$

Defining the following variables

$$\begin{aligned} a_n &= u_{\text{R}}^4(n) + u_{\text{I}}^4(n) \\ b_n &= 4(y_{\text{R}}(n)u_{\text{R}}^3(n) + y_{\text{I}}(n)u_{\text{I}}^3(n)) \\ c_n &= 6(y_{\text{R}}^2(n)u_{\text{R}}^2(n) + y_{\text{I}}^2(n)u_{\text{I}}^2(n)) \\ d_n &= 4(y_{\text{R}}^3(n)u_{\text{R}}(n) + y_{\text{I}}^3(n)u_{\text{I}}(n)) \\ e_n &= y_{\text{R}}^4(n) + y_{\text{I}}^4(n) - r_4 \end{aligned} \quad (16)$$

we obtain the coefficients as

$$\begin{aligned} &\beta_8 = \sum_{n\in\mathcal{N}} a_n^2,\ \beta_7 = 2\sum_{n\in\mathcal{N}} a_n b_n \\ &\beta_6 = \sum_{n\in\mathcal{N}} b_n^2 + 2a_n c_n,\ \beta_5 = 2\sum_{n\in\mathcal{N}} a_n d_n + b_n c_n \\ &\beta_4 = \sum_{n\in\mathcal{N}} 2a_n e_n + 2b_n d_n + c_n^2 \\ &\beta_3 = 2\sum_{n\in\mathcal{N}} c_n d_n + b_n e_n,\ \beta_2 = \sum_{n\in\mathcal{N}} d_n^2 + 2c_n e_n, \\ &\beta_1 = 2\sum_{n\in\mathcal{N}} d_n e_n,\ \beta_0 = \sum_{n\in\mathcal{N}} e_n^2. \end{aligned} \quad (17)$$

The optimal step size must be one real root of the following seventh-order univariate polynomial

$$\sum_{i=1}^{8} i\beta_i t^{i-1} = 0. \quad (18)$$

Finding the roots of a univariate seventh-order polynomial is not difficult. Since the coefficients of the polynomial are real-valued, there are four possible cases for the property of the roots. The first case is that (18) has a real root and three pairs of complex conjugate roots. In this case, the minimizer is the unique real root. The other three cases are: 1) three real roots and two pairs of complex conjugate roots; 2) five real roots and one pair of complex conjugate roots; 3) seven real roots. For the latter three cases, the optimal step size is the real root that corresponding to the minimum objective function value. The computational cost for computing the coefficients in (17) is $\mathcal{O}(N)$. Once the coefficients of (18) are obtained, the complexity of the root finding is merely $\mathcal{O}(1)$. Therefore, the dominant cost is still for computing the gradient in (12) at each iteration, which is $\mathcal{O}(LN)$. The steps for implementing the POBE for $p = 4$ with optimal step size is listed in Algorithm 1.

**Algorithm 1** POBE for $p = 4$ with optimal step size

**Input:** Received samples $\{x(n)\}_{n=0}^{P-1}$ and equalizer length $L$.
  **Initialize:** Choose $\boldsymbol{w}$ randomly.
  **while** convergence not attained **do**
    Form $\boldsymbol{x}_n = [x(n), \cdots, x(n-L+1)]^\top$ for $n \in \mathcal{N}$ with $\mathcal{N} = \{L-1, \cdots, P-1\}$.
    Compute equalizer output: $y(n) = \boldsymbol{w}^{\text{H}}\boldsymbol{x}_n$ for $n \in \mathcal{N}$.
    Compute $\nabla f(\boldsymbol{w})$ by (12) and $u(n) = -\nabla f(\boldsymbol{w})^{\text{H}}\boldsymbol{x}_n$.
    Calculate the polynomial coefficients by (16) and (17).
    Determine the step size $t^\star$ by finding the roots of (18).
    Update $\boldsymbol{w} \leftarrow \boldsymbol{w} - t^\star\nabla f(\boldsymbol{w})$
  **end while**

## 4. SIMULATION RESULTS

Denoting the composite channel-equalizer response as $g(n) = h(n) * w(n)$, the following quantified ISI

$$\text{ISI} = \frac{\sum_n |g(n)|^2 - \max_n |g(n)|^2}{\max_n |g(n)|^2} \tag{19}$$

is used as the performance measure of an equalizer. Smaller ISI implies better equalization. If ISI $= 0$, then the channel is perfectly equalized and the transmitted symbols are exactly recovered up to a delay and a scalar.

We use a typical FIR voice-band communication channel, which is the same as that in [8]. The POBE in Algorithm 1 is compared with the super-exponential algorithm (SEA) [15], CMA [3], WFOS [14], GCMA [7, 8] with $p = 4$, and MMA [11]. The WFOS is a variant of CMA in which the optimal step size is automatically determined at each iteration. The CMA, GCMA, and MMA use a fixed step size $4 \times 10^{-3}/N$ to guarantee the convergence. The equalizer length is set to $L = 21$ for all methods. The input data symbol are randomly generated from the 16-QAM constellation. The white Gaussian noise is added such that the signal-to-noise ratio (SNR) is 20 dB.

Fig. 1 displays the constellations of the transmitted and received symbols, and the equalizer outputs when $N = 600$. It is seen that the outputs of the SEA, CMA and WFOS have a global phase rotation while the GCMA, MMA, and POBE eliminate the phase ambiguity.

Fig. 2 plots the ISI versus the number of iterations with $N = 100$ and $N = 600$, respectively. We see that the existing five equalizers diverges while the POBE converges when the number of sample blocks is small ($N = 100$). When $N$ is relatively large ($N = 600$), all methods converge. Apparently, the POBE converges much faster than the other five equalizers. Moreover, the POBE attains a lower ISI, which indicates that it achieves a more accurate equalization. Fig. 3 shows the ISI versus $N$, where the ISI is an average of 100 independent trials. Again, the POBE can attain lower ISIs, achieving a better equalization performance.

## 5. CONCLUSION

To overcome the drawbacks of the existing blind equalizers, the POBE based on minimization of eighth-order multivariate polynomial is presented. The POBE can achieve automatic carrier phase recovery thanks to the sensitivity of the loss function to phase rotation. We propose a gradient descent method using optimal step size. The POBE gets rid of the nuisance to tune the step size because the optimal step size is easily obtained by root finding of a seventh-order univariate polynomial. The POBE noticeably speeds up the convergence rate and attains a much lower ISI. In the case of small sample number, it avoids from divergence. Simulation results verify the advantages of POBE.

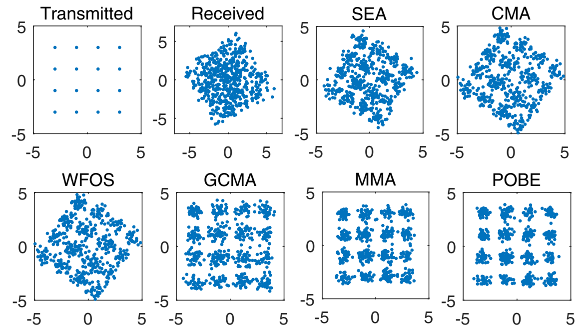


**Fig. 1**. Constellations of signals.

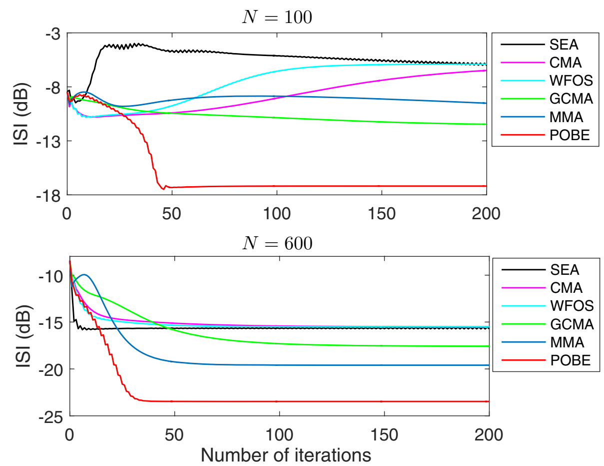


**Fig. 2**. ISI versus number of iterations.

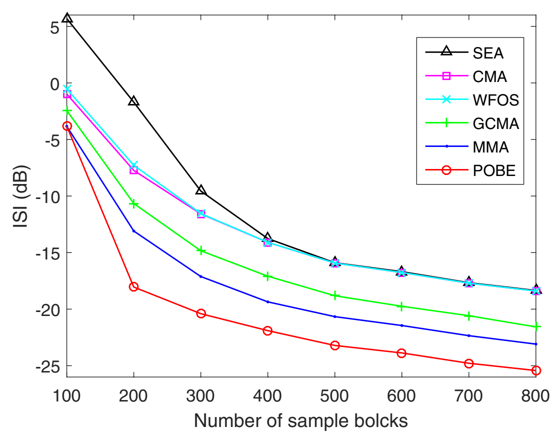


**Fig. 3**. ISI versus number of sample blocks.

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