An Efficient Online Estimation Algorithm for Evolving Quantum States

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Abstract— **In this paper, we propose an online optimization algorithm for estimating state density in free evolution quantum systems from noisy continuous weak measurements. The problem is formulated via sparsity-promoting semidefinite programming, and an online quantum state estimation algorithm is developed based on online proximal gradient and alternating direction multiplier method. The proposed algorithm is computationally efficient and further features high robustness to measurement noise. The merits of the approach are illustrated by numerical experiments in 1-, 2-, 3-, and 4-qubit systems.**

I. INTRODUCTION

Quantum state estimation (QSE) is the procedure to obtain and process quantum measurements for reconstructing the quantum state. The problem of state estimation can be transformed into an optimization problem and solved by numerical methods [1]. QSE is the basis of closed-loop feedback control of quantum state which aims to achieve high-precision quantum state manipulation. An n-qubit system can be fully expressed as a density matrix $\rho \in \mathbb{C}^{d\times d}$ ($d = 2^n$), that is positive semidefinite and unit-trace Hermitian [2]. The most commonly used approach for QSE is termed quantum state tomography (QST). QST is based on strong (projective) measurements, which lead to a destruction of the quantum state [3]. As a consequence, strong measurements are not deemed suitable for real-time QSE.

Weak measurements provide an alternative for estimating the quantum state. A measurement is characterized as weak when it has little impact on the measured system. Continuous weak measurements (CWM) were first developed by Silberfarb et al. [4]. In contrast to strong (projective) measurements, CWM can be applied to the evolving quantum system due to their incomplete destruction characteristic [5]. Therefore, it is beneficial to use CWM to measure the state of the evolving quantum system online.

Traditional learning algorithms are offline in that they require to process the *entire* dataset, at each iteration, whence they are unsuitable for real-time processing of big data. For this reason, stochastic optimization techniques that process only a fraction of the available data, per iteration, have been developed [6], [7]. Such methods can be invoked for online learning, for example tracking the state of a dynamical system, and this is the path we undertake in this paper.

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Yang et al. [8] developed an online state estimation method for the single qubit quantum system by resorting to CVX [9] for solving a sequence of problems (for each instance of weak measurement). However, CVX is an offline optimization tool by nature, and furthermore is generally restricted to small-/medium- size problems (as we also illustrate in our comparative experiments).

In this paper, we target to solve the problem of online estimation of evolving quantum states from continuous weak measurements. We propose a novel online quantum state estimation algorithm by using a combination of the Alternating Direction Method of Multipliers (ADMM) and Online Proximal Gradient (OPG) method adapted to the optimization problem of evolving quantum state estimation subject to quantum constraints. ADMM is an effective tool for dealing with separable objective functions and structured regularization, which can lead to distributed and parallel numerical solvers [10]; OPG is an typical online optimization algorithm based on the proximal gradient method [11]. A combination of both techniques for online learning was introduced in [12]. Our proposed algorithm is applied to estimating evolving density matrices, and is tested in 1-, 2-, 3-, and 4-qubit systems.

This paper is organized as follows. Sec. II introduces the continuous weak measurement model for an n-qubit open quantum system. The online quantum state estimation algorithm is proposed in Sec. III. Numerical experiments are carried out in Sec. IV, while Sec. V concludes the paper.

II. OPEN QUANTUM SYSTEM MODEL WITH CONTINUOUS WEAK MEASUREMENTS

The process of online estimation of a quantum state is depicted in Fig.1; it consists of two steps (notation q^{-1} represents unit delay operator). The first step comprises a continuous weak measurement (CWM), which includes: 1) introducing a coupling relationship between the probe system P and the target system S, to form a joint system, and 2) using the measurement operator M to carry a weak measurement at each sampling time, thus obtaining a noisy measurement y which contains information about the true density matrix ρ. The second step of the online estimator aims to reconstruct the quantum state $\hat{\rho}_k$ and measurement noise $\hat{e}_k$ simultaneously at each instant k, using also previously estimated values (thus serving as an adaptive filter for tracking the state evolution). The goal of the estimator that we propose is to efficiently estimate states of the dynamic quantum system online.

The evolution equation of an n-qubit open quantum system

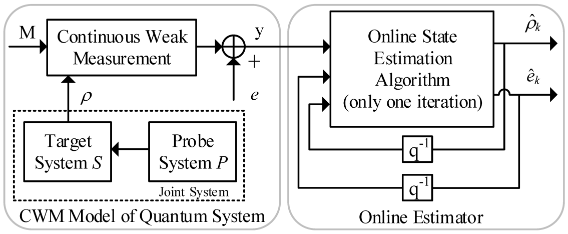

Fig. 1. Online quantum state estimation process.

S is given by:

$$\begin{aligned}&\rho(t+\Delta t)-\rho(t)=-\tfrac{i}{\hbar}[H_n,\rho(t)]\Delta t\\&+\sum\left[L_n\rho(t)L_n^\dagger-\tfrac{1}{2}\left(L_n^\dagger L_n\rho(t)+\rho(t)L_n^\dagger L_n\right)\right]\Delta t,\end{aligned} \tag{1}$$

where $\rho(t)$ is the density matrix at t-time; Δt is a short interaction time interval; $\hbar$ is set equal to 1; $H_n \in \mathbb{C}^{d\times d}$ is the Hamiltonian representing the total energy of the system; $L_n \in \mathbb{C}^{d\times d}$ is a bounded operator pertaining to the Lindblad interaction; $L_n^\dagger$ denotes the conjugate transposition of L_n.

The corresponding approximation discrete evolution equation of (1) is obtained as [13]:

$$\rho_{k+1}=\sum_{i=1}^{2^n}A_i(\Delta t)\rho_k A_i(\Delta t)^\dagger, \tag{2}$$

where $k = 1, 2, \ldots, N$ represents the sampling times; $A_i(\Delta t) \in \mathbb{C}^{d\times d}$ $(i = 1\ldots 2^n)$ are the system evolution operators and can be constructed as:

$$\begin{aligned}A_1(\Delta t)&=\underbrace{m_0(\Delta t)\otimes\ldots\otimes m_0(\Delta t)\otimes m_0(\Delta t)}_{n},\\A_2(\Delta t)&=\underbrace{m_0(\Delta t)\otimes\ldots\otimes m_0(\Delta t)\otimes m_1(\Delta t)}_{n},\\&\vdots\\A_{2^n}(\Delta t)&=\underbrace{m_1(\Delta t)\otimes\ldots\otimes m_1(\Delta t)\otimes m_1(\Delta t)}_{n},\end{aligned} \tag{3}$$

in which $m_0(\Delta t) := I-(L_1^\dagger L_1/2+iH_1)\Delta t$ and $m_1(\Delta t) := L_1\sqrt{\Delta t}$ [14], where $H_1 = H_0 + u_x H_x$, (H_0 is the free Hamiltonian, H_x is the control Hamiltonian and u_x is the external regulate value); $L_1 = \xi\sigma$, where ξ is the system interaction strength, and σ denotes the Pauli matrix which can be selected as one of $\{\sigma_x, \sigma_y, \sigma_z, I\}$, with $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, and I is the 2×2 identity matrix.

In view of (2), the discrete evolution of the measurement operators can be described by :

$$M_{k+1}=\sum_{i=1}^{2^n}A_i(\Delta t)M_k A_i(\Delta t)^\dagger, \tag{4}$$

where $M_k \in \mathbb{C}^{d\times d}$ for each k.

A measurement in the actual system is acquired by the evaluation of the operator M acting on the quantum state ρ, namely $y = \mathrm{tr}(M^\dagger\rho)$. In [4], the construction method of measurements is the evolving measurement operator acting on the initial state, which means that the state to be estimated is always the initial state. When the evolutionary measurement operators $\{M_i\}_{i=1}^k$ are inverted relative to the quantum state ρ_k at time k, the outcome is consistent with the construction method in [4]. Thus, in order to directly estimate the latest quantum state ρ_k , the sequence of measurements $\{y_i\}_{i=1}^k$ (where y_i is a scalar) can be obtained as:

$$\begin{aligned}&y_i=\mathrm{tr}(M_{k-i+1}^\dagger\rho_k)+e_i=\mathrm{vec}(M_{k-i+1})^\dagger\,\mathrm{vec}(\rho_k)+e_i,\\&i=1,\ldots,k,\end{aligned} \tag{5}$$

in which $\mathrm{vec}(X)$ represents the vector concatenation of matrix X's columns into a (column) vector (note that this operation is invertible, in that the matrix can be reconstructed uniquely from the vector concatenation of its columns); e_i represents the meaurement noise; and $\mathrm{tr}(\cdot)$ denotes the trace of a matrix which is the sum of its diagonal entries.

During real-time system operation, the entire dataset is not available in advance but rather meaurements are obtained in a streaming fashion. The online estimation that we develop uses the k-th measurement y_k (along with previous estimate $\hat{\rho}_{k-1}$) to obtain $\hat{\rho}_k$, and can be deemed an online tracking filter for state evolution ρ_k.

III. ONLINE STATE ESTIMATION ALGORITHM FOR QUANTUM STSTEM BASED ON OPG-ADMM

A. OPG-ADMM

The optimization problem of separable objective functions with linear constraints is of the form:

$$\begin{aligned}\underset{x\in\mathcal{X},z\in\mathcal{Z}}{\text{minimize}}\quad & f(x)+g(z)\\ \text{s.t.}\quad & Ax+Bz=c,\end{aligned} \tag{6}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ are the decision variables; $A \in \mathbb{R}^{p\times n}, B \in \mathbb{R}^{p\times m}, c \in \mathbb{R}^p$; f and g are convex closed and proper functions; $\mathcal{X}$ and $\mathcal{Z}$ are closed convex sets.

For the purpose of online estimation, we leverage the computational framework OPG-ADMM for solving problem (6) that was proposed in [12]. The algorithm comprises two steps: 1) using the augmented Lagrangian for (6): $\mathrm{L}(x,z,\lambda) = f(x) + g(z) + (\alpha/2)\left\|Ax + Bz - c - \lambda/\alpha\right\|_2^2$, ($\lambda \in \mathbb{R}^p$ is the Lagrangian multiplier, $\alpha > 0$ is the penalty parameter, and we have used completion of squares to obtain a compact form of the augmented Lagrangian), the original problem is split into two separate subproblems with respect to f and g; 2) utilizing OPG method as the update rule for f-sub-problem (i.e., using subgradient to linearize f and perform a proximal gradient update).

At the k-th instant, OPG-ADMM solves two separate sub-problems over x and z, respectively, then updates the Lagrange multiplier λ. The iterates are given by:

$$\begin{cases}x_k=\arg\min\limits_{x\in\mathcal{X}}\Big\{f_{k-1}^{\mathrm{T}}x+\dfrac{\alpha}{2}\left\|Ax+Bz_{k-1}-c-\lambda_{k-1}/\alpha\right\|_2^2\\ \qquad\qquad +(1/2\eta_k)\left\|x-x_{k-1}\right\|_{P_k}^2\Big\},\\ z_k=\arg\min\limits_{z\in\mathcal{Z}}\Big\{g(z)+\dfrac{\alpha}{2}\left\|Ax_k+Bz-c-\lambda_{k-1}/\alpha\right\|_2^2\Big\},\\ \lambda_k=\lambda_{k-1}-\alpha\left(Ax_k+Bz_k-c\right),\end{cases} \tag{7}$$

where $f_{k-1} \in \partial f(x)|_{x=x_{k-1}}$ is a subgradient of $f(x)$ at x_{k-1}; $\eta_k > 0$ is the proximal gradient step size; $\|\cdot\|_P^2$ is defined via $\|x\|_P^2 := x^{\mathrm{T}} P x$, where P is a positive definite matrix.

B. ONLINE STATE ESTIMATION FOR QUANTUM SYSTEM (OSEQ)

In order to alleviate the computational burden, as required for online processing, while making full use of weak measurements so as to achieve higher estimation accuracy, a sliding window containing multiple measurements is adopted. Notice that the sliding window of measurements is dynamic and contains the most recent measurements:

$$b_k = \begin{cases} (y_1, \cdots, y_{k-1}, y_k)^{\mathrm{T}}, & k < l, \\ (y_{k-l+1}, \cdots, y_{k-1}, y_k)^{\mathrm{T}}, & k \geq l. \end{cases} \tag{8}$$

The parameter l is the size of the sliding window. When the number of obtained measurements is less than l, the size of the window is equal to the number of instants the system has evolved. Otherwise, the size of window is fixed to l (containing the l most recent measurements). The update strategy of the sliding window is First-In-First-Out (FIFO), which allows the online stream of measurements to be incorporated into the model, while gradually removing old measurements. By the time the sliding window is full of measurements (i.e., has length l), the algorithm is expected to reach high estimation accuracy, and subsequently continue tracking the state evolution. It follows directly from (5) that we may write the measurements in a given window as $b_k = \mathcal{A}_k \operatorname{vec}(\rho_k) + e_k$, where

$$\mathcal{A}_k = \begin{cases} (\operatorname{vec}(M_k), \ldots, \operatorname{vec}(M_2), \operatorname{vec}(M_1))^{\dagger}, & k < l, \\ (\operatorname{vec}(M_l), \ldots, \operatorname{vec}(M_2), \operatorname{vec}(M_1))^{\dagger}, & k \geq l. \end{cases} \tag{9}$$

Note that in this setting b_k is used for estimating ρ_k, in view of (5). In particular, $\mathcal{A}_k$ remains *unchanged* when the number of obtained measurements is greater than or equal to l.

The online QSE problem can be formulated as a sequence of convex optimization problems:

$$\begin{aligned} \underset{\hat{\rho}}{\text{minimize}} \quad & \|\mathcal{A}_k \operatorname{vec}(\hat{\rho}) - b_k\|_2^2 \\ \text{s.t.} \quad & \hat{\rho} \succeq 0, \operatorname{tr}(\hat{\rho}) = 1, \end{aligned} \tag{10}$$

where $\hat{\rho} \in \mathbb{C}^{d \times d}$ denotes the estimated density matrix; convex set $C : \{\hat{\rho} \succeq 0, \operatorname{tr}(\hat{\rho}) = 1\}$ means the quantum state density matrix constraints.

As an alternative to (10), we may consider the unconstrained problem:

$$\underset{\hat{\rho}}{\text{minimize}} \quad I_C(\hat{\rho}) + (1/2\gamma)\|\mathcal{A}_k \operatorname{vec}(\hat{\rho}) - b_k\|_2^2, \tag{11}$$

where $\gamma > 0$ is a regularization parameter; $I_C(\hat{\rho})$ is the indicator function of a set of quantum state constraints, and is defined as: $I_C(\hat{\rho}) = \begin{cases} 0 & \text{if } \hat{\rho} \succeq 0, \operatorname{tr}(\hat{\rho}) = 1, \\ \infty & \text{otherwise.} \end{cases}$

By introducing the auxiliary variable $\hat{e}$, problem (11) is recast as:

$$\begin{aligned} \underset{\hat{\rho} \in C, \hat{e}}{\text{minimize}} \quad & 0 + (1/2\gamma)\|\hat{e}\|_2^2 \\ \text{s.t.} \quad & \mathcal{A}_k \operatorname{vec}(\hat{\rho}) + \hat{e} = b_k. \end{aligned} \tag{12}$$

We invoke OPG-ADMM as a numerical solver for problem (12), whence the online quantum state estimation problem can be split into two sub-problems, as elaborated in what follows. In other words, we estimate the density matrix at time k, and the noise corresponding to the k-th window of measurements, using the sampling matrix $\mathcal{A}_k$ and measurements b_k alongside previously obtained estimates pertaining to the estimates time $k-1$. The subproblems of $\hat{\rho}$ and $\hat{e}$ are carried out separately, and sequentially, followed by an update of the Lagrange multiplier vector. Note that index k represents the sampling times in which we apply the continuous measurement as well as the estimation updates; thus, we perform a *single* iteration of OPG-ADMM for (12) so as to devise a computationally minimalistic method suitable for real-time implementation. The method performs the following updates, at instant k:

$$\begin{aligned} \hat{\rho}_k =& \arg\min_{\hat{\rho} \in C} \Big\{ \frac{\alpha}{2} \|\mathcal{A}_k \operatorname{vec}(\hat{\rho}) + \hat{e}_{k-1} - b_k - \lambda_{k-1}/\alpha\|_2^2 \\ & + (1/2\eta_k) \|\operatorname{vec}(\hat{\rho} - \hat{\rho}_{k-1})\|_{P_k}^2 \Big\}, \end{aligned} \tag{13a}$$

$$\begin{aligned} \hat{e}_k =& \arg\min_{\hat{e}} \big\{ (1/2\gamma)\|\hat{e}\|_2^2 \\ & + \frac{\alpha}{2} \|\mathcal{A}_k \operatorname{vec}(\hat{\rho}_k) + \hat{e} - b_k - \lambda_{k-1}/\alpha\|_2^2 \big\}, \end{aligned} \tag{13b}$$

$$\lambda_k = \lambda_{k-1} - \alpha (\mathcal{A}_k \operatorname{vec}(\hat{\rho}_k) + \hat{e}_k - b_k), \tag{13c}$$

where $\eta_k > 0$ is the step size parameter, and $P_k \succ 0$ is the proximity parameter.

In the following, we explicitly provide efficient methods for solving the two optimization problems for updating primal variables $\hat{\rho}_k, \hat{e}_k$ in (13a), (13b) (the dual update being straightforward, cf. (13c)).

Update for $\hat{\rho}_k$: In the subproblem for $\hat{\rho}_k$ with $\hat{e} \equiv \hat{e}_{k-1}$ and $\lambda \equiv \lambda_{k-1}$ fixed, there are three components: the constraint set for $\hat{\rho}$, the quadratic penalty term of augmented Lagrangian, and the proximal term. The measurement matrix $\mathcal{A}_k$ involved in the quadratic penalty term makes an analytic solution for $\hat{\rho}_k$ cumbersome. In order to resolve this issue, we choose an appropriate parameter for the proximal term as $P_k = \tau I - \alpha \eta_k \mathcal{A}_k^{\dagger} \mathcal{A}_k$ ($\tau > 0$ is a constant), whence the quadratic term $(\alpha/2) \|\operatorname{vec}(\hat{\rho} - \hat{\rho}_{k-1})\|_{\mathcal{A}_k^{\dagger}\mathcal{A}_k}$ (adding and subtracting $\mathcal{A}_k \hat{\rho}_{k-1}$ inside the norm at the last term of (13a)) is canceled. In order for P_k to be positive definite (i.e., to define a norm) we may select:

$$\eta_k = \tau / (\alpha \lambda_{\max} + \mathrm{c}), \tag{14}$$

where $\lambda_{\max}$ is the largest eigenvalue of $\mathcal{A}_k^{\dagger} \mathcal{A}_k$, and $\mathrm{c} > 0$ is a small constant (we use $c = 0.1$ in our experiments).

Note that from the definition of $\mathcal{A}_k$, cf. (9), it holds that $\mathcal{A}_k \equiv (\operatorname{vec}(M_l), \cdots, \operatorname{vec}(M_1))^{\dagger}$ is a fixed matrix for $k \geq l$. As a consequence, the maximum eignevalue used for η_k needs only be computed l times (where l is the size of the window). After cancelation, the proximal term remaining is represented as $(\tau/2\eta_k)\,\|\operatorname{vec}(\hat{\rho}-\hat{\rho}_{k-1})\|^2$. This term is crucial for yielding a tracking capability to our method, since it requires the new estimate be 'close' to the previous one (i.e., adds memory to the adaptive filter, cf. Fig. 1).

Subsequently, the subproblem for $\hat{\rho}_k$ is equivalent to: $\arg\min_{\hat{\rho}\in C}\left\{\langle \operatorname{vec}(\hat{\rho}-\hat{\rho}_{k-1}), \alpha\mathcal{A}_k^{\dagger}(\mathcal{A}_k \operatorname{vec}(\hat{\rho}_{k-1})+\hat{e}_{k-1}-b_k-\lambda_{k-1}/\alpha)\rangle + \frac{\tau}{2\eta_k}\|\operatorname{vec}(\hat{\rho}-\hat{\rho}_{k-1})\|^2\right\}$. We define the intermediate variable (for completion of squares):

$$\begin{aligned}\operatorname{vec}(\tilde{\rho}_k) = \operatorname{vec}(\hat{\rho}_{k-1}) - \frac{\alpha\eta_k}{\tau}\mathcal{A}_k^{\dagger}\left(\mathcal{A}_k \operatorname{vec}(\hat{\rho}_{k-1})\right.\\ \left.+\hat{e}_{k-1}-b_k-\lambda_{k-1}/\alpha\right),\end{aligned} \tag{15}$$

to obtain a compact form as:

$$\hat{\rho}_k = \arg\min_{\hat{\rho}\in C}\|\hat{\rho}-\tilde{\rho}_k\|_F^2, \tag{16}$$

where $\|\cdot\|_F$ is the squared Frobenius norm (the sum of squared absolute values of the entries).

Therefore, (16) is equivalent to the following constrained problem (projection to the constraint set):

$$\begin{aligned}\hat{\rho}_k =& \arg\min_{\hat{\rho}}\left\|\hat{\rho}-\left(\tilde{\rho}_k+\tilde{\rho}_k^{\dagger}\right)/2\right\|_F^2\\ &\text{s.t.}\qquad \operatorname{tr}(\hat{\rho})=1, \hat{\rho}\succeq 0.\end{aligned} \tag{17}$$

Problem (17) is a semidefinite program that can be solved using interior-point methods. Instead, we may devise a direct method using spectral decomposition [15]. From unitary diagonalization of the Hermitian matrix $(\tilde{\rho}_k+\tilde{\rho}_k^{\dagger})/2 = U\tilde{\Lambda}U^{\dagger}$, where $U\in\mathbb{C}^{d\times d}$ is a unitary matrix, and $\tilde{\Lambda} = \operatorname{diag}(\tilde{\lambda}_1,\cdots,\tilde{\lambda}_d)$ is a diagonal matrix with entries the eignevalues of $(\tilde{\rho}_k+\tilde{\rho}_k^{\dagger})/2$. In addition, the optimal solution can be written as $\hat{\rho}=U\hat{\Lambda}U^{\dagger}$, where $\hat{\Lambda}$ is obtained from [15]:

$$\begin{aligned}\hat{\Lambda} =& \arg\min_{\Lambda}\|\Lambda-\tilde{\Lambda}\|_F^2\\ &\text{s.t.}\qquad \operatorname{tr}(\Lambda)=1, \Lambda\succeq 0.\end{aligned} \tag{18}$$

Since the optimal solution is diagonal, the problem boils down to projection on the probability simplex, which can be efficiently computed in a finite number of (at most d) steps.

Update for $\hat{e}_k$: The subproblem for $\hat{e}_k$ is an unconstrained quadratic program ($\hat{\rho}\equiv\hat{\rho}_k$ and $\lambda\equiv\lambda_{k-1}$ are fixed) that admits an analytical solution directly from first-order optimality conditions (setting the gradient to zero) as:

$$\hat{e}_k = \left(\gamma\alpha/(1+\gamma\alpha)\right)\left(\lambda_{k-1}/\alpha - \mathcal{A}_k \operatorname{vec}(\hat{\rho}_k)+b_k\right). \tag{19}$$

The proposed method for Online State Estimation for Quantum systems (OSEQ) is summarized in Alg. 1: it is parameterized with three positive constants $\alpha,\gamma,\tau>0$ that are design parameters.

We recap that a key ingredient of OSEQ is the proximal term $\frac{1}{2\eta_k}\|\operatorname{vec}(\hat{\rho}-\hat{\rho}_{k-1})\|_{P_k}^2$ in the subproblem for the density matrix, which embodies online trackability and accelerated convergence. Moreover, the total computational complexity for each step of OSEQ (i.e., estimation for a given window) is $\mathcal{O}(d^3+ld^2)$ where $\mathcal{O}(d^3)$ is the cost for eigendecomposition required for updating the density and $\mathcal{O}(ld^2)$ the cost for matrix-vector multiplication $\mathcal{A}_k \operatorname{vec}(\hat{\rho})$.

Algorithm 1 OSEQ

Require: Initial estimates $\hat{\rho}_1$, $\hat{e}_1=0$, $\lambda_1=0$; tuning parameters α, γ, $\tau>0$; window size $l\in\mathbb{Z}_+$.

for $k=1,2,\ldots$ **do**
  Obtain measurement b_k;
  if $k\le l$ **then**
    Calculate the gradient step η_k according to (14);
  else
    $\mathcal{A}_k=\mathcal{A}_{k-1}$, $\eta_k=\eta_{k-1}$;
  end if
  Calculate $\tilde{\rho}_k$ according to (15);
  Run eigenvalue decomposition on $(\tilde{\rho}_k+\tilde{\rho}_k^{\dagger})/2$ as $Udiag\{\tilde{\lambda}_i\}U^{\dagger}$;
  Project eigenvalues $\{\tilde{\lambda}_i\}$ to the probability simplex to obtain $\hat{\Lambda}$
  Update $\hat{\rho}_k$= $U\hat{\Lambda}U^{\dagger}$;
  Update $\hat{e}_k$ according to (19);
  Update λ_k according to (13c);
end for

IV. SIMULATIONS

In this section, numerical experiments are carried out to assess the performance of OSEQ for online quantum state estimation. The experiment assesses the performance of OSEQ with regards to: a) estimation accuracy and b) computational efficiency. We compare our proposed algorithm with the method proposed in [8] termed CVX-OQSE (as it obtains a solution to optimization problem (11), using CVX) applied to 1-, 2-, 3-, and 4-qubit systems, respectively.

In experiments, the initial state density matrix of the true n-qubit system is chosen as $\rho_1^n = \underbrace{\rho_1\otimes\ldots\otimes\rho_1}_{n}$, $\rho_1 = [0.5,(1-i)/(\sqrt{8});(1+i)/(\sqrt{8}),0.5]$, and the initial estimate is selected as $\hat{\rho}_1^n = \underbrace{\hat{\rho}_1\otimes\ldots\otimes\hat{\rho}_1}_{n}$, $\hat{\rho}_1=[0,0;0,1]$; the initial measurement operator is $M_1^n = \underbrace{\sigma_z\otimes\ldots\otimes\sigma_z}_{n}$ (in both cases we use the superscript n to indicate the number of qubits, ranging in $\{1,2,3,4\}$ in our experiments); for $m_0(\Delta t)$ and $m_1(\Delta t)$ which constructs the discrete evolution equation of n-qubit system (2), $L_1=\xi\sigma_z$, $H_1=\sigma_z+u_x\sigma_x$. The system parameters are set as: $\xi=0.7$, $u_x=1$, $N=500$ and SNR $=30$dB. The parameters of OSEQ are selected as: $\gamma=0.1$, $\tau=10$, $c=0.1$. For $n=1,2,3,4$, we respectively choose $\alpha=2,10,12,15$, and set the size of the sliding window is as $l=13,16,30,100$ (for 1-, 2-, 3- and 4-qubit systems, the number of decision variables is 4, 16, 64 and 256, respectively, so the window size needs to be increased). Simulations are conducted in MATLAB R2016a, running in Inter Core i7-8750M CPU, clocked at 2.2GHz, with a memory of 16GB.

Fidelity is used to measure estimation accuracy and is defined as:

$$fidelity(k) = \operatorname{tr}\left(\sqrt{\sqrt{\hat{\rho}_k}\rho_k\sqrt{\hat{\rho}_k}}\right), \tag{20}$$

TABLE I

ESTIMATION ACCURACY AND ESTIMATION TIME REQUIRED FOR OSEQ AND CVX-OQSE FOR $n = 1, 2, 3, 4$.

	Qubit	1	2	3	4
OSEQ	Accuracy (%)	100	100	99.99	99.84
	Time (sec)	0.145	0.191	0.473	1.841
CVX-OQSE	Accuracy (%)	100	60-95	50-95	Null
	Time (sec)	142.4	168.5	213.3	Null

where $\hat{\rho}_k$ is the estimated density matrix and ρ_k is the true density matrix.

Fig. 2 shows the evolution of $fidelity(k)$ for OSEQ and CVX-OQSE in different qubit systems (note that CVX-OQSE is unable to estimate the density matrix of an 4-qubit system under measurement noise, so there is no fidelity curve for CVX-OQSE in this case in Fig. 2(b)). Table I reports the accuracy and time required for OSEQ and CVX-OQSE, where accuracy refers to the final fidelity value.

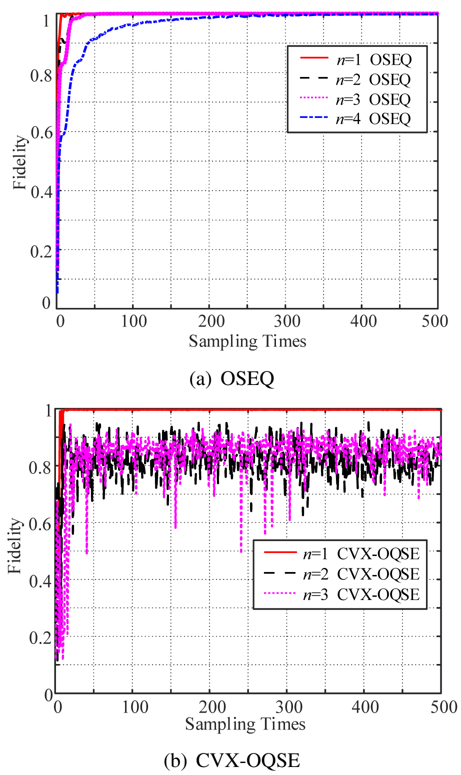

Fig. 2. Online estimation of quantum state in different qubit systems. Red solid, black dashed, pink dotted, and blue dot-dashed lines represent the fidelity time series for 1-, 2-, 3-, and 4-qubit systems, respectively.

From Fig. 2 and Table I, we conclude that QSEQ can efficiently and robustly estimate the quantum state online. The estimation accuracy of OSEQ in 1-, 2-, 3-, and 4-qubit systems gradually increases and stabilizes, with stable estimation accuracy equal to 100%, 100%, 99.99%, and 99.84%, respectively. CVX-OQSE only achieves stable estimation accuracy in 1-qubit system, and this comes at a cost of up to three orders of magnitude higher run-time (which may be limiting for real-time applications). We recall that OSEQ only applies one update step (for primal and dual variables) at each sampling time, while CVX-OQSE seeks to solve the optimization problem (11) via an interior point method that requires a larger number of iterations. We assess that OSEQ constitutes a promising method for real-time quantum state estimation.

V. CONCLUSIONS

We have devised a new method for online quantum state estimation from noisy linear measurements obtained concurrently with the state evolution. In order to obtain an adaptive filter for accurately tracking the state evolution we have: 1) adopted a sliding overlaping window approach to process the data in a streaming fashion, 2) cast state estimation as a semidefinite program, and 3) devised an efficient iterative solution with tunable trackability based on OPG-ADMM. Numerous experiments are supportive of the potential merits of the method as a real-time solution for state estimation in multi-qubit quantum systems.

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