Nonparametric Adaptive Value-at-Risk Quantification Based on the Multiscale Energy Distribution of Asset Returns

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***Abstract*—Quantifying risk is pivotal for every financial institution, with the temporal dimension being the key aspect for all the well-established risk measures. However, exploiting the frequency information conveyed by financial data, could yield improved insights about the inherent risk evolution in a joint time-frequency fashion. Nevertheless, the great majority of risk managers make no explicit distinction between the information captured by patterns of different frequency content, while relying on the full time-resolution data, regardless of the trading horizon. To address this problem, a novel value-at-risk (VaR) quantification method is proposed, which combines nonlinearly the time-evolving energy profile of returns series at multiple frequency scales, determined by the predefined trading horizon. Most importantly, our proposed method can be coupled with any quantile-based risk measure to enhance its performance. Experimental evaluation with real data reveals an increased robustness of our method in efficiently controlling under-/over-estimated VaR values.**

***Index Terms*—Risk quantification, adaptive value-at-risk, time-scale decompositions, energy distribution**

I. Introduction

In the financial world, risk management is a key process for the identification, analysis, and alleviation of uncertainty in investment decisions. To this end, the design of efficient risk measurement methods plays a central role in the risk management pipeline of financial institutions, such as banks, investment funds, and insurance companies.

From an economic viewpoint, the heterogeneity of market participants (e.g. buyers, sellers, banks, federal governments, etc.) justifies the simultaneous utilization of several periodicities. In a seminal work [1], it was highlighted that more attention should be devoted to the process of choosing the basic unit of time: an empirical study which is based on a yearly rate of returns will yield different results from one that employs a monthly rate of returns. This difference is not a result of inconsistency or contradiction, but is due to selecting an inappropriate division of the period studied.

Despite these findings, the great majority of the well-established risk measures, and particularly value-at-risk (VaR), expected shortfall (ES), and expectiles, are based on the time-varying behavior of a returns series by considering its full time resolution. As a result, they ignore completely the contribution of the individual intrinsic periodicities (i.e., *frequencies*), that can be more relevant to a predetermined trading horizon. Given this remark, the key question is whether all the low- and high-frequency components that constitute the original returns series are relevant for risk quantification at a given trading horizon, or if by considering only a subset of them may entail more meaningful information.

To address this problem, in this paper we propose a novel *nonparametric adaptive VaR quantification* method, which is based on the inherent time-varying energy distribution of returns series. More specifically, our proposed approach enables a risk manager to focus precisely only on those frequency components that contain the most relevant information for a specific trading horizon. To this end, the relevance of each frequency component (or, equivalently *time resolution*) is quantified in terms of its contribution to the overall energy content of the returns series.

The rest of the paper is organized as follows: Section II refers to related prior studies. In Section III, our energy-based VaR quantification method is analyzed, while Section IV demonstrates its predictive accuracy on a set of stock indexes. Finally, Section V summarizes the main outcomes and gives directions for further extensions.

II. Relation to prior work

In risk quantification, wavelet analysis has gained an increasing interest, since it enables capturing the time-varying features of risk, while unraveling the risk dynamics at distinct frequencies. In particular, a multiresolution approach has been employed in [2]–[4] to estimate the systematic risk in a capital asset pricing model by decomposing the variance of market returns at multiple time scales. A wavelet-based counterpart of beta was proposed in [5] as an alternative measure of market risk A combination of continuous and discrete wavelet transforms with traditional financial models (e.g. DCC-EGARCH and copulas) was employed in [6], [7] to improve the accuracy of VaR and ES estimates of oil-exchange rate and oil-stock market portfolios after noise is removed from the original data.

Despite the increased efficiency of these methods, they mainly rely on a scale-by-scale analysis of variance and cross-spectra covariance among distinct assets, while also considering all the decomposition levels. Doing so, they rather provide empirical results with respect to the explanatory information conveyed by each frequency for a given trading horizon and for various asset classes, instead of exploiting directly the most relevant subset of frequencies for a given trading horizon, as is the case of our method. Furthermore, the intermediate step of smoothing (denoising) the wavelet coefficients (ref. [7]) distorts the high-frequency information, which is important for short-term risk management. Instead, our method does not perform any denoising, thus preserving all the necessary information for a given trading horizon. In addition, the need for estimating accurately model parameters (e.g. in GARCH-like models) or for choosing specific marginal distributions to accurately fit copulas, as in [6], [7], entails the danger of inaccurate risk measurement due to imprecise parameters estimation or inferior fitting performance of the chosen marginal distributions. To alleviate this issue, our method does not involve any model selection or parameters estimation procedures. As such, it better adapts to the returns series behavior, by avoiding model misfitting errors. Most importantly, our proposed framework is generic and can be coupled with any of the existing quantile-based risk measures to enhance their performance by i) automatically determining the optimal subset of relevant frequencies for a given trading horizon and ii) nonlinearly combining the statistics, specifically the quantiles, of the distinct multiresolution series to calculate the overall VaR.

III. Energy-based VaR Quantification

In the following, we denote asset prices by p_t, where, for convenience, we assume that the index t indicates a daily frequency. Subsequently, $\mathbf{p} = [p_1, \ldots, p_N] \in \mathbb{R}^N$ denotes a discrete-time series of N prices observed at times $\{t_1, \ldots, t_N\}$. Typically, risk quantification employs the returns of an investment or trading strategy. Hereafter, our proposed VaR quantification method employs continuously compounded returns, r_t, over a time period of T_p time units, which are defined as follows,

$$r_t \doteq \ln(p_t) - \ln(p_{t-T_p}) \ , \ t = T_p + 1, \ldots, N \ . \qquad (1)$$

For instance, when $T_p = 1$, r_t corresponds to daily returns, whereas for $T_p = 5$ and $T_p = 25$, r_t corresponds to weekly and monthly returns, respectively, computed on daily data.

A. Conventional Risk Measures

Volatility, that is, the standard deviation of returns, is the main measure of risk in most financial applications. However, volatility is likely to systematically underestimate risk, especially when the probability of extreme returns is high and the distribution of $\mathbf{r}$, defined in (1), deviates significantly from the Gaussian. The next most commonly used risk measure is the *value-at-risk* (VaR). VaR is a single summary statistical measure of risk, it is distribution independent, and gives a measure of losses as a result of "typical" market movements. Let r_t denote the random variable of compounded returns, given by (1), for a given time period, and $c \in (0, 1)$ be a confidence level. The α-level VaR, with $\alpha = 1 - c$, is defined as the number $\mathrm{VaR}_t(\alpha)$ such that

$$\Pr\left(r_t \leq -\mathrm{VaR}_t(\alpha)\right) = \alpha \ . \qquad (2)$$

In other words, $-\mathrm{VaR}_t(\alpha)$ is the αth quantile of the distribution of returns. Notice that we follow the convention that VaR is a positive number, thus the minus sign is put to express losses. The trading horizon (a.k.a. holding period) is the time interval over which losses may occur. This is usually one day, but can be more or less depending on the particular needs of a risk manager. Those who actively trade their portfolios may use a one-day trading horizon, but longer horizons are more realistic for institutional investors and non-financial corporations. Nevertheless, the identification of the probability distribution of returns is the most difficult and critical part in VaR estimation. The conventional approach is to estimate the quantile by using past observations (historical simulation) or fit a statistical model. For instance, under a normality assumption for the distribution of returns, the α-level VaR is given by

$$\mathrm{VaR}_t(\alpha) = -z_\alpha \sigma_{r_t} \ , \qquad (3)$$

where z_α is the αth quantile of the standard normal distribution (e.g. for $c = 95\%$, $\alpha = 0.05$ and $z_\alpha = -1.65$) and σ_{r_t} is the volatility of returns over the predetermined time period.

To address the time scaling of volatility and VaR, when the time period T_p, over which the returns are computed, differs from the trading horizon T_H, a commonly used method is the square-root-of-time rule. In particular, let σ_{T_p} be the volatility estimated over T_p time units and σ_{T_H} be the target volatility over a trading horizon of T_H time units. Then, given that T_p and T_H are expressed in the same time unit, we have $\sigma_{T_H} = \sqrt{T_H/T_p}\sigma_{T_p}$.

B. Multiresolution Analysis of Returns Series

To overcome the limitations of the widely-used orthogonal discrete wavelet transform (DWT), the maximal overlap DWT (MODWT [8, Ch. 5] has been proposed as an efficient alternative. Unlike the DWT, MODWT is defined naturally for all sample sizes N (and not only for powers of two), it offers an increased resolution at coarser scales, and it is invariant to translations. Furthermore, in contrast to DWT, the MODWT does not downsample after filtering the data at each frequency (hereafter called *scale*). This information redundancy is important for our proposed method, since it improves the subsequent statistical inference due to the increased sample size.

Having fixed the wavelet filter and a maximum number of decomposition scales, J, the outcome of a J-level MODWT applied on $\mathbf{r} \in \mathbb{R}^N$ is a $(J+1) \times N$ matrix, $\mathbf{C} = [\mathbf{d}_1 \ \mathbf{d}_2 \ \cdots \ \mathbf{d}_J \ \mathbf{a}_J]^\mathsf{T}$, whose first J rows contain the detail coefficients vectors $\mathbf{d}_j \in \mathbb{R}^N$ at scales 2^{j-1}, $j = 1, \ldots, J$, and the last, $(J+1)$th, row contains the approximation coefficients vector $\mathbf{a}_J \in \mathbb{R}^N$ at scale 2^J and beyond. In particular, $\mathbf{d}_j$

contains frequency information that corresponds to a time resolution between $[2^j, 2^{j+1})$ time units. For example, for daily data and a 4-level MODWT, $\mathbf{d}_1$ is associated with a time resolution of $[2, 4)$ days, $\mathbf{d}_3$ corresponds to $[8, 16)$ days, while $\mathbf{a}_4$ captures the smooth variations for a time resolution of 16 days and beyond.

At the core of our proposed approach is the inherent relation between the concepts of *risk* and *energy distribution*. First, we notice that the variance has been the most famous moment-based measure of total risk in segmented markets. Then, in physics, the kinetic energy, E_k, of an object is the energy that it possesses due to its motion, and depends on the mass, m, and the speed, v, of the object,

$$E_k \doteq (mv^2)/2 \ . \tag{4}$$

The analogue of mass for financial prices can be thought of as the value of an asset or a portfolio. Moreover, the notion of speed is related precisely to the price variation per time unit, or, in other words, to the returns. In turn, the square of returns is used as a proxy of volatility, thus giving a direct relation between the variance of returns (i.e., the square of volatility) with their energy. The *preservation of energy* property of the MODWT [8, Sec. 5.4] induces that

$$E_{\mathbf{r}} = \sum_{j=1}^{J} E_{\mathbf{d}_j} + E_{\mathbf{a}_J} \ , \tag{5}$$

where, from a signal processing perspective, the energy of a time series $\mathbf{x} \in \mathbb{R}^N$ is simply defined as its squared ℓ_2 norm, that is, $E_{\mathbf{x}} \doteq \|\mathbf{x}\|_2^2$.

Given the wavelet decomposition $\mathbf{C}$, a *multiresolution analysis* (MRA) is the process of inverting the wavelet transform, in order to reconstruct a time-domain vector from a vector of transform coefficients. For a J-level MODWT, an approximation (low-resolution) series of returns $\mathbf{r}_{A_J} \in \mathbb{R}^N$ is reconstructed from the Jth-level approximation coefficients $\mathbf{a}_J$, along with a set of J detail (higher-resolution) returns series $\mathbf{r}_{D_j} \in \mathbb{R}^N$, reconstructed from the corresponding detail coefficients $\mathbf{d}_j$, $j = 1, \ldots, J$. The MRA of $\mathbf{r}$ can be represented by the matrix

$$\mathbf{R} = [\mathbf{r}_{D_1} \ \mathbf{r}_{D_2} \ \cdots \ \mathbf{r}_{D_J} \ \mathbf{r}_{A_J}]^{\mathsf{T}} \in \mathbb{R}^{(J+1)\times N} \ . \tag{6}$$

It holds that the original returns series $\mathbf{r}$ is recovered exactly as a linear combination of the J-level MRA, that is,

$$\mathbf{r} = \sum_{j=1}^{J} \mathbf{r}_{D_j} + \mathbf{r}_{A_J} \ . \tag{7}$$

The performance of a wavelet-based risk quantification method depends on the choice of the specific wavelet filter. The critical tradeoff to be found concerns the choice of a suitable filter with an appropriate length. In particular, increasing the filter length yields improved data fitting, but also severely influences the boundary conditions during the circular filtering. Our experimental evaluation showed that the Fejer-Korovkin wavelet (`'fk'`) of order 6 yielded a high performance, in terms of predictive accuracy of our energy-based VaR measure, for several distinct indexes. An alternative could be the concatenation of multiple fixed wavelet dictionaries to possibly mitigate the optimal filter selection issue, though at the price of an increased computational expense, given the high redundancy already induced by a single MODWT. Nevertheless, the automatic selection of the optimal wavelet is still an open task.

C. Proposed VaR as a Nonlinear Combination of Quantiles

In practice, the problem of estimating and comparing risk for distinct assets or trading strategies is highly nontrivial, since the underlying distribution of market prices and returns for the various assets is unknown. Furthermore, the task of forecasting risk, especially for longer trading horizons, which is also more realistic for institutional investors and non-financial corporations, is much more demanding since we are trying to estimate events that occur rarely.

Our nonparametric VaR quantification method employs the energy distribution of the transform coefficients to determine the amount of risk which is relevant to a given trading horizon. More specifically, for a given trading horizon of T_H time units (e.g. days, weeks, months), the maximum decomposition level J_ρ, and accordingly the maximum wavelet scale $\tau_{J_\rho} = 2^{J_\rho - 1}$, to account for estimating the risk is given by the following proposition,

Proposition 1: For a given trading horizon of T_H time units and a J-level MODWT, the maximum decomposition level J_ρ ($1 \leq J_\rho \leq J$) to account for estimating the risk is given by $J_\rho = \lfloor \log_2(T_H) \rfloor + 1$, where $\lfloor \cdot \rfloor$ denotes the floor function.

The intuition behind this rule of thumb is that a risk manager is interested in assessing the risk associated with a subset of time resolutions (i.e., frequencies), determined by the specific trading horizon. Moreover, to account for the nonuniform size of the consecutive intervals, as the decomposition level increases (e.g. levels 1, 2, and 3 correspond to time resolution intervals of $[2, 4)$, $[4, 8)$, and $[8, 16)$ time units, respectively), we include an additional decomposition level, which contains information at the next lower time resolution from the one determined by the specific trading horizon.

For convenience, we assume $J_\rho < J$, thus in the following we omit the incorporation of the Jth-level approximation series $\mathbf{r}_{A_J}$. From (7), the original returns series can be approximated by

$$\hat{\mathbf{r}} \approx \sum_{j=1}^{J_\rho} \mathbf{r}_{D_j} \ . \tag{8}$$

A straightforward extension is to consider a weighted version of this approximation, as follows,

$$\hat{\mathbf{r}}_w \approx \sum_{j=1}^{J_\rho} w_j \mathbf{r}_{D_j} \ , \tag{9}$$

for some weights $w_j \in \mathbb{R}$, $j = 1, \ldots, J_\rho$. From (9) and the approximate normality assumption for the MRA series, we deduce that $\hat{\mathbf{r}}_w$ is also approximately normally distributed with parameters

$$\mu_{\hat{\mathbf{r}}_w} = \sum_{j=1}^{J_\rho} w_j \mu_{\mathbf{r}_{D_j}} \ , \ \sigma^2_{\hat{\mathbf{r}}_w} = \sum_{j=1}^{J_\rho} w_j^2 \sigma^2_{\mathbf{r}_{D_j}} \ , \tag{10}$$

where $\mu_{\mathbf{r}_{D_j}}$ is the sample mean of $\mathbf{r}_{D_j}$ and $\sigma^2_{\mathbf{r}_{D_j}}$ its sample variance.

Let $\hat{q}_\alpha$ denote the αth quantile of $\hat{\mathbf{r}}_w$. A lower quantile approximation of $\hat{q}_\alpha$, with $\alpha \in (0, 1/2)$, is given by [9],

$$\hat{q}_\alpha \approx \mu_{\hat{\mathbf{r}}_w} - z_{1-\alpha} \sigma_{\hat{\mathbf{r}}_w} \ , \tag{11}$$

where $z_{1-\alpha}$ is the $(1-\alpha)$th quantile of the standard normal distribution. Similarly, the lower αth quantiles of the individual MRA series are approximated by

$$\hat{q}_{\alpha;\mathbf{r}_{D_j}} \approx \mu_{\mathbf{r}_{D_j}} - z_{1-\alpha} \sigma_{\mathbf{r}_{D_j}} \ , \ j = 1, \dots, J_\rho \ . \tag{12}$$

By solving (12) in terms of $\sigma_{\mathbf{r}_{D_j}}$ and substituting in (10) and (11), we obtain the following approximation for the lower αth quantile of $\hat{\mathbf{r}}_w$, as a function of the lower αth quantiles of its constituent MRA returns series,

$$\hat{q}_\alpha \approx \sum_{j=1}^{J_\rho} w_j \mu_{\mathbf{r}_{D_j}} - \left[\sum_{j=1}^{J_\rho} w_j^2 \left(\mu_{\mathbf{r}_{D_j}} - \hat{q}_{\alpha;\mathbf{r}_{D_j}} \right)^2 \right]^{1/2} \ . \tag{13}$$

Concerning the choice of the weights, in our implementation we adopt an energy-based weighting scheme by setting $w_j = E_{\mathbf{d}_j} / E_{\mathbf{r}}$, $j = 1, \dots, J_\rho$. That is, the jth weight is equal to the relative energy associated with the wavelet coefficients at level j. Doing so, the decomposition scales with a higher contribution to the overall energy, or, equivalently, to the total risk, of the returns series are granted a higher weight.

Having estimated the αth quantile, $\hat{q}_\alpha$, our proposed α-level energy-based VaR is given by

$$\mathrm{VaR}_{\mathrm{QNL},t}(\alpha) = -\hat{q}_\alpha \ . \tag{14}$$

We emphasize again that the critical advantage of employing our MRA-based approach is that the distinct time resolutions (i.e., scales) are computed based on the original full-resolution series. This is in contrast to the conventional approaches, which first subsample the original series to form a lower resolution dataset according to the predefined trading horizon, thus discarding significant information, prior to estimating the VaR. Notice also that, although nonlinear, the quantiles' combination expressed by (13) does not account for potential correlations between the distinct scales. To this end, more complex quantile regression methods can be utilized, which is left as a separate thorough study.

IV. Experimental Evaluation

In this section, we illustrate with an empirical evaluation on two major stock indexes, the advantages of using our proposed energy-based VaR as a measure of risk in periods of financial turmoil, against the traditional VaR measures. We provide empirical evidence showing that our energy-based VaR adapts better to longer trading horizons and provides an improved responsiveness to extreme events during the financial crisis, as opposed to the conventional VaR with regulatory coverage levels. Specifically, we consider the daily adjusted closing prices of S&P500 and DAX, two of the major stock indexes worldwide. Our data cover the period from January 1, 2001, to December 31, 2010.

Our proposed energy-based VaR, hereafter denoted by $\mathrm{VaR}_{\mathrm{QNL}}$, is compared with the conventional methods based on historical simulation and the normal distribution [10], denoted by $\mathrm{VaR}_{\mathrm{HS}}$ and $\mathrm{VaR}_{\mathrm{Gauss}}$, respectively. The window size is fixed to 250 days, the Fejer-Korovkin wavelet of order 6 (`fk6`) at 5 decomposition scales is used for the MODWT, and three trading horizons are compared, namely, $T_H \in \{5, 10, 15\}$ days to estimate VaR at the $c = 97.5\%$ confidence level (i.e., $\alpha = 0.025$), which is dictated by the Basel Committee on Banking Supervision (https://www.bis.org/publ/bcbs265.pdf). The prediction accuracy is evaluated in terms of the ratio of VaR violations, $R_V \in \mathbb{R}$, which is defined as follows,

$$R_V = \frac{\sum_{t=1}^{N} \mathbb{1}\left(r_t \leq -\mathrm{VaR}_t(\alpha)\right)}{N \cdot \alpha} \ , \tag{15}$$

where $\mathbb{1}(\cdot)$ denotes the indicator function. A risk measure is considered to be accurate if the value of R_V approximates 1.

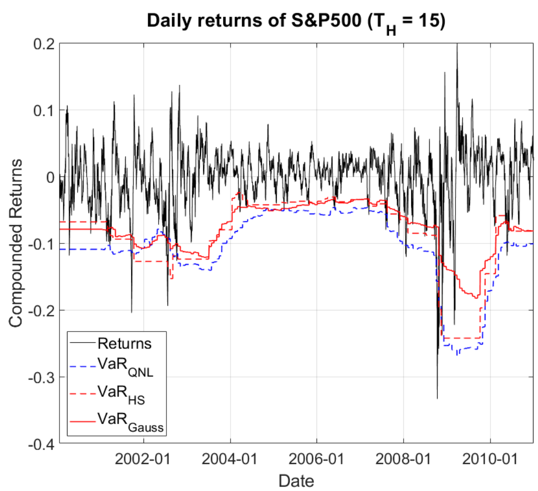
Table I summarizes the performance of the three VaR measures applied on the two market indexes. The historical method yields a slightly improved performance for a small trading horizon $T_H = 5$, while our QNL method is inferior to the other two. The reason is that for a small T_H, the cutoff level J_ρ given by Proposition 1 is very small, thus the approximation given by (9) is quite rough. However, as the trading horizon increases, which also makes risk measurement a highly challenging task, our proposed energy-based VaR measure outperforms significantly the conventional VaR measures for both indexes, since it exploits precisely the information that is relevant to the given trading horizon, while also achieving a better approximation to the original returns series $\mathbf{r}$.

TABLE I: Performance evaluation, in terms of R_V, of the three VaR measures applied on S&P500 and DAX ($T_H \in \{5, 10, 15\}$, $\alpha = 0.025$).

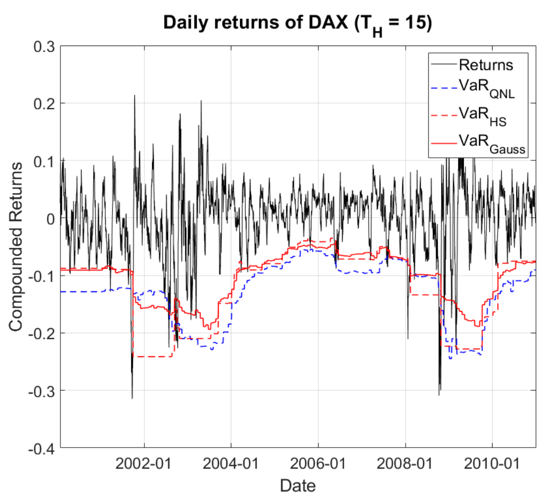
S&P500			
	R_V		
	$T_H = 5$	$T_H = 10$	$T_H = 15$
$\mathrm{VaR}_{\mathrm{QNL}}$	1.362	**0.813**	**0.756**
$\mathrm{VaR}_{\mathrm{HS}}$	**1.145**	1.626	1.803
$\mathrm{VaR}_{\mathrm{Gauss}}$	1.608	1.727	1.919
DAX			
	R_V		
	$T_H = 5$	$T_H = 10$	$T_H = 15$
$\mathrm{VaR}_{\mathrm{QNL}}$	1.389	**1.004**	**1.049**
$\mathrm{VaR}_{\mathrm{HS}}$	**1.217**	1.277	1.624
$\mathrm{VaR}_{\mathrm{Gauss}}$	1.346	1.736	1.868

Fig. 1(a) shows the returns series for S&P500, along with the estimated VaR using the three methods and for $T_H = 15$ and $\alpha = 0.025$. As it can be seen, our proposed VaR measure achieves a better tradeoff between the ratio of VaR violations (see Table I) and the proximity to the extreme losses, when compared against the other two well-established VaR estimation methods. This difference, which is more prominent as the trading horizon increases, can be attributed to the fact that a larger trading horizon incorporates an increased number of frequency components that are better captured by our energy-based VaR measure.

Similar results are obtained for the DAX index, as shown in Fig. 1(b). As in the case of S&P500, the performance of our proposed $\mathrm{VaR_{QNL}}$ measure improves against $\mathrm{VaR_{HS}}$ and $\mathrm{VaR_{Gauss}}$ for a longer trading horizon, in terms of the tradeoff between the ratio of VaR violations and the proximity to the extreme losses. As before, this behavior is attributed to the fact that our energy-based VaR employs only the relevant time resolutions for each trading horizon, in order to calculate the nonlinear combination of the individual α-level quantiles.



(a) S&P500



(b) DAX

Fig. 1: Daily returns of S&P500 and DAX for the period 1/1/2001–31/12/2010 and estimated VaR using $\mathrm{VaR_{QNL}}$, $\mathrm{VaR_{HS}}$ and $\mathrm{VaR_{Gauss}}$ for $T_H = 15$ and $\alpha = 0.025$.

V. Conclusions and Future Work

This paper introduced a novel VaR quantification method based on the energy distribution of returns in the MODWT domain. First, the optimal subset of decomposition scales was identified based on their relative energy contribution for a given trading horizon. Then, a nonlinear combination of the individual quantiles of the selected scales was calculated, yielding the overall energy-based VaR. The experimental evaluation on real stock indexes revealed the superior performance of our VaR measure, in terms of achieving a better ratio between the expected and observed number of VaR violations, when compared against the widely used historical and Gaussian-based VaR measures, especially for longer trading horizons.

In our current implementation, the MODWT is applied using a fixed wavelet filter. However, we expect that the accuracy in capturing more complex patterns could increase by designing an adaptive wavelet, which better fits the structure of a given returns series. Furthermore, we are interested in generalizing the rule of thumb for selecting the subset of scales that are relevant to a given trading horizon, instead of employing only consecutive scales, as we do now. Finally, we envision that the performance of our proposed QNL method could be enhanced by accounting for potential correlations among the distinct MRA series at different decomposition scales.

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