Signal Analysis Using Local Polynomial Approximations

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***Abstract*—Local polynomial approximations represent a versatile feature space for time-domain signal analysis. The parameters of such polynomial approximations can be computed by efficient recursions using autonomous linear state space models and often allow analytical solutions for quantities of interest. The approach is illustrated by practical examples including the estimation of the delay difference between two acoustic signals and template matching in electrocardiogram signals with local variations in amplitude and time scale.**

***Index Terms*—localized polynomials, localized feature space, delay estimation, time-scale estimation, local signal approximation, autonomous linear state space models**

I. Introduction

Estimating time-domain parameters such as time shifts and time scales of signals are old problems. Signal scaling in time, for example due to Doppler shifts, need to be estimated for radar and sonar applications [1]. Similarly, in the biomedical field, time scale estimates are fundamental in ultrasound imaging [2]. Furthermore, time shift or delay estimations are crucial for time of arrival computations, either in acoustic signals, where the interaural time delay is essential for source localization [3], or in radar or sonar systems for object localization [4].

Existing methods for the estimation of such parameters can be split into two groups: sample-based methods and feature-space methods. Presumably, the most common sample-based method for time delay estimation is cross-correlation [5], or, if the goal is extended to time scale estimation, methods based on complex ambiguity functions [6]. However, the precision of the delay estimate is commonly restricted by the sampling rate and a new evaluation is needed with each time delay or time scale considered. More efficient adaptive algorithms that can tackle these problems exist, but they are restricted to narrowband signals [1], [7].

Feature-space methods such as the traditional Fourier transform, the fractional Fourier transform, or the Hilbert transform [8], [9], [2] involve transforming the signals to a space spanned by sinusoids. However, working in such a space only allows discrete frequency resolution, which, in turn, smears time-domain signal features.

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Another class of feature-space methods are based on polynomials. The substitution or interpolation of sample-based signals by local polynomials was already proposed in the 1940s, along with the well-known B-splines [10]. For some time, this approach was adopted mainly in computer-aided design and in 3-D surface modeling [11]. Later on, splines have been advancing into the broader field of signal processing [12]. More recently, signal analysis based on recursive localized fitting of polynomials and/or sinusoids have been proposed in [13]–[17], where the efficiency of the recursive computations is based on autonomous (i.e., input-free) linear state space models (ALSSMs).

In this paper, we propose the idea, implicitly suggested by [13]–[17], to use locally fitted polynomials as feature spaces for various signal processing tasks. Such polynomial feature spaces are attractive for a number of reasons. First, as observed in [13]–[17], the mapping to the local feature space can be computed by efficient recursions. Second, quantities of interest can often be determined analytically. Third, scaling in time does not change the degree of the local polynomial representations. Forth, polynomials are suitable for multistage processing where feature-space polynomials are themselves locally approximated by polynomials. Fifth, when polynomial signal approximations are used in quadratic cost terms, the terms are often conveniently also in polynomial form and efficiently minimized using analytical or standard numerical methods. To realize this idea of using polynomial approximations for estimating time-domain parameters, we review the recursive computations for transforming signals to the proposed feature space. We also provide the analytical tools necessary to manipulate such feature-space polynomials and demonstrate the versatility of this approach with multiple examples.

In Section II of this paper, we describe the recursive ALSSM method for efficient transformation of a sample-based signal to our feature space. In Section III, we introduce a vector notation for feature-space polynomials and algebraic rules for efficient manipulation of such polynomials. Section IV concludes with three exemplary applications: fitting polynomials of different degrees, time delay estimation between two acoustic signals, and template matching with electrocardiogram signals; the latter two examples both involve real data. We focus on examples with only up to three channels here, however, due to the structure of the transform used and the low complexity of our method, the generalization to a large number of channels is straightforward and remains computationally attractive.

II. Transformation to Local Polynomial Feature Space Using ALSSMs

Prior to any further processing, we transform the given scalar signal $y = [y_1, \dots, y_K]^\mathsf{T} \in \mathbb{R}^K$ at each time index $k \in \{1, \dots, K\}$, where $K \gg 1$, into a feature space $X \in \mathbb{R}^Q$. The features are the coefficients of a polynomial of the degree $Q-1$, which locally approximates a windowed segment of the signal around k. The approximation is done by minimizing the squared error cost

$$J_a^b(k, x, \gamma) = \sum_{i=k+a}^{k+b} \gamma^{|i-k|} \big(y_i - p_{i-k}(x)\big)^2 \ , \tag{1}$$

where $a, b \in \mathbb{Z}, a < b$ define the window boundaries, $\gamma \in (0, 1]$ is an exponential window weight that minimizes the effect of samples further away from k on the cost and where $p_i(x)$ is the polynomial function in i with coefficient vector $x = [\lambda_0, \dots, \lambda_{Q-1}]^\mathsf{T} \in \mathbb{R}^Q$,

$$p_i(x) = \lambda_0 + \lambda_1 i + \ldots + \lambda_{Q-1} i^{Q-1} \ . \tag{2}$$

Without loss of generality, we assume that $a \leq 0$ and $b > 0$. Note that in the case of infinite windows (i.e., $a \to -\infty$), $\gamma < 1$ must additionally hold for stability. More intricate window weights and individual sample weights can be applied in the cost function. This is further discussed in [14], [15]. The coefficients of the polynomial approximation are thus given by

$$\hat{x}_k = \operatorname*{argmin}_{x \in \mathbb{R}^Q} J_a^b(k, x, \gamma) \ . \tag{3}$$

Representation of the polynomial function by an autonomous linear state space model (ALSSM) leads to efficient recursions for the polynomial approximation at each time step. The ALSSM corresponding to $p_i(x)$ with coefficients x is

$$x_{i+1} = A x_i \tag{4}$$
$$p_i(x) = c^\mathsf{T} x_i \ , \tag{5}$$

where the states $x_i \in \mathbb{R}^Q$, $A \in \mathbb{R}^{Q \times Q}$, and $c \in \mathbb{R}^Q$ for a system of order Q. The initial state is $x_0 = x$. For the example of $Q = 4$, the system matrices are

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$
$$c = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T} \ , \tag{7}$$

with generalization to any order Q given in [15]; or in the computationally more robust form using the binomial basis in [14].

Then, $p_i(x) = c^\mathsf{T} A^i x$ holds, which we insert in the cost (1) to obtain

$$J_a^b(k, x, \gamma) = J_a^0(k, x, \gamma) + J_1^b(k, x, \gamma) \tag{8}$$
$$= \textstyle\sum_{i=k+a}^{k} \gamma^{k-i} (y_i - c^\mathsf{T} A^{i-k} x)^2$$
$$+ \textstyle\sum_{i=k+1}^{k+b} \gamma^{i-k} (y_i - c^\mathsf{T} A^{i-k} x)^2 \ . \tag{9}$$

The first summation term in (8), which pertains to the past samples, can be written as

$$J_a^0(k, x, \gamma) = \kappa_k^{a,0} - 2x^\mathsf{T} \xi_k^{a,0} + x^\mathsf{T} W_k^{a,0} x \ , \tag{10}$$

where the parameters

$$\kappa_k^{a,0} = \textstyle\sum_{i=k+a}^{k} \gamma^{k-i} y_i^2 \in \mathbb{R} \tag{11}$$
$$\xi_k^{a,0} = \textstyle\sum_{i=k+a}^{k} \gamma^{k-i} (A^\mathsf{T})^{i-k} c y_i \in \mathbb{R}^Q \tag{12}$$
$$W_k^{a,0} = \textstyle\sum_{i=k+a}^{k} \gamma^{k-i} (A^\mathsf{T})^{i-k} c c^\mathsf{T} A^{i-k} \in \mathbb{R}^{Q \times Q} \tag{13}$$

are computed by the forward recursions

$$\kappa_k^{a,0} = \gamma \kappa_{k-1}^{a,0} + y_k^2 - \gamma^{-a+1} y_{k-1+a}^2 \tag{14}$$
$$\xi_k^{a,0} = \gamma A^{-\mathsf{T}} \xi_{k-1}^{a,0} + c y_k - \gamma^{-a+1} (A^\mathsf{T})^{a-1} c y_{k-1+a} \tag{15}$$
$$W_k^{a,0} = \gamma A^{-\mathsf{T}} W_{k-1}^{a,0} A^{-1} + c c^\mathsf{T} - \gamma^{-a+1} (A^\mathsf{T})^{a-1} c c^\mathsf{T} A^{a-1} \tag{16}$$

initialized by $\kappa_0^{a,0} = 0$, $\xi_0^{a,0} = 0$, $W_0^{a,0} = 0$. For an infinite window $a \to -\infty$, we have $\gamma^{-a+1} \to 0$ and the recursions (14)-(16) simplify accordingly.

Similarly, the second summation term of (8), which pertains to the future samples, is expressed as

$$J_1^b(k, x, \gamma) = \kappa_k^{1,b} - 2x^\mathsf{T} \xi_k^{1,b} + x^\mathsf{T} W_k^{1,b} x \ , \tag{17}$$

where

$$\kappa_k^{1,b} = \textstyle\sum_{i=k+1}^{k+b} \gamma^{i-k} y_i^2 \in \mathbb{R} \tag{18}$$
$$\xi_k^{1,b} = \textstyle\sum_{i=k+1}^{k+b} \gamma^{i-k} (A^\mathsf{T})^{i-k} c y_i \in \mathbb{R}^Q \tag{19}$$
$$W_k^{1,b} = \textstyle\sum_{i=k+1}^{k+b} \gamma^{i-k} (A^\mathsf{T})^{i-k} c c^\mathsf{T} A^{i-k} \in \mathbb{R}^{Q \times Q} \tag{20}$$

are computed by the backward recursions

$$\kappa_k^{1,b} = \gamma \kappa_{k+1}^{1,b} + \gamma y_{k+1}^2 - \gamma^{b+1} y_{k+b+1}^2 \tag{21}$$
$$\xi_k^{1,b} = \gamma A^\mathsf{T} \xi_{k+1}^{1,b} + \gamma A^\mathsf{T} c y_{k+1} - \gamma^{b+1} (A^\mathsf{T})^{b+1} c y_{k+b+1} \tag{22}$$
$$W_k^{1,b} = \gamma A^\mathsf{T} W_{k+1}^{1,b} A + \gamma A^\mathsf{T} c c^\mathsf{T} A - \gamma^{b+1} (A^\mathsf{T})^{b+1} c c^\mathsf{T} A^{b+1} \tag{23}$$

which are initialized by $\kappa_K^{1,b} = 0$, $\xi_K^{1,b} = 0$, $W_K^{1,b} = 0$.

The total cost remains a quadratic function in x,

$$J_a^b(k,x,\gamma) = \kappa_k^{a,0} + \kappa_k^{1,b} - 2x^\mathsf{T}(\xi_k^{a,0} + \xi_k^{1,b}) + x^\mathsf{T}(W_k^{a,0} + W_k^{1,b}) x \tag{24}$$

with the coefficients estimate $\hat{x}_k$ that solves (3) given by

$$\hat{x}_k = (W_k^{a,0} + W_k^{1,b})^{-1} (\xi_k^{a,0} + \xi_k^{1,b}) \ . \tag{25}$$

III. Manipulating Polynomials in Localized Feature Space

In the previous section of this paper, we transformed a sample-based signal into a localized feature space spanned by polynomial coefficients. In this section, we introduce a vector notation for such polynomials and summarize algebraic rules for efficiently manipulating them [17].

A. Definition and Properties of the Vector Exponent Notation

Let z be a scalar in $\mathbb{R}$ and $q = [q_1, \ldots, q_Q]^\mathsf{T}$ a vector in $\mathbb{N}_0^Q$ with $Q \in \mathbb{N}$. We define the function $(z, q) \mapsto z^q$ as the element-wise power of z, i.e.,

$$z^q \triangleq [z^{q_1}, \ldots, z^{q_Q}]^\mathsf{T} \ . \tag{26}$$

Note that the elements of q are neither ordered nor unique.

A univariate polynomial in z with coefficient vector $\alpha \in \mathbb{R}^Q$ and exponent vector q as in (26) expresses as

$$\alpha^\mathsf{T} z^q \ \in \mathbb{R} \ . \tag{27}$$

A bivariate polynomial with additional variable $w \in \mathbb{R}$ and additional exponent vector $r \in \mathbb{N}_0^R$ and a joint coefficient vector $\tilde{\alpha} \in \mathbb{R}^{QR}$ expresses as

$$\tilde{\alpha}^\mathsf{T}(z^q \otimes w^r) \ \in \mathbb{R} \ , \tag{28}$$

with $\otimes$ denoting the Kronecker product.

B. A Calculus for Polynomials in Vector Exponent Notation

Mathematical operations with polynomials such as sum, product, and power often simplify to a linear transformation of the polynomial coefficient and/or exponent vector with the help of the vector exponent notation. In the following, we give a brief summary of the transformation rules for such common operations. The full derivation, including rules for multivariate polynomials, are given in [17].

1) Sum of two polynomials of same exponent: Let $\alpha, \beta \in \mathbb{R}^Q$ be the coefficient vectors of two polynomials in vector exponent notation with common exponent vector $q \in \mathbb{N}_0^Q$. Then, the sum of the two polynomials is

$$\alpha^\mathsf{T} z^q + \beta^\mathsf{T} z^q = (\alpha + \beta)^\mathsf{T} z^q \ . \tag{29}$$

2) Product of two polynomials of same exponent: Given the coefficients $\alpha, \beta \in \mathbb{R}^Q$ and the exponent vector q,

$$(\alpha^\mathsf{T} z^q)(\beta^\mathsf{T} z^q) = \tilde{\alpha}^\mathsf{T} z^{\tilde{q}} \tag{30}$$

with

$$\tilde{\alpha} = \alpha \otimes \beta \ \in \mathbb{R}^{Q^2} \tag{31}$$

$$\tilde{q} = Mq \ \in \mathbb{N}_0^{Q^2} \ , \tag{32}$$

where

$$M = (I_Q \otimes \mathbb{1}_Q) + (\mathbb{1}_Q \otimes I_Q) \ , \tag{33}$$

and where I_Q denotes the identity matrix of size Q and $\mathbb{1}_Q = [1, \ldots, 1]^\mathsf{T} \in \mathbb{R}^Q$.

3) Integral of a polynomial: Given the coefficients $\alpha \in \mathbb{R}^Q$,

$$\int (\alpha^\mathsf{T} z^q) dz = \alpha^\mathsf{T} \int z^q dz = \tilde{\alpha}^\mathsf{T} z^{\tilde{q}} \tag{34}$$

with the substitutes

$$\tilde{\alpha} = \Lambda\alpha \ \in \mathbb{R}^Q \tag{35}$$

$$\tilde{q} = q + \mathbb{1}_Q \ \in \mathbb{N}_0^Q \ , \tag{36}$$

and $\Lambda^{-1} = \mathrm{Diag}(\tilde{q}) \in \mathbb{R}^{Q \times Q}$, the diagonal matrix of $\tilde{q}$.

4) Definite integral: The integral over finite interval $[a, b]$ is

$$\int_a^b (\alpha^\mathsf{T} z^q) dz = (\Lambda\alpha)^\mathsf{T}(b^{\tilde{q}} - a^{\tilde{q}}) \tag{37}$$

using Λ and $\tilde{q}$ as in (35) and (36), respectively.

IV. Practical Examples

In this section, we give three examples using localized feature space polynomials as proposed in Section II and III.

A. Approximating a Polynomial by a Polynomial of Lower Degree

A common task is to fit a lower-degree polynomial $\beta^\mathsf{T} z^r$ to a given higher-degree polynomial $\alpha^\mathsf{T} z^q$ over the interval $[a, b]$ such that the squared error is minimal ($\beta \in \mathbb{R}^R$, $r \in \mathbb{N}_0^R$, $\alpha \in \mathbb{R}^Q$, $q \in \mathbb{N}_0^Q$). Thus, we minimize the cost function

$$J(\alpha, \beta) = \int_a^b \left[\alpha^\mathsf{T} z^q - \beta^\mathsf{T} z^r\right]^2 dz \tag{38}$$

over β, which yields

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \left(\alpha^\mathsf{T} A^\mathsf{T} C A \alpha - 2\beta^\mathsf{T} B^\mathsf{T} C A \alpha + \beta^\mathsf{T} B^\mathsf{T} C B \beta\right) \tag{39}$$

with the fixed matrices $A \in \mathbb{R}^{(Q+R) \times Q}$, $B \in \mathbb{R}^{(Q+R) \times R}$ and $C \in \mathbb{R}^{(Q+R) \times (Q+R)}$, derived by repetitive (and somewhat tedious) application of the rules given in Section III-B. The full derivation is given in the Appendix of this paper. Note that the cost in (39) is of a quadratic form in β and is minimal for any β satisfying the linear equality $B^\mathsf{T} C A \alpha = B^\mathsf{T} C B \beta$. Figure 1 illustrates an example of such a fit.

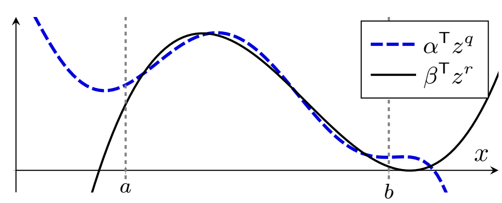

Fig. 1. Fit of a 3rd-degree polynomial ($\beta^\mathsf{T} z^r$, solid line) to a given 7th-degree polynomial ($\alpha^\mathsf{T} z^q$, dashed line) over the interval $[a, b]$, using (39).

B. Time Delay Estimation

In a second example, we observe two signals, where the second signal shows a variably delayed or jittery version of the first signal, as illustrated in Fig. 2. Delayed and jittered signals are often observed in transmission systems with multiple physical pathways of variable length. To estimate the time delay between two channels, we exploit our localized feature space. We first locally approximate the signals of both channels by higher degree polynomials according to Section II. We then minimize the squared error between two corresponding polynomials while allowing a variable time shift, which leads to an estimate of the local time delay.

Let $\alpha_k^\mathsf{T} z^q \in \mathbb{R}$ and $\beta_k^\mathsf{T} z^q \in \mathbb{R}$ be two polynomials in $z \in \mathbb{R}$ of degree $Q = \max(q) \in \mathbb{N}$, which locally approximate the two signals around index k. We consider the cost function

$$J_k(s) = \int_a^b \left[\alpha_k^\mathsf{T}(z - \frac{1}{2}s)^q - \beta_k^\mathsf{T}(z + \frac{1}{2}s)^q\right]^2 dz \ , \tag{40}$$

which is the squared error between the two polynomials over the interval (a, b), when a time shift $s \in \mathbb{R}$ is applied. Minimizing the cost over s, yields an estimate of s.

After repetitive (and, again, somewhat tedious) application of the calculus outlined in Section III-B, (40) transforms into a polynomial in s in the elementary form

$$J_k(s) = \Big(A(\alpha_k \otimes \alpha_k) - B(\alpha_k \otimes \beta_k) + C(\beta_k \otimes \beta_k)\Big)^{\mathsf{T}} s^{\tilde{q}} \;, \tag{41}$$

with $\tilde{q} = M \cdot [1, \ldots, Q+1]^{\mathsf{T}} \in \mathbb{R}^{(Q+1)^2}$, M as in (33), and fixed A, B, $C \in \mathbb{R}^{(Q+1)^2 \times (Q+1)^2}$. Note that A, B, and C are independent of α and β and, thus, fully precomputable [17].

Finally, minimizing (41) boils down to finding a (local) minimum in the univariate polynomial (41) in s. In practice, the feasible shift is often limited by physical bounds, and, thus, can be determined efficiently using standard methods, e.g. using gradient descent.

Note that, if a wider observation interval than that spanned by a single polynomial is desired, multiple cost functions can be accumulated and jointly minimized. Hence, for a rectangular observation window of length $2\Delta+1$, the accumulated cost is

$$\bar{J}_k(s) = \textstyle\sum_{i=k-\Delta}^{k+\Delta} J_i(s) \tag{42}$$

which has to be minimized over s. Note that (42) remains of the same polynomial degree as (41) and, thus, the minimization complexity is independent of the selected window length.

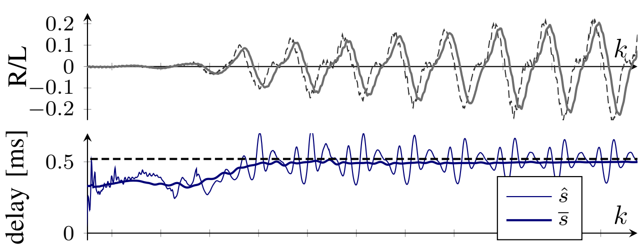

Fig. 2. Top: two-channel acoustic signal from left (L, solid line) and right (R, dashed line) ear with interaural time delay. Bottom: $\hat{s}$ shows the local time delay estimate of corresponding local polynomial fits, while $\overline{s}$ is the averaged version of $\hat{s}$ using a rectangular window according to (42). The *true delay* (dashed black line), calculated according to head geometry and speed of sound for an azimuth of 45°, is approximately $0.52\,$ms.

Figure 2 demonstrates the results of an interaural time delay estimation using polynomials of degree 4. The data is generated by the convolution of the International Speech Test Signal [18] with a head-related impulse response recorded at 45° azimuth and 0° elevation from the CIPIC database [19].

C. Time & Amplitude Scaling

When a moving source emits a field, such as an electric or magnetic one, and this field is measured by fixed sensors, then we will observe signals of a variable time and amplitude scale, depending on the source speed and source strength. To estimate source parameters such as speed and strength from such observations is a common problem [20], [21].

In the following example, we use real electrocardiogram (ECG) signals measured with electrodes inside the esophagus, which is close to the heart's atrium. Analyzing such ECG signals is a problem of the kind described above: with every heart beat the heart gets depolarized which sends a moving electrical wave front along the heart's surface, producing a transient electrical field and inducing the well-known ECG signals. The amplitudes of such ECG signals are proportional to the source strength while ECG pulse durations are inversely proportional to the depolarization front speed.

To estimate both parameters, source strength and source speed, we first generate a time- and amplitude-normalized reference signal using a physical heart model, before we match the time and amplitude scale of the reference signal to the observed real signals. For efficiency, we work in the local polynomial feature space of the signals.

Let $\alpha \in \mathbb{R}^Q$ be the coefficients of a polynomial $\alpha^{\mathsf{T}} z^q$ with exponent vector $q \in \mathbb{N}_0^Q$ and of degree $\max(q)$, approximating the reference signal shape over the interval $[a, b]$, $a, b \in \mathbb{R}$. Furthermore, let $\beta_k \in \mathbb{R}^Q$ be the coefficient vector of the polynomial $\beta_k^{\mathsf{T}} z^q$ which locally approximates the observed signal around time index k. We then consider the cost

$$J_k(\lambda, \eta) = \int_a^b \Big[\lambda \alpha^{\mathsf{T}} (\eta z)^q - \beta_k^{\mathsf{T}} z^q\Big]^2 dz \;, \tag{43}$$

where $\lambda \in \mathbb{R}$ is the (unknown) amplitude scaling and $\eta \in \mathbb{R}$ the (unknown) time dilation of the signal. This cost becomes minimal when the amplitude and time scale of the two polynomials match best, yielding an estimate for λ and η.

We note that the cost function (43) is the integral of a quadratic form of a (higher-degree) polynomial and is thus a multivariate polynomial in λ and η. Therefore, applying the calculus outlined in Section III-B transforms (43) to

$$J_k(\lambda,\eta) = A(\beta_k \otimes \beta_k) - 2\lambda\big(B(\alpha \otimes \beta_k)\big)^{\mathsf{T}} \eta^q + \lambda^2 \big(C(\alpha \otimes \alpha)\big)^{\mathsf{T}} \eta^{Mq} \tag{44}$$

with fixed $A \in \mathbb{R}^{Q^2}$, $B \in \mathbb{R}^{Q \times Q^2}$, and $C \in \mathbb{R}^{Q^2 \times Q^2}$, and M as in (33). Note that (44) is a multivariate polynomial of degree 2 in λ and of degree $(\max(q))^2$ in η. Full derivation of (44) is given in detail in [17, Section 6.3].

We obtain a closed-form solution for the minimization over λ by setting the first derivative of (44) to zero,

$$\hat{\lambda}_k = \underset{\lambda}{\operatorname{argmin}}\, J_k(\lambda, \eta) = \frac{(\alpha \otimes \beta_k)^{\mathsf{T}} B^{\mathsf{T}} \eta^q}{(\alpha \otimes \alpha)^{\mathsf{T}} C^{\mathsf{T}} \eta^{Mq}} \;. \tag{45}$$

Finally, the substitution of (45) into (44) yields the univariate rational function

$$\hat{\eta}_k = \underset{\eta}{\operatorname{argmin}}\, J_k(\hat{\lambda}_k, \eta) = \underset{\eta}{\operatorname{argmin}} \frac{[(B(\alpha \otimes \beta_k))^{\mathsf{T}} \eta^q]^2}{(C(\alpha \otimes \alpha))^{\mathsf{T}} \eta^{Mq}} \;, \tag{46}$$

which needs to be minimized. In our example, we used a plain grid search for minimization, since the search interval of η is narrow due to physical constraints, and processing speed was more important than precision in our application.

Figure 3 displays an example of a real esophageal ECG signal of an atrial wave with the matched reference signal after applying amplitude and time scaling.

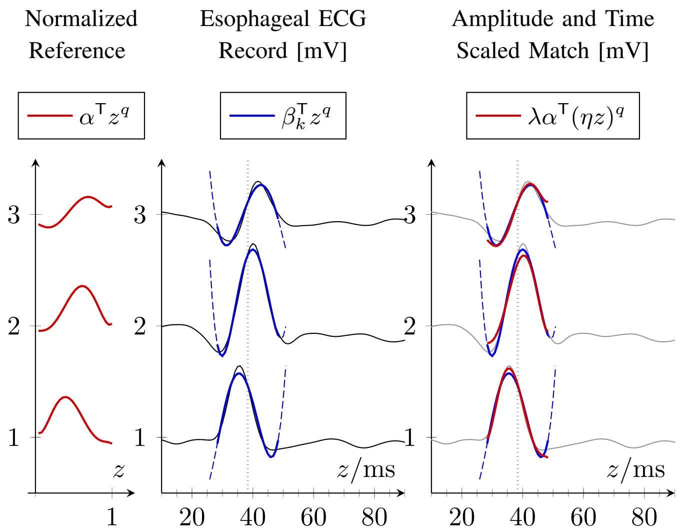

Fig. 3. Left: 5th-degree polynomial approximation of a 3-channel normalized reference signal of the cardiac electrical activity (red lines). The reference signal was generated using a physical heart model. Middle: 3-channel recording of an atrial wave in an esophageal ECG signal (gray) with local polynomial approximation (blue line). Right: match of time- and amplitude-scaled reference signal (red) to local polynomial approximation of ECG signal (blue)

V. Conclusion

This paper proposes a feature space of localized polynomials for parameter estimation problems. Transformation to feature space is efficiently done by means of recursions that result from the parameterization of polynomials by ALSSMs. The local signal form is fully encoded in the polynomial coefficients, which provides a framework for solving different estimation problems analytically with sub-sample resolution.

Appendix

Derivation of (39) according to [16]:

$$\begin{aligned}
&\int_a^b \left[\alpha^\mathsf{T} z^q - \beta^\mathsf{T} z^r\right]^2 dz \\
&= \int_a^b \left[(A\alpha - B\beta)^\mathsf{T} z^s\right]^2 dz \\
&= \int_a^b \left((A\alpha - B\beta)\otimes(A\alpha - B\beta)\right)^\mathsf{T} z^{Ms} dz \\
&= \left(\Lambda\left[(A\alpha - B\beta)\otimes(A\alpha - B\beta)\right]\right)^\mathsf{T} \overbrace{(b^{Ms+1} - a^{Ms+1})}^{c} \\
&= \Big((A\alpha\otimes A\alpha) - (A\alpha\otimes B\beta) - (B\beta\otimes A\alpha) + (B\beta\otimes B\beta)\Big)^\mathsf{T} \Lambda^\mathsf{T} c \\
&= \alpha^\mathsf{T} A^\mathsf{T} C A\alpha - 2\beta^\mathsf{T} B^\mathsf{T} C A\alpha + \beta^\mathsf{T} B^\mathsf{T} C B\beta \qquad (47)
\end{aligned}$$

with M according to (33), exponent vector

$$s = \begin{bmatrix} q \\ r \end{bmatrix} \in \mathbb{R}^{Q+R} ,$$

fixed transformation matrices

$$A = \begin{bmatrix} I_Q \\ 0_{R\times Q} \end{bmatrix} \in \mathbb{R}^{(Q+R)\times Q} , \; B = \begin{bmatrix} 0_{Q\times R} \\ I_R \end{bmatrix} \in \mathbb{R}^{(Q+R)\times R} ,$$

and $\mathrm{vec}\,(C) \equiv \Lambda^\mathsf{T} c$, with unity matrix I and zero matrix 0 of indicated dimensions.

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