Time Encoding Using the Hyperbolic Secant Kernel

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***Abstract*—We investigate the problem of reconstructing signals with finite rate of innovation from non-uniform samples obtained using an integrate-and-fire system. We assume that the signal is first filtered using the derivative of a hyperbolic secant as a sampling kernel. Timing information is then obtained using an integrator and a threshold detector. The reconstruction method we propose achieves perfect reconstruction of streams of K Diracs at arbitrary time locations, or equivalently piecewise constant signals with discontinuities at arbitrary time locations, using as few as 3K+1 non-uniform samples.**

***Index Terms*—Non-uniform sampling, time encoding, integrate-and-fire, finite rate of innovation, hyperbolic secant function.**

I. Introduction

Time encoding is a sampling paradigm inspired by the behaviour of biological spiking neurons which, when compared to classical sampling, can achieve low power consumption and a lighter demand on storage due to a reduced number of samples. Time-based sampling theory has recently experienced a renewed interest with several authors proving the capabilities of time encoding to sample and reconstruct bandlimited signals [1]–[6], signals that belong to shift-invariant subspaces [7], [8], and more recently, signals with finite rate of innovation (FRI) such as streams of pulses and piecewise constant signals [9]–[11]. This latter class of signals is of particular interest for applications ranging from time-of-flight technology, to neuromorphic cameras [12], ultrawideband (UWB) communications [13] and processing of neuronal signals [14].

The authors in [10] propose a framework that achieves reconstruction of bursts of Diracs of same-sign amplitudes from timing information obtained using a time encoding machine based on an integrator and threshold detector. The sufficient conditions for perfect estimation require the separation between the input Diracs, as well as the trigger mark of the threshold detector to be sufficiently small, in order to achieve the desired density of output samples. This method has also been extended to an iterative algorithm that operates with bursts of Diracs of arbitrary amplitudes and locations in [15].

In this paper, we propose a mathematical framework to achieve perfect reconstruction of streams of Diracs of arbitrary locations from time encoded information. This framework is based on adapting the algorithm proposed in [16], which reconstructs FRI signals from non-uniform samples, to the problem of reconstruction from samples generated by an integrate-and-fire system. Specifically, we propose a system where the signal is first filtered with a novel sampling kernel, the derivative of a hyperbolic secant, and then samples are generated using an integrate-and-fire system. We show that the samples generated by such a system can be related back to the original non-uniform sampling framework in [16] and thus the locations of the input Diracs can be retrieved perfectly.

Using the hyperbolic secant has some of the same advantages as exponential splines presented in [10]. In particular, the number of samples produced by a signal with a finite number of innovations is also finite. For example, when sampling a piecewise constant signal, samples are only generated around discontinuities. However, there are also further advantages. First, the kernel is smooth whereas exponential splines are non-smooth. In the latter case this means that output spikes must all fall within a certain region to allow reconstruction. Second, an arbitrary number of innovations can be reconstructed without modifying the kernel but by using a sufficient number of samples. In the case of exponential splines, we must use splines of higher order or multiple channels of low order splines to achieve the same result.

This paper is organised as follows. Section II gives a summary of the algorithm proposed in [16], which achieves perfect reconstruction of FRI signals from non-uniform samples. Then, in Section III we introduce our time encoding acquisition system based on integrate-and-fire. We also present an extension to the algorithm in [16] to reconstruct a stream of Diracs from time-based samples obtained using just such an integrate-and-fire system. In Section IV, we further extend this algorithm to the case of truncated sampling kernels as well as unknown starting conditions. The experimental results in Section V demonstrate the perfect retrieval of input Diracs and piecewise constant signals from timing information obtained using the integrate-and-fire encoding strategy. Finally, we present concluding remarks in Section VI.

II. Non-uniform sampling and reconstruction of FRI signals using the hyperbolic secant kernel

A generalised sampling process can be viewed as the system in Fig. 1. The input signal is first filtered with a kernel, $\varphi(t)$, and then measured at a set of points in time, $\tau_n \in \mathbb{R}$. In the non-uniform sampling case, these samples are unequally spaced in time.

In this paper we focus on sampling and reconstructing FRI signals [17]. First let us consider the archetypal FRI signal, a sum of K scaled and shifted Diracs:

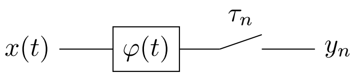

Fig. 1: A generalised sampling process

$$x(t) = \sum_{k=1}^{K} x_k \delta(t - t_k) \; . \tag{1}$$

Therefore, the output of the sampling process in Fig. 1 is:

$$y_n = y(\tau_n) = \sum_{k=1}^{K} x_k \varphi(\tau_n - t_k) \; . \tag{2}$$

The ability to perfectly reconstruct a signal sampled in this manner depends on the choice of sampling kernel. Non-uniform sampling and perfect reconstruction using a hyperbolic secant kernel was first shown in [16]. Here we will provide a brief review of this method as it forms the basis for the integrate-and-fire method presented in Section III.

The hyperbolic secant can be expressed in terms of exponentials, as follows:

$$\varphi(t) = \operatorname{sech}(at) = \frac{2e^{at}}{1 + e^{2at}} \; , \tag{3}$$

where $a \in \mathbb{R}$. Substituting Eq. (3) into Eq. (2) shows that when $\varphi(t) = \operatorname{sech}(at)$ each sample is the quotient of two polynomials evaluated at τ_n:

$$\begin{aligned} y_n = y(\tau_n) &= \sum_{k=1}^{K} x_k \operatorname{sech}(a\tau_n - at_k) \\ &= \sum_{k=1}^{K} \frac{2x_k e^{-at_k} e^{a\tau_n}}{1 + e^{-2at_k} e^{2a\tau_n}} \\ &= \frac{e^{a\tau_n} P(e^{2a\tau_n})}{Q(e^{2a\tau_n})} \; , \end{aligned} \tag{4}$$

where $P(x)$ and $Q(x)$ are polynomials of order $K - 1$ and K respectively. Furthermore, information of the locations, t_k, of the input Diracs is contained in the $Q(x)$ polynomial through the factorisation:

$$Q(x) = \prod_{k=1}^{K} \left(1 + e^{-2at_k} x\right) \; . \tag{5}$$

By multiplying both sides of (4) by $Q(x)$ we can form the following system of linear equations:

$$Q(e^{2a\tau_n}) y_n = a^{a\tau_n} P(e^{2a\tau_n}) \tag{6}$$

$$y_n \sum_{k=0}^{K} q_k e^{2ka\tau_n} = \sum_{k=0}^{K-1} p_k e^{(2k+1)a\tau_n} \; , \tag{7}$$

where q_k and p_k are the coefficients of the polynomials $Q(x)$ and $P(x)$ respectively. This can be expressed in the following matrix form:

$$\underbrace{\begin{bmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_N \end{bmatrix}}_{\mathbf{Y}} \underbrace{\begin{bmatrix} 1 & e^{2a\tau_1} & \dots & e^{2Ka\tau_1} \\ 1 & e^{2a\tau_2} & \dots & e^{2Ka\tau_2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{2a\tau_N} & \dots & e^{2Ka\tau_N} \end{bmatrix}}_{\mathbf{V}} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_K \end{bmatrix} = \underbrace{\begin{bmatrix} e^{a\tau_1} & e^{3a\tau_1} & \dots & e^{(2K-1)a\tau_1} \\ e^{a\tau_2} & e^{3a\tau_2} & \dots & e^{(2K-1)a\tau_2} \\ \vdots & \vdots & \ddots & \vdots \\ e^{a\tau_N} & e^{3a\tau_N} & \dots & e^{(2K-1)a\tau_N} \end{bmatrix}}_{\mathbf{W}} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{K-1} \end{bmatrix} \tag{8}$$

This system $\mathbf{YVq} = \mathbf{Wp}$ can be solved up to a multiplicative constant when $N \geq 2K$. Using the coefficients of $Q(x)$ we can then compute its roots, ζ_k. The locations of the input Diracs are then retrieved using:

$$t_k = \frac{\ln(-\zeta_k)}{2a} \; . \tag{9}$$

By substituting values of t_k and τ_n into Eq. (2) we can now form a system of linear equations, which can be solved to retrieve the magnitudes, x_k, of the input Diracs.

III. Time encoding and reconstruction using an integrate-and-fire system

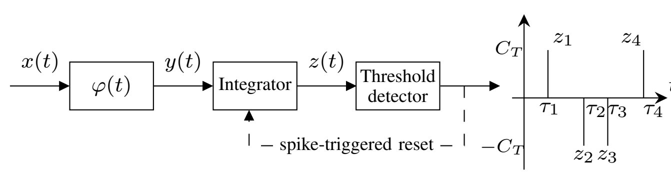

Fig. 2: Time encoding machine using integrate-and-fire mechanism

The integrate-and-fire system illustrated in Fig. 2 is the same as that used in [10]. The input signal $x(t)$, is filtered with the kernel $\varphi(t)$, to give $y(t)$ and then integrated. The integral $z(t)$ is then compared to a positive and a negative threshold value, $\pm C_T$.

At times τ_n, where n is positive integer, $z(t)$ reaches the threshold value and the amplitude, $z_n = \pm C_T$, is recorded along with the time τ_n. The integral $z(t)$ is then reset to zero.

The system in Fig. 2 is similar in form to the sampling process depicted in Fig. 1 but differs in that the device used to measure the filtered signal at points τ_n is replaced with an integrator, a threshold detector, and a mechanism to reset the integrator to zero. The timing of samples generated by this system is therefore dependent on the input signal and not predetermined like in the classical uniform and non-uniform sampling cases. Furthermore, the amplitude of the samples generated, z_m, are related to but not equivalent to the samples in the generalised sampling process, y_n. In what follows we

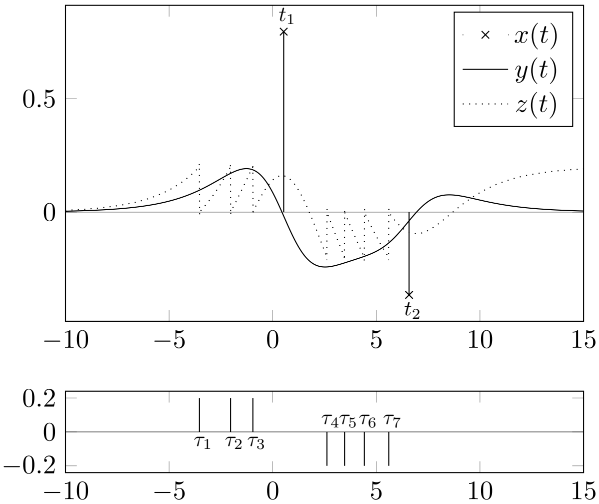

Fig. 3: Example intermediate stages and output of an integrate-and-fire time encoding machine. Input signal, $x(t)$ (top). Filtered signal, $y(t)$ (top solid). Integral with resets $z(t)$ (top dashed). Resulting time encoded samples (bottom).

show how the output samples, z_n, from such an integrate-and-fire system can be related to the samples, y_n, in the non-uniform sampling case reviewed in Section II. Thus we also demonstrate perfect reconstruction using an integrate-and-fire time encoding.

As shown in [10], a relationship between the output spikes, z_n, and the corresponding input signal, $x(t)$, is given by:

$$z_n = z(\tau_n) = \int_{\tau_{n-1}}^{\tau_n} y(t)dt = \langle x(t), \psi_{\theta_n}(t)\rangle \ , \quad (10)$$

where $\langle x, y\rangle$ denotes the inner product between x and y and:

$$\psi_{\theta_n}(t) = \int_{\tau_{n-1}-t}^{\tau_n - t} \varphi(\gamma)d\gamma \ . \quad (11)$$

Substituting Eq. (1) into Eq. (10) we find an expression for z_n in terms of ψ_{θ_n}, t_k, and x_k:

$$\begin{aligned} z_n &= \left\langle \sum_{k=1}^{K} x_k \delta(t - t_k), \psi_{\theta_n}(t) \right\rangle \\ &= \sum_{k=1}^{K} x_k \psi_{\theta_n}(t_k) \end{aligned} \quad (12)$$

With an appropriate choice of kernel we can create a linear system of equations from Eq. (12) and using the non-uniform sampling technique in Section II. First we select the derivative of a hyperbolic secant as our kernel:

$$\varphi(t) = \frac{d}{dt}\operatorname{sech}(at) = -a \operatorname{sech}(at)\tanh(at) \ , \quad (13)$$

which gives

$$\psi_{\theta_n}(t) = \operatorname{sech}(a\tau_n - at) - \operatorname{sech}(a\tau_{n-1} - at) \ . \quad (14)$$

Also by defining $\tau_0 = -\infty$ such that $z_1 = \int_{-\infty}^{\tau_1} y(t)dt$ we find that

$$\psi_{\theta_1}(t) = \operatorname{sech}(a\tau_1 - at) \ . \quad (15)$$

From Eq. (14) and Eq. (15) it can clearly be seen that:

$$\sum_{n=1}^{m} \psi_{\theta_n}(t) = \operatorname{sech}(a\tau_m - at) \ . \quad (16)$$

Now taking the cumulative sum of samples, z_n, we can define a set of modified samples, y_n:

$$\begin{aligned} y_n = \sum_{m=1}^{n} z_m &\overset{(a)}{=} \sum_{m=1}^{n}\sum_{k=1}^{K} x_k \psi_{\theta_m}(t_k) \\ &\overset{(b)}{=} \sum_{k=1}^{K} x_k \operatorname{sech}(a\tau_n - at_k) \end{aligned} \quad (17)$$

where (a) follows from substituting in Eq. (12) and (b) follows from changing the order of summation and substituting in Eq. (16). Equation (17) is equivalent to the non-uniform sampling case in Eq. (4). Therefore, it is clear that we can fully retrieve the innovation parameters of $x(t)$ using the reconstruction method reviewed in Section II and the linear system of equations in Eq. (8). This can be achieved when we have at least $2K$ samples, y_n.

We further note that the system in Eq. (8) is always solvable. Assuming the case using a minimum number of samples $N = 2K$, the product $\mathbf{YV}$ has $2K$ rows and $K+1$ columns. Therefore, in order to have full rank, a maximum of $K-1$ rows of $\mathbf{YV}$ may be zero. The number of zero rows corresponds to the number of cumulative samples, y_n, that are zero. These samples are equivalent to the samples of a sum of scaled and shifted hyperbolic secants in Eq. (4). From Eq. (4) we can see that $y(t)$ has the same zeros as the polynomial $P(x)$. This polynomial is of order K and therefore has $K-1$ roots. Consequently, assuming that each sample is unique, only $K-1$ samples can be zero. $\mathbf{YV}$ is thus guaranteed to be full rank. The argument extends to $N = 2K + L$ for $L > 1$ where at most $K - 1 + L$ samples can be zero.

Therefore, it is clear that in order to reconstruct the input signal the integrate-and-fire system must generate at least $2K$ unique samples. Sufficient conditions on the input signal can be found to guarantee at least $2K$ samples in the case where all input Diracs are of the same sign. Without loss of generality, consider the case of all positive input Diracs. The integral $y(t)$ is expressed in Eq. (2). For a single Dirac ($K = 1$), we can ensure $2N+1$ samples are generated by making sure the maximum of $y(t)$ is above $(N+1)C_T$. This means that the integral is reset and a positive sample generated at least $N+1$ times before it reaches the maximum. After the maximum the integral is reset and a negative sample generated at least N

times. Increasing K, that is adding more positive Diracs, only serves to increase the amplitude of the maximum. Therefore, we can guarantee at least $2K+1$ output samples when at least one of the input Diracs has an amplitude of at least $(K+1)C_T$.

IV. Reconstruction with unknown starting conditions

The aforementioned method for reconstruction relies on knowledge of the starting conditions of the time encoding machine. In this section we relax this assumption and present a reconstruction scheme tailored to the following practical considerations. Our knowledge of the starting conditions of the encoder can be lacking in three main ways:

1. *The encoder may have spiked one or more times before recording started* - The method in Section III requires that we know and use the first output spike, τ_1, otherwise Eq. (16) cannot be realised. Ideally we would like to be able to reconstruct using a number of different subsets of the generated samples and not just the first $2K$ spikes.

2. *The kernel may be truncated in some way resulting in an initial bias to $z(t)$* - This paper has considered kernels of infinite support. In principal there is no reason why truncated or similarly windowed kernels could not be used subject to certain conditions on the input signal. However, truncating the kernel introduces a (negative) bias into the integral $y(t)$ which must be taken into account when reconstructing.

3. *The integral $z(t)$ may have started with a non-zero value* - This is important in the case where we want to perform reconstruction of localised portions of the input signal. For example, reconstruction of streams of bursts of Diracs was demonstrated in [10] using kernels of compact support. The first spike to follow a burst of Diracs in the input signal was influenced not only by the current burst but also by the residual integral remaining from the previous burst. This was accounted for by realising that the influence of the previous burst only affects the first spike to follow the current burst and not the subsequent spikes. Since our method requires the first sample be used, a different approach is needed to account for the residual integral.

All three of these cases are equivalent to adding a constant term to the first sample we use. For the first and third case we show this by modifying the expression for the first sample, z_1, from Eq. (10) with a residual integral, b, and consider samples starting from $n = M$ for $M > 0$ (e.g. if $M = 2$ then we are discarding the first output sample). The first sample, z_M, is therefore:

$$\begin{aligned} z_M = \pm C_T &= \int_{\tau_{M-1}}^{\tau_M} y(t)dt \\ &= \int_{\tau_0}^{\tau_M} y(t)dt - \int_{\tau_0}^{\tau_{M-1}} y(t)dt \\ &= \int_{\tau_0}^{\tau_M} y(t)dt - \underbrace{\left(b + \sum_{n=1}^{M-1} z_n\right)}_{c} \end{aligned} \tag{18}$$

The derivation then follows in the same way as Eq. (10) except we define $\psi_{\theta_M}(t) = \int_{\tau_0-t}^{\tau_M-t} \varphi(\gamma)d\gamma = \operatorname{sech}(a\tau_M - at)$ such that:

$$\sum_{l=M}^{n} \psi_{\theta_l}(t) = \operatorname{sech}(a\tau_n - at) \;, \tag{19}$$

and

$$z_M = \left(\sum_{k=1}^{K} x_k \psi_{\theta_M}(t_k)\right) - c \;. \tag{20}$$

It can be seen from Eq. (18) and (20) that estimating using samples starting at M is equivalent to treating z_M as the first sample and adding a constant term. Taking the cumulative sum of samples starting from M now yields an expression with a constant:

$$y_n = \sum_{l=M}^{M+n-1} z_l \tag{21}$$

which gives,

$$\begin{aligned} y_n + c &= \sum_{k=1}^{K} \sum_{l=1}^{M+n-1} \psi_{\theta_l}(t_k) \\ &= \sum_{k=1}^{K} \operatorname{sech}(a\tau_n - at_k) \;. \end{aligned} \tag{22}$$

which is equivalent to Eq. (17) with a constant term added to the left hand side. The linear system in Eq. (8) can now be modified to take this constant into account:

$$(\mathbf{Y} + c\mathbf{I})\mathbf{V}\mathbf{q} = \mathbf{W}\mathbf{p} \;. \tag{23}$$

Given knowledge of the constant, c, Eq. (23) can be solved up to a multiplicative constant using $2K$ samples. In the case when c is unknown, the system can be solved using $3K + 1$ samples by linearising the problem as follows:

$$\begin{bmatrix}\mathbf{YV} & \mathbf{V}\end{bmatrix} \begin{bmatrix}\mathbf{q} \\ c\mathbf{q}\end{bmatrix} = \mathbf{W}\mathbf{p} \;. \tag{24}$$

V. Experimental results

Samples of two input Diracs generated using an integrate and fire process are plotted in Fig. 4 alongside the estimated signal reconstructed using the samples. It can be seen that the reconstruction is identical within numerical precision.

Reconstruction of sums of scaled and shifted Diracs can be further extended to the case of piecewise constant signals. Let $x(t)$ be a piecewise constant signal:

$$x(t) = \sum_{k=1}^{K} x_k u(t - t_k) \;, \tag{25}$$

where $u(t)$ is the Heaviside function. By defining a new sampling kernel,

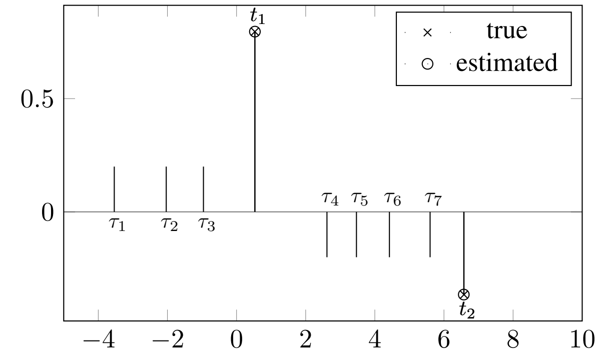

Fig. 4: Reconstruction of input Diracs from output samples generated by integrate-and-fire method.

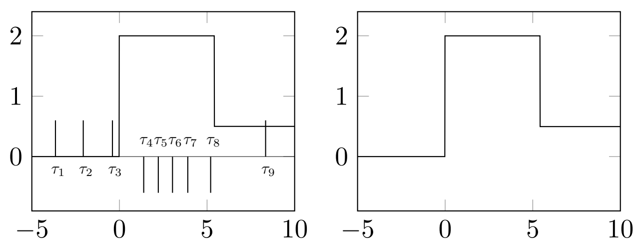

Fig. 5: Reconstruction of piecewise constant signal from output samples generated by integrate-and-fire method.

$$\varphi'(t) = \frac{d\varphi(t)}{dt} = a^2 \left(\tanh^2(at)\,\mathrm{sech}(at) - \mathrm{sech}^3(at)\right) \ , \tag{26}$$

we can reformulate the problem of sampling a piecewise constant signal into the problem of sampling a sum of scaled and shifted Diracs:

$$y(t) = \left(x * \frac{d\varphi}{dt}\right)(t) \overset{(a)}{=} \left(-\frac{dx}{dt} * \varphi\right)(t) \ ,$$

where (a) follows from integration by parts and

$$\frac{dx}{dt} = \sum_{k=1}^{K} x_k \delta(t - t_k) \ . \tag{27}$$

Sampling a piecewise constant using the filter in Eq. (26) is therefore equivalent to sampling a sum of scaled and shifted Diracs with our original filter in Eq. (3). An example of sampling and reconstructing a piecewise constant signal is shown in Fig. 5.

VI. Conclusion

In this paper we have presented a method for generating time encodings using the hyperbolic secant as a sampling kernel in conjunction with an integrate-and-fire system. In the case of FRI signals, such as piecewise constant signals and sums of scaled and shifted Diracs, we have shown that the input signal can be perfectly reconstructed from the samples generated by such a system. In particular, in the ideal case where we know the starting conditions of the system, a signal with $2K$ innovations can be reconstructed from as few as $2K$ samples. This assumption was then relaxed to allow reconstruction from $3K + 1$ samples without knowledge of the starting conditions and in the case where the kernel is truncated or windowed.

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