

Shuffling for understanding multifractality, application to asset price time series

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Abstract—Multifractal analysis has become a standard signal processing tool successfully used to model scale-free temporal dynamics in many applications, very different in nature. This is notably the case in financial engineering where, after Mandelbrot’s seminal contributions, multifractal models have been used since the late 90ies to describe temporal fluctuations in asset prices. However, what exact features of temporal dynamics are actually encoded in multifractal properties remains generally only partially understood. In finance, notably, multifractality is associated to the burstiness of the returns, yet its relation to trends (signs of the returns) or volatility (modulus of the returns) remains unclear. Comparing the estimated multifractal properties of well-controlled synthetic multifractal processes to those of surrogate data, obtained by applying random permutations (shuffling) either to signs, or to modulus, or to both, of increments of original data, permits to better understand what aspects of temporal dynamics are captured by multifractality. The same procedure applied to a large dataset of asset prices entering the composition of the Eurostoxx600 index permits to evidence a simple and solid relation between multifractality and volatility as well as a weaker and complicated relation to returns.

Index Terms—Multifractality, scale-free temporal dynamics, asset prices, finance, shuffling, surrogate.

I. INTRODUCTION

Context. Self-similarity has been massively used in many applications very different in nature to model scale-free temporal dynamics, that is, temporal dynamics that are not governed by any preferred time scale, in signals (cf. e.g., [1]). Self-similarity (or its asymptotic counterpart, long range dependence (LRD)) essentially models a slow power law decay of the covariance function at large lags [13], thus indicating that correlations amongst samples of data remain significant even for samples very far apart [1]. LRD is quantified by departures of the Hurst parameter H from $1/2$. Multifractality complements and enriches LRD by introducing versatility in the modeling of scale-free dynamics at all statistical orders. It accounts for potentially different scaling exponents, when self-similarity actually implies that scale-free dynamics at any finite statistical order are controlled uniquely by H . In finance,

it is well documented that asset prices show no temporal correlation ($H = 0.5$, the efficient market hypothesis (EMH)) but have some structured scale-free dependencies, sometimes quantified by the fact that volatility (squared returns) show LRD with $H > 0.5$, which prompted for the use of multifractality to model asset price fluctuations [6], [8].

However, what aspects of temporal dynamics are actually encoded by multifractality remains not very well understood. Despite being crucial for a better understanding of the underlying mechanisms governing temporal dynamics, this open question has rarely been studied theoretically and barely investigated on real-world data. For instance, in finance, multifractality has been related to the stage of development of financial markets and EMH [8], [16] or to the way volatility relaxes after endogenous or exogeneous shocks [4], [14] but it remains an open question to assess whether multifractality encodes the temporal fluctuations in return signs or in volatility.

Related work. At the theoretical level, it was proven that, for fractional Brownian motion, the reference process for modeling scale-free dynamics, LRD is associated with the fluctuations of the signs of its increments, while their modulus only shows short-term dependencies [13]. To the best of our knowledge, similar theoretical results do not exist for multifractal processes and the question remains barely studied empirically.

In a seminal contribution [12], Mandelbrot introduced multifractality in the modeling of asset price time series to account for burstiness, departures from Gaussianity, large size fluctuations and scale-free dynamics. This paved the way to numerous studies consistently reporting multifractal dynamics in financial time series [1], [3], [5], [10]. It renewed to some extent financial engineering, notably by suggesting to replace the Markov Switching Model used to describe the fluctuations of volatility, which is by construction based on the existence of a few typical time scales to be identified, by a multifractal switching model [9]. However, the understanding of what mechanisms in pricing are actually encoded in multifractality

remains an open question and an obvious preliminary step for actual use in financial engineering.

Goals, contributions and outline. Making use of *random permutation* (or *shuffling*) procedures, that consist in *destroying* a targeted property of data while *preserving others unaltered*, the present work aims to provide a better understanding of whether multifractality encodes the temporal dynamics of the signs or of the modulus in a multifractal time series, both at a general level and in the precise context of asset price analysis. To that end, Section II restates the key notions for theoretical and practical multifractal analysis, recalls the paradigm model of the fractional Brownian motion in multifractal time and defines the shuffling procedures used here. Section III reports Monte Carlo simulations comparing the multifractal properties estimated from a large number of independent copies of fractional Brownian motion in multifractal time to those estimated from surrogate versions of such data, obtained from the application of several different shuffling procedures. This permits to evidence relations between, on one hand, LRD and the temporal dynamics of signs, and on other hand, multifractality and the temporal dynamics of the modulus. Section IV compares the multifractal properties of a large dataset of financial times series consisting of daily prices for 244 assets contributing to the Eurostoxx600 index. Result shows that multifractality is strongly associated with the temporal dynamics of volatility.

II. MULTIFRACTAL ANALYSIS AND SHUFFLING

A. Multifractal analysis: Theory

Local regularity. Multifractal analysis aims at characterizing the fluctuations along time of the local regularity of a signal $X(t)$, cf., e.g., [17]. Local regularity can be quantified by pointwise exponents, the most commonly used being the Hölder exponent, $h(t) > 0$, defined as follows: X is said to be in $C^\alpha(t)$, $\alpha \geq 0$, if there exist a polynomial P_t with $\deg(P_t) < \alpha$ and a constant $C > 0$ such that: $|X(t+a) - P_t(t+a)| \leq C|a|^\alpha$ when $|a| \rightarrow 0$. The Hölder exponent consists of the largest such α : $h(t) \triangleq \sup\{\alpha : X \in C^\alpha(t)\} \geq 0$. In essence, the larger $h(t)$, the smoother X around t , and conversely, the closer $h(t)$ to 0, the more irregular X at t .

Multifractal spectrum. Despite being based on a local measure of regularity, multifractal theory does not, however, base analysis on the time evolution of the function $h(t)$ itself, but instead aims to provide a global and geometrical description of the temporal dynamics of X with the so-called multifractal spectrum $\mathcal{D}(h)$. It is defined as the collection of Hausdorff dimensions \dim_H of the sets of points $t \in \mathbb{R}$ where $h(t)$ takes the same value h [17]: $\mathcal{D}(h) \triangleq \dim_H(t : h(t) = h)$. The multifractal spectrum $\mathcal{D}(h)$ can thus be considered as an efficient summary for the temporal dynamics of X .

B. Multifractal analysis: formalism

The practical estimation procedure for $\mathcal{D}(h)$ from data is referred to as the *multifractal formalism*, and requires the use of *multiscale quantities*, that match the pointwise exponent chosen to quantify regularity. It is well documented that

measuring the Hölder exponent calls for the use of *wavelet leaders* defined as follows [17]. Let $d_X(j, k) = 2^{-j/2} \langle \psi_{j,k} | X \rangle$ denote the discrete wavelet transform (DWT) coefficients of X , with ψ the mother wavelet, an oscillating reference pattern, characterized by its number of vanishing moments N_ψ , a positive integer defined as $\psi \in C^{N_\psi-1}$ and $\forall n = 0, \dots, N_\psi - 1$, $\int_{\mathbb{R}} t^n \psi(t) dt \equiv 0$ and $\int_{\mathbb{R}} t^{N_\psi} \psi(t) dt \neq 0$. Further, ψ is chosen such that the collection of dilated and translated templates $\{\psi_{j,k}(t) = 2^{-j/2} \psi(2^{-j}t - k)\}_{(j,k) \in \mathbb{Z}^2}$ forms an orthonormal basis of $\mathcal{L}^2(\mathbb{R})$ [11].

Wavelet leaders are defined as local suprema of wavelet coefficients: $\ell_X(j, k) \triangleq \sup_{2^{j'} k' \in 3\lambda_{j,k}} |d_X(j', k')|$, with $\lambda_{j,k} = [k2^j, (k+1)2^j]$ the dyadic interval of size 2^j and $3\lambda_{j,k} \triangleq \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}$ the union of $\lambda_{j,k}$ with its 2 neighbors. It can be shown that wavelet leaders reproduce Hölder exponents in the limit of fine scales, $L_X(j, k) \sim C2^{jh(t)}$ as $2^j \rightarrow 0$ for $t = 2^j k$ and hence that [17]:

$$\frac{1}{n_j} \sum_{k=1}^{n_j} L_X(j, k)^q \sim K_q 2^{j\zeta(q)}, \quad 2^j \rightarrow 0. \quad (1)$$

Eq. 1 serves for the estimation of the so-called *scaling exponents* $\zeta(q)$ by linear regressions [17]. In turn, the function $\zeta(q)$ can be related to $\mathcal{D}(h)$ via a Legendre transform:

$$\mathcal{L}(h) = \inf_q (1 + qh - \zeta(q)) \quad (2)$$

which provides an upper bound estimate, $\mathcal{L}(h) \geq \mathcal{D}(h)$ for large classes of processes [17].

C. Multifractal analysis: Practice

Prior to multifractal analysis, it is common to consider the sum of the squared wavelet coefficients, closely related to the Fourier spectrum and covariance function of X [1], [2]. For processes with scale free and/or multifractal dynamics, this sum, $S_2(j)$, behaves, over a wide range of scales, as [1]:

$$S_2(j) = \frac{1}{n_j} \sum_{k=1}^{n_j} d_X^2(j, k) \sim F_2 2^{j2H}, \quad (3)$$

where H is often referred to as the *Hurst* or *self-similarity* or *long-memory* exponent of X , and can be related to the second-order multifractal scaling exponent as $2H = \zeta(2)$.

For most well-behaved multifractal processes, the scaling exponents can be approximated by truncating the polynomial expansion, $\zeta(q) = \sum_{p \geq 1} c_p q^p / q!$, whose coefficients c_p can be theoretically related to the cumulants of order p of $C_p(j)$ of the variables $\log_2 L_X(j, k)$ as [17]:

$$C_p(j) = c_p^0 + c_p j. \quad (4)$$

Function $C_1(j)$ conveys information mostly driven by the covariance function of the increments Y of process X and is hence very much related to the function $\log_2 S_2(j)$ [17]. Along the same line, $2H = \zeta(2)$ implies that $H = c_1 + c_2 + 2c_3/2 + 3c_4/4 + \dots$. For most multifractal processes, $\forall p \geq 2$, $c_1 \gg c_p$, and hence that $H \simeq c_1$. Conversely, functions $C_p(j)$ for $p \geq 2$ convey information on temporal dynamics

beyond second order statistics, i.e., not already encoded in the covariance function. When the $c_p \neq 0$, for some $p \geq 2$, such information is referred to as the multifractality of X . Moreover, Eq. 4 serves for the estimation of c_p by linear regressions [17]. In turns, it has been shown that $\mathcal{D}(h)$ can be efficiently approximated as [2]:

$$\mathcal{D}(h) = 1 + \frac{c_2}{2} \left(\frac{h - c_1}{c_2} \right)^2 - \frac{c_3}{6} \left(\frac{h - c_1}{c_2} \right)^3 + \dots \quad (5)$$

Descriptors $c_1, c_2, c_3, c_4, \dots$, corresponding respectively to location of the maximum, width, asymmetry, flatness, \dots , of $\mathcal{D}(h)$ thus summarize the temporal dynamics of X .

D. Scale-free Models

Fractional Brownian motion (fBm). Defined as the only Gaussian self-similar process with stationary increments, fBm, $B_H(t)$, has its entire statistics controlled by the unique parameter $0 < H < 1$ [13]. This thus implies that $\zeta(q) = qH$ and $D(h) = \delta(h - H)$.

Fractional Brownian motion in multifractal time (MF-fBm). Introduced by Mandelbrot for a better modeling of asset price fluctuations [12], MF-fBm is defined by composing an fBm B_{H_0} with a multifractal measure A , $\mathcal{B}_{H_0, A}(t) \triangleq B_{H_0}(A(t))$. For such processes, $\zeta(q)$, $D(h)$ and shape parameters $c_1, c_2, c_3, c_4, \dots$ depend jointly on H_0 and on parameters entering the definition of A [7], [17]. The Hurst parameter associated with $\mathcal{B}_{H_0, A}$ can be defined as $H = \zeta(2)/2$.

E. Shuffling procedures

To understand what aspects of temporal dynamics are actually encoded in multifractal properties, the estimated multifractal properties of original data X , summarized by H, c_1, c_2, c_3, c_4 , will be compared to those of *surrogate* copies, obtained by applying several different *shuffling* procedures to the increment process Y of X , $Y(t) = X(t+1) - X(t)$.

Procedure *Sign-Shuffling* consists in creating a surrogate process $Y^{(S)}$ by rewriting $Y(t) = \text{sgn}(Y(t)) \times |Y(t)|$ and randomly permuting (or *shuffling*) $\text{sgn}(Y(t))$ while preserving unaltered moduli: $Y^{(S)}(t) \triangleq \text{shuffle}(\text{sgn}(Y(t)) \times |Y(t)|)$.

Procedure *Modulus-Shuffling* consists in creating a surrogate process $Y^{(M)}$ by shuffling variables $|Y(t)|$ while preserving unaltered signs: $Y^{(M)}(t) \triangleq \text{sgn}(Y(t) \times \text{shuffle}(|Y(t)|))$.

Procedure *Full-Shuffling* consists in creating a surrogate process $Y^{(F)}$ by shuffling both signs and moduli: $Y^{(F)}(t) \triangleq \text{shuffle}(\text{sgn}(Y(t)) \times \text{shuffle}(|Y(t)|))$.

Surrogate processes $X^{(S)}, X^{(M)}$ and $X^{(F)}$ are constructed as the primitives of $Y^{(S)}, Y^{(M)}$ and $Y^{(F)}$, respectively.

III. SHUFFLING SYNTHETIC MULTIFRACTAL PROCESSES

A. Monte Carlo simulations set-up

The three shuffling procedures described above are applied to 1000 independent copies of MF-fBm, defined from fBm and Compound Poisson cascades, used as multifractal measures [7]. Long time series of 2^{18} samples are used, to obtain statistically relevant results hence permitting a discussion not confused by estimation performance issues. Multifractal

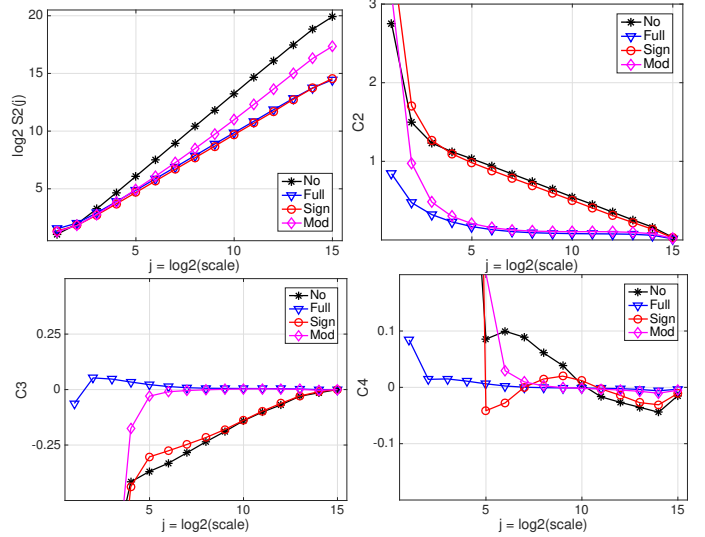


Fig. 1. **Self-similarity, multifractality and shuffling.** Functions $\log_2 S_2(j)$ (top left), $C_2(j)$ (top right), $C_3(j)$ (bottom left) and $C_4(j)$ (bottom right) estimated from MF-fBm compared to those obtained on shuffled surrogates. Original data X (black stars), Sign-Shuffled $X^{(S)}$ (red circles), Modulus-Shuffled $X^{(M)}$ (pink diamonds), Full-Shuffled $X^{(F)}$ (blue triangles). Averages over 1000 independent copies.

TABLE I
ESTIMATED MULTIFRACTAL PARAMETERS (AND CONFIDENCE INTERVALS) FOR MF-fBm COMPARED TO THOSE OBTAINED FROM SHUFFLED SURROGATES. AVERAGES OVER 1000 INDEPENDENT COPIES.

	H	c_2	c_3	c_4
Data	0.70 (0.01)	-0.140 (0.002)	0.054 (0.003)	-0.026 (0.006)
F	0.49 (0.01)	-0.007 (0.001)	-0.001 (0.002)	-0.001 (0.001)
S	0.50 (0.01)	-0.135 (0.002)	0.052 (0.003)	-0.013 (0.004)
M	0.66 (0.01)	-0.006 (0.001)	-0.000 (0.002)	-0.003 (0.001)

analysis is conducted using least asymmetric orthonormal wavelets with $N_\psi = 3$ vanishing moments. Other choices yield equivalent results. Synthesis and analysis procedures were devised by ourselves and will be made publicly available¹. Estimation results, obtained as averages across independent copies, are reported for one set of multifractal attributes (cf. Table I), equivalent results are yet obtained with other sets of multifractal attributes. Because function $C_1(j)$ conveys an information almost equivalent to that in $\log_2 S_2(j)$, it is not plotted in Fig. 1 (nor later in Figs. 2 and 3) for space reasons. Also, estimates of c_1 , close to those of H , are not reported in Table I (nor later in Table II).

B. Self-similarity, multifractality and shuffling

Shuffling and scale invariance. Fig. 1 shows that estimated functions $C_p(j)$ and $\log_2 S_2(j)$ for original data essentially consist of straight lines as functions of octave j , as expected from Eqs. 3-4 for exactly scale-free synthetic processes. Deviations from straight lines observed at the finest scales, even for exactly scale-free and multifractal processes, are due to a bias

¹<http://www.ens-lyon.fr/PHYSIQUE/Equipe3/MultiFracs/>

induced by wavelet leaders: By definition, the computation of wavelet leaders requires wavelet coefficients at finer scales, which are not available because of discretization. This has been documented elsewhere [17] and is no longer discussed here. Importantly, Fig. 1 shows that functions $\log_2 S_2(j)$ and $C_p(j)$ consist of straight lines as in Eqs. 3-4 also for shuffled data, thus indicating that *scale-free dynamics still exist in all shuffled time series*, and raising the question of whether LRD or multifractal properties were modified or not.

Shuffling, multifractality and LRD. Comparing now the functions $C_p(j)$, for $p \geq 2$, in Fig. 1, clearly shows that i) they are not altered by Sign-Shuffling, and that ii) they are systematically altered by Modulus-Shuffling and Full-Shuffling and transformed into functions that no longer depend on j . This is confirmed in Table I that shows that c_p , for $p \geq 2$ of $X^{(S)}$ are essentially similar to those of X , while those of $X^{(M)}$ and $X^{(F)}$ are close to 0. These empirical facts clearly show that *multifractality is destroyed by shuffling moduli* of the increments of X but not by shuffling signs.

Moreover, inspection of the functions $\log_2 S_2(j)$ in Fig. 1 shows that all forms of shuffling affect the second order statistics of the process (the correlation structure). However, a careful examination of the plots shows that in the limit of coarse scales $2^j \rightarrow +\infty$, the impact of Modulus-Shuffling vanishes. This is confirmed by Table I, reporting estimates obtained for coarse scales and showing that estimated H for $X^{(M)}$ are much closer to those of X than the estimates from $X^{(S)}$ which are closer to 0.5.

This leads to the following conclusions. First, Sign-Shuffling impacts LRD for multifractal processes, a result highly reminiscent of that proving that Sign-Shuffling impacts LRD for self-similar processes (cf. [13]). Second, Modulus-Shuffling impacts multifractality in an intricate way: the impacts concentrates mostly at fine scales and vanishes in the asymptotic limit of coarse scales.

IV. MULTIFRACTALITY IN FINANCE

A. Dataset and analysis

The dataset analyzed here consists of a collection of $p = 211$ daily asset price time series chosen within the basket defining the Eurostoxx600 index, recorded over 12 years, from December, the 14th, 2001 to January, the 24th, 2013, for a total sample size of $n = 2900$. This selection constitutes a relevant basket that could potentially be used in practice by investment firms. An example of such time series is provided in Fig. 2.

Multifractal analysis is applied to these time series and to the surrogate time series obtained by the shuffling procedures, with the same parameter setting as that described for Monte Carlo simulations in Section III-A above.

B. Results and discussion

For illustration and pedagogy, Fig. 2 displays, for an arbitrarily chosen asset, the price time series and compares the functions $\log_2 S_2(j)$ and $C_p(j)$ and Legendre spectra $\mathcal{L}(h)$ for the original data and shuffled surrogates. It shows that the functions $C_p(j)$ for $p \geq 2$ obtained after Sign-Shuffling

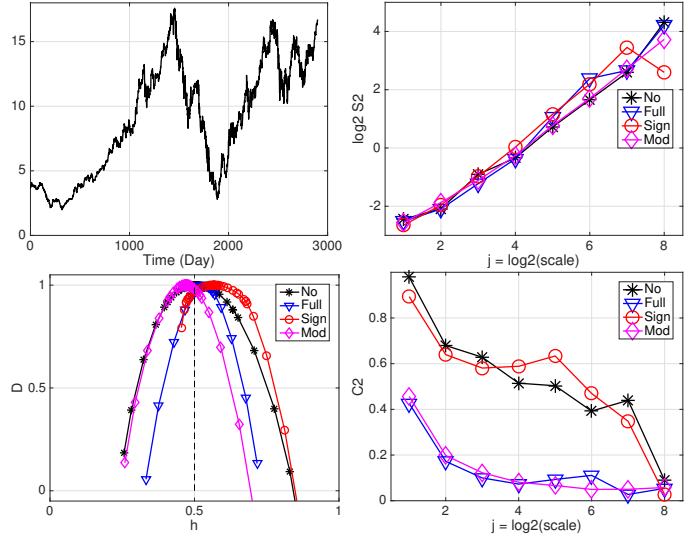


Fig. 2. **Self-similarity and multifractality on one asset price.** Time series (top left), $\log_2 S_2(j)$ (top right), estimated Legendre spectrum $\mathcal{L}(h)$ (bottom left), $C_2(j)$ (bottom right). Original data X (black stars), Sign-Shuffled $X^{(S)}$ (red circles), Modulus-Shuffled $X^{(S)}$ (pink diamonds), Full-Shuffled $X^{(F)}$ (blue triangles).

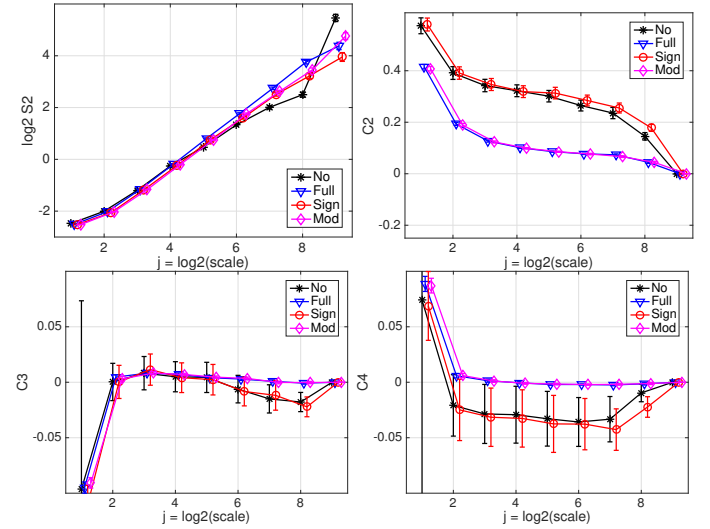


Fig. 3. **Self-similarity and multifractality in Eurostoxx basket.** Functions $\log_2 S_2(j)$ (top left), $C_2(j)$ (top right), $C_3(j)$ (bottom left) and $C_4(j)$ (bottom right) averaged over 211 Eurostoxx asset price time series and compared to those obtained as averages on shuffled surrogates. Original data X (black stars), Sign-Shuffled $X^{(S)}$ (red circles), Modulus-Shuffled $X^{(S)}$ (pink diamonds), Full-Shuffled $X^{(F)}$ (blue triangles).

very well match those of the original data, while the functions $C_p(j)$ for $p \geq 2$ obtained after Modulus-Shuffling are strongly modified and no longer depend on scales 2^j , when the biased short time scales are disregarded. On this example, one sees that Modulus-Shuffling suppresses multifractality while Sign-Shuffling does not. Fig. 2 also clearly shows that functions $\log_2 S_2(j)$ and $C_p(j)$ are not altered by any form of shuffling: this is because the original time series show no correlation structure, shuffling can hence not destroy it.

TABLE II
ESTIMATED MULTIFRACTAL PARAMETERS (AND CONFIDENCE INTERVALS)
FOR EUROSTOXX ASSET PRICE TIMES SERIES COMPARED TO THOSE
OBTAINED FROM SHUFFLED SURROGATES

	H	c_2	$100 \times c_3$	$10 \times c_4$
Data	0.47 (0.01)	-0.095 (0.008)	-0.26 (0.96)	0.298 (0.144)
F	0.46 (0.01)	-0.029 (0.002)	-0.33 (0.06)	-0.018 (0.007)
S	0.43 (0.01)	-0.092 (0.008)	-0.47 (0.82)	0.284 (0.141)
M	0.48 (0.01)	-0.029 (0.002)	-0.33 (0.06)	-0.018 (0.007)

To draw conclusion at the entire basket level, averages (and confidence intervals) for functions $\log_2 S_2(j)$ and $C_p(j)$ are computed over assets (after a suitable normalization of increment powers, an operation that does not alter time dynamics), as if they were independent realizations of one same financial process. This approach is common in *econophysics* and relies on the existence of universal behaviors in economic and financial systems [15]. Fig. 3 compares basket averaged functions $\log_2 S_2(j)$ and $C_p(j)$, $p = 2, 3, 4$ for the original data and for shuffled surrogates.

Fig. 3 clearly shows that functions $C_p(j)$ for $p \geq 2$ obtained after Sign-Shuffling almost perfectly superimpose with those of the original data. Conversely, it also shows that functions $C_p(j)$ for $p \geq 2$ obtained after Modulus-Shuffling are strongly altered and are close to constant with respect to scales 2^j , when the biased short time scales are disregarded. This is further confirmed in Table II that reports estimated multifractal parameters c_2 , c_3 and c_4 from the averaged $C_p(j)$, $p = 2, 3, 4$ for original and shuffled data. It can be seen that estimates of Sign-Shuffled surrogates and original data are in close agreements while estimates of Modulus-Shuffled surrogates strongly differ and are close to 0. Therefore, Sign-Shuffling leaves multifractality unaltered while Modulus-Shuffling quasi-totally destroys it, and, remarkably, at any statistical order.

Fig. 3 also shows that functions $\log_2 S_2(j)$ are essentially not altered by any form of shuffling. Further, Table II confirms that estimated H are essentially identical for original and shuffled data. In addition, these estimated H are found close to 0.5 thus confirming the absence of long memory and even the absence of any autocorrelation in asset price fluctuations, which can hence not be altered by shuffling. Incidentally, this also validates that shuffling per se does not create any spurious form of long memory (nor of multifractality).

Table II also shows that the asset price times series considered here display a significant multifractality (large c_2) that tends to involve only even cumulants ($c_3 \simeq 0$ while c_4 significantly departs from 0). Interestingly, this indicates that multifractality does not account for any asymmetry in such data, reinforcing the relation to modulus, hence to volatility.

V. CONCLUSIONS AND PERSPECTIVES

The present work explored the relations between multifractality and the time dynamics of signs and modulus in synthetic and financial time series. Using the state-of-the-art wavelet leader based multifractal formalism, and conducting Monte Carlo simulations over synthetic copies of reference multi-

fractal processes (fractional Brownian motion in multifractal time), and by comparing estimated multifractal properties of original and shuffled surrogates, we have established a clear and generic relation between multifractality and the temporal dynamics of the increment moduli. Conversely, we have shown that multifractality is not quantifying the temporal dynamics of increment signs. It has also been shown that the temporal dynamics of increment signs is coded in long memory, while the possible impact of the temporal dynamics of increment moduli on long memory concentrates on short time scales and asymptotically vanishes at coarse scales.

For asset price time series, the achieved results unambiguously indicate that multifractality is not related to the temporal dynamics of the returns but is clearly tied to the temporal dynamics of volatility. These findings are of interest for financial engineering at large, e.g., suggesting that multifractality can not be used for return prediction, but could be involved in volatility forecasting.

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