ESPRIT Angles-of-Arrival Estimation with Missing Sensor Data

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***Abstract*—This paper describes a procedure for Angles-of-Arrival (AoA) estimation for a uniform linear array (ULA) with missing sensors. The novelty of the approach is that, rather than using AoA estimates obtained from contiguous subarrays, we use estimates for the corresponding signal subspaces. The report shows that the ESPRIT invariance equations for each contiguous subarray define an operator that "propagates" the signal subspace beyond the physical array. Care needs to be taken to ensure the same bases are used for each subarray. The estimates of the signal subspaces for the missing array elements are appropriately combined to yield an estimate of the signal subspace for the complete ULA. The paper only addressed the case of one missing sensor, but the approach can be readily generalised. Simulations show that the proposed method yields AoA estimates which are very close to those obtained if there was no missing sensor, in contrast to the case where the measurements from the missing sensor were zero. In order to appropriately combine the estimates for the missing signal subspace terms, the report assessed the accuracy of signal subspace estimation as a function of the number N of ULA elements. Simulations indicate that, in the examples considered, the variance of these estimates decreases only as $N^{1/3}$ which is surprising given that the variance of AoA estimates (at least for one source) decrease as N^3. The paper suggests that further study of this empirical result is warranted.**
***Index Terms*—Sensor array signal processing, subspace methods.**

I. Introduction

Subspace based angles-of-arrival (AoA) estimation methods are now in common use in sensor array signal processing applications. In particular ESPRIT algorithms [1], [2] are of considerable interest to researchers and practitioners. ESPRIT algorithms exploit a particular kind of invariance property which arises due to particular geometric relationships between two selected sub-arrays. In standard ESPRIT (S-ESPRIT) there are two identical (non-overlapping) subarrays which have a fixed spatial offset. In Unitary ESPRIT (U-ESPRIT), the uniform structure of a ULA (or URA) can be exploited to obtain a similar invariance property to S-ESPRIT. In U-ESPRIT, overlapping sub-arrays are used. U-ESPRIT applies a specific transformation to the array data model which yields a *real valued* array manifold matrix. This has some computational benefits but its main application is in uniform rectangular arrays where correct pairing between elevation and azimuth AoA estimates is automatically achieved [2]. In this report, we utilise the same specific overlapping subarrays as for ULA U-ESPRIT, but we don't apply the U-ESPRIT data transformation.

In practical problems, "missing" sensors can arise. This may be due to sensor failure, or the sensor array may not have the complete uniform structure of a ULA. In either case, our approach is to try and estimate the signal subspace data corresponding to missing sensor(s) so that ULA U-ESPRIT can be applied. The approach presented here relies on there being at least one pair of ULA subarrays for which U-ESPRIT can be applied. An estimate of the "missing" components of the signal subspace is then obtained. After this processing step, U-ESPRIT can be applied to the complete array.

In the literature, there are two kinds of array interpolation problems. The first constructs the "virtual array" manifold matrix corresponding to a specified number of AoAs, usually spanning some sector of physical space. A fixed transformation matrix is applied to the actual array manifold response at the set of chosen AoAs, and is chosen to minimise the square error with respect to the virtual array responses (see e.g. [3], [4], [5], [6]). Although these approaches differ from that presented here, they are motivated by a similar idea – to transform an (irregular) array to a regular one (e.g. ULA) where specific algorithms (e.g. root-MUSIC or ESPRIT) that require this geometry can then be applied.

The second kind of approach uses some signal information to improve the interpolation. For example, [7] weights the approximation error between the actual and virtual array responses with the conventional beamformer (CBF) response, whilst [8] use the CBF response to identify AoA regions where source energy is localised. The idea is that the approximation is then made better over the range of AoAs where incident sources lie. A high resolution subspace method can then be applied to the virtual array (a ULA with root-MUSIC in the case of [7], ESPRIT in the case of [8]).

Perhaps the closest related work in the literature to our approach is that of [9] where, among other problems, the issue of interpolating a linear array to replace missing sensor data is accomplished using a combination of forwards and

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backwards (spatially across the array) linear predictors. The conventional beamformer is applied to the virtual array data. Two approaches are considered - one whereby the missing sensor data is estimated, the other where missing terms in the sample covariance matrix (due to missing sensors) are estimated. In some sense, the latter approach resembles more the approach presented here, which we now briefly describe.

Our approach is essentially based on estimating the missing components in the *signal subspace*. ESPRIT algorithms essentially determine a co-ordinate transformation matrix between the signal subspace components corresponding to two specified subarrays. Our main idea is that provided we estimate this transformation matrix through estimation of the array covariance matrix for all available sensors, we can use the transformation matrix to then estimate the missing signal subspace components due to missing sensors. Our method does not directly estimate AoAs from the incomplete sensor data, nor does it estimate the missing sensor data *per se* as in [9]. We are not aware of any similar approach, so claim some novelty in our idea.

It should also be noted that "ESPRIT-like" methods for AoA estimation when array invariance properties are not met (e.g. [10]), or where there are multiple invariances (e.g. [11]) have also been proposed. However these methods don't use array interpolation ideas.

Due to space limitations here, in this paper we don't conduct a detailed comparison with other methods for dealing with missing sensors, including the works cited above. However such a comparison is suggested as useful for ongoing work.

II. Sensor Array Signal Model

In this section, we revise the narrowband sensor array signal model for a uniform linear array (ULA) with uncorrelated (spatially uniform) sensor noise.

We consider a ULA with element spacing δ wavelengths located on the x axis with sensor n located at $x = n\delta\lambda$, for $n = 0, \ldots, N-1$. Consider a uniform narrowband plane wave incident on the array at angle θ with respect to the normal of the array. The steering vector associated with the incident signal has elements

$$a_n(\theta) = e^{2\pi i\, n\, \delta\, \sin(\theta)} \ , \qquad (1)$$

for $n = 0, \ldots, N-1$. Our baseband signal model assumes $M < N$ incident signals on the array with AoAs $\theta_m, m = 1, \ldots, M$. The resulting model for the signals at sensor n at sample instant t is

$$x_n(t) = \sum_{m=1}^{M} a_n(\theta_m)\, s_m(t) + v_n(t) \ .$$

Stacking the sensor samples $x_n(t)$ into a $N \times 1$ vector $x(t)$ and similarly for $v_n(t)$, and stacking the source signals $s_m(t)$ into a $M \times 1$ vector $s(t)$ we obtain

$$x(t) = A(\underline{\theta})\, s(t) + v(t) \ ,$$

where $A(\underline{\theta}) = A(\theta_1, \ldots, \theta_M)$ is a $N \times M$ matrix having $a(\theta_m)$ as its column m. We assume that the source signals are realisations of temporally uncorrelated zero-mean w.s.s. random processes with covariance $R_{\mathbf{s}}$, assumed full rank M. We assume that the sensor noise is a zero-mean, temporally uncorrelated w.s.s. process with (spatial) covariance $\sigma^2 I$. We can "stack" T snapshots $x(t_n), n = 1, \ldots, T$ sideways into a $N \times T$ matrix X and similarly for $s(t)$ and $v(t)$ giving the data model $X = A(\underline{\theta})\, S + V$. The spatial covariance data matrix is given by

$$R_{\mathbf{x}} = A(\underline{\theta})\, R_{\mathbf{s}}\, A(\underline{\theta})^H + \sigma^2 I \ .$$

Let $R_{\mathbf{x}} = \Phi\, \Lambda\, \Phi^H$ be the eigenvalue decomposition, where Φ is unitary and Λ is diagonal with strictly positive elements on its diagonal. The signal subspace is the space spanned by the eigenvectors (columns of Φ) corresponding to the M largest eigenvalues. If the eigenvalues are ordered in decreasing value, then we denote $E_s = [\phi_1 \ \cdots \ \phi_M]$. A key result is that the columns of E_s and those of $A(\underline{\theta})$ span the same linear subspace - the signal subspace.

III. Array Invariance and ULA U-ESPRIT

In this section, we briefly review ULA U-ESPRIT [2]. We define two sub-arrays, the first consisting of elements $0, \ldots, N-2$, and the second consisting of elements $1, \ldots, N-1$. Whilst the term "Unitary ESPRIT" refers to a (unitary) data pre-processing step resulting in real-valued subspaces, we'll retain the terminology here although we work directly with a complex version. The main difference here, compared to S-ESPRIT [1], is that elements are shared between subarrays. Define the two $(N-1) \times N$ subarray selection matrices

$$J_1 = \left[\ I_{N-1} \quad 0_{N-1}\ \right] \ , \quad J_2 = \left[\ 0_{N-1} \quad I_{N-1}\ \right] \ ,$$

where O_{N-1} is the $(N-1)$-dimensional (column) vector consisting of all zeros. From the form of the ULA steering vector (1),

$$J_2\, A(\underline{\theta}) = J_1\, A(\underline{\theta})\, D(\underline{\theta}) \ , \qquad (2)$$

where $D(\underline{\theta}) = \operatorname{diag}\left(e^{2\pi i\delta \sin(\theta_1)}, \ldots, e^{2\pi i\delta \sin(\theta_M)}\right)$. Under the full rank assumption on $R_{\mathbf{s}}$, the (M-dimensional) range spaces of the steering matrix $A(\underline{\theta})$ and the array data covariance matrix subspace E_s coincide, so there is a $M \times M$ non-singular matrix T such that $A(\underline{\theta}) = E_s T$ where the columns of E_s span the signal subspace. These columns are the eigenvectors of $R_{\mathbf{x}}$ corresponding to the M largest eigenvalues. Thus, in (2),

$$J_2\, E_s = J_1\, E_s\, \underbrace{\left(T\, D(\underline{\theta})\, T^{-1}\right)}_{\Psi} \ . \qquad (3)$$

As described in [14], when the signal subspace is not known exactly and is derived from the sample covariance estimate, this equation only holds approximately and needs to be solved using least-squares or total least-squares techniques resulting in an estimate $\hat{\Psi}$ of Ψ. It should be noted that the eigenvalues of Ψ are precisely the diagonal elements of $D(\underline{\theta})$ which contain the source signals' AoAs. So we can estimate the source signal AoAs by finding the eigenvalues of $\hat{\Psi}$. We can also think of the matrix Ψ as being an "operator" which "shifts" the signal subspace corresponding to sub-array one onto the co-ordinates of the signal subspace corresponding to sub-array two in an analogous manner to the matrix $D(\underline{\theta})$ which "shifts" the corresponding subspaces in the co-ordinates specified by the columns of $A(\underline{\theta})$.

IV. ULA ESPRIT WITH A MISSING SENSOR

This paper is concerned with the case where there is a single missing sensor in the array. The case where there are more than one missing sensor can be similarly considered by obvious generalisation. Suppose initially, that $N \geq 7$ and that the missing sensor, labelled $N_m \in \{3, \ldots, N-4\}$. We define two subarrays - the first, designated the *left subarray* consists of sensors $\{0, \ldots, N_m - 1\}$; the second, designated the *right subarray* consists of sensors $\{N_m + 1, \ldots, N-1\}$. Under the assumed conditions, both the left and right subarrays have a minimum of three elements and are each ULAs. Thus ULA U-ESPRIT can be applied independently to each of the left and right subarrays. Within each of the left and right subarrays, two subarrays will be constructed. For the left subarray, define subarray one to consist of elements $\{0, \ldots, N_m - 2\}$ and subarray two to consist of elements $\{1, \ldots, N_m - 1\}$. For the right subarray, define subarray three to consist of elements $\{N_m + 1, \ldots, N-2\}$ and subarray four to consist of elements $\{N_m + 2, \ldots, N-1\}$. Figure 1 shows the subarray labelling.

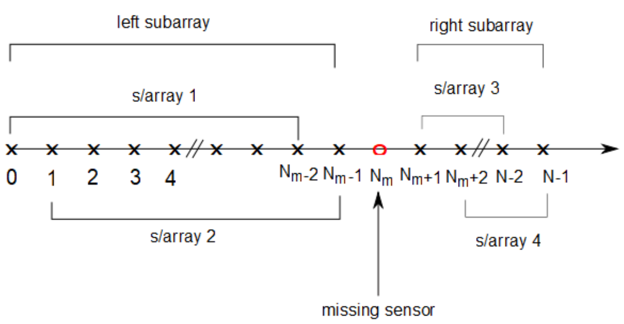


Fig. 1. ULA with a missing sensor - sub-array labelling.

Now denote the selection matrices for the left and right subarrays as $J_L \in \mathbb{R}^{N_m \times N}$ and $J_R \in \mathbb{R}^{(N-N_m+1)\times N}$ respectively, i.e.

$$J_L = \left[\begin{array}{cc} I_{N_m} & 0_{N_m, N-N_m} \end{array} \right] ,$$
$$J_R = \left[\begin{array}{cc} 0_{N-N_m-1, N_m+1} & I_{N-N_m-1} \end{array} \right] ,$$

where $0_{n,m}$ is the zero matrix of size $n \times m$. Denote by $x_L(t) = J_L\, x(t)$, and $x_R(t) = J_R\, x(t)$, the left and right subarray data for snapshot t respectively.

Now, the key idea here is that we need to compute the combined signal subspace for the left and right subarrays together. Firstly, we compute the sample covariance matrix for all available data,

$$\hat{R}_{LR} = \frac{1}{T} \sum_{t=1}^{T} \left[\begin{array}{c} x_L(t) \\ x_R(t) \end{array} \right] \left[\begin{array}{cc} x_L(t)^H & x_R(t)^H \end{array} \right] .$$

Now assuming that the number of signals $M < N-1$, then we can compute the corresponding signal subspace estimate $\hat{E}_{LR}$ as the $(N-1) \times M$ matrix having as its columns the eigenvectors of $\hat{R}_{LR}$ corresponding to the M largest eigenvalues. Then it will hold that there is a $M \times M$ matrix $\tilde{T}$ such that (approximately)

$$A(\underline{\theta}) = \left[\begin{array}{c} A_L(\underline{\theta}) \\ A_R(\underline{\theta}) \end{array} \right] = \hat{E}_{LR}\, \tilde{T} = \left[\begin{array}{c} \hat{E}_L \\ \hat{E}_R \end{array} \right] \tilde{T} .$$

where $A_L(\underline{\theta}) = J_L\, A(\underline{\theta})$, $A_R(\underline{\theta}) = J_R\, A(\underline{\theta})$, $\hat{E}_L = J_L\, \hat{E}_{LR}$, and $\hat{E}_R = J_R\, \hat{E}_{LR}$.

Define the subarray selection matrices K_1, K_2, K_3, K_4, as

$$K_1 = \left[\begin{array}{cc} I_{N_m-1} & 0_{N_m-1} \end{array} \right] ,$$
$$K_2 = \left[\begin{array}{cc} 0_{N_m-1} & I_{N_m-1} \end{array} \right] ,$$
$$K_3 = \left[\begin{array}{cc} I_{N-N_m-2} & 0_{N-N_m-2} \end{array} \right] ,$$
$$K_4 = \left[\begin{array}{cc} 0_{N-N_m-2} & I_{N-N_m-2} \end{array} \right] ,$$

where $K_1, K_2 \in \mathbb{R}^{(N_m-1)\times N_m}$ and $K_3, K_4 \in \mathbb{R}^{(N-N_m-2)\times(N-N_m-1)}$. From sec. III, we then have the invariance equations (regarded as approximations)

$$K_2\, \hat{E}_L = K_1\, \hat{E}_L\, \hat{\Psi}_{1,2}$$
$$K_3\, \hat{E}_R = K_4\, \hat{E}_R\, \hat{\Psi}_{4,3} , \qquad (4)$$

where $\hat{\Psi}_{1,2}, \hat{\Psi}_{4,3} \in \mathbb{C}^{M \times M}$. We regard $\hat{\Psi}_{1,2}$ as an operator in the signal subspace co-ordinates which "shifts" to the right one sensor. Similarly, We regard $\hat{\Psi}_{4,3}$ as an operator in the signal subspace co-ordinates which "shifts" to the left one sensor. We'll use these matrices to construct an estimate for the signal subspace terms corresponding to the missing sensor. It's worth observing that in the absence of noise, (thus dropping the circumflex), $\Psi_{1,2} = \tilde{T}\, D(\underline{\theta})\, \tilde{T}^{-1} = \Psi_{4,3}^{-1}$.

Let e_m^T denote the $1 \times M$ row vector forming the components of the signal subspace corresponding to the missing sensor, then we define an estimate $\hat{e}_m^T$ by (with obvious abuse of notation),

$$\hat{e}_m^T = \frac{\hat{E}_L(N_m - 1, :)\, \hat{\Psi}_{1,2} + \hat{E}_R(0, :)\, \hat{\Psi}_{4,3}}{2} . \qquad (5)$$

Thus we construct the estimate for the signal subspace terms for the missing sensor by propagating the values for the sensors either side using the shift matrices and averaging. It's possible to do this because the construction of the estimated signal subspace $\hat{E}_{LR}$ means that the co-ordinates for the signal subspaces for the left and right subarrays are the same. Thus we can construct

$$\hat{E}_s = \begin{bmatrix} \hat{E}_L \\ \hat{e}_m^T \\ \hat{E}_R \end{bmatrix} ,$$

and apply ULA ESPRIT as outlined in sec. III.

V. Numerical Simulations

In order to assess the accuracy of the interpolation method described in sec. IV, we simulated a scenario with two incident source signals at angles $\theta_1 = -20°, \theta_1 = 30°$ with equal SNR. All simulations averaged over 10,000 realisations. In each case, the number of data snapshots per batch is $T = 100$. In the simulation we chose a ULA with $N = 11$ elements. The missing sensor was $N_m = 5$ giving a left subarray with 5 elements and right subarray with 5 elements. Figure 2 shows the estimated RMS error in AoA estimation (errors for each source are averaged) for four different ESPRIT algorithms : (i) the complete subarray (i.e. no missing sensor - this provides the benchmark), (ii) the full array but setting the data for the missing sensor to zero, (iii) the average of the left subarray alone and the right subarray alone, and (iv) using the interpolation method described in sec. IV.

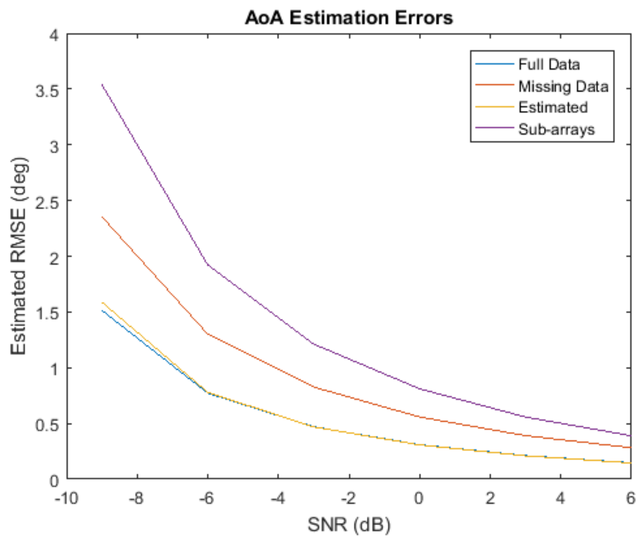
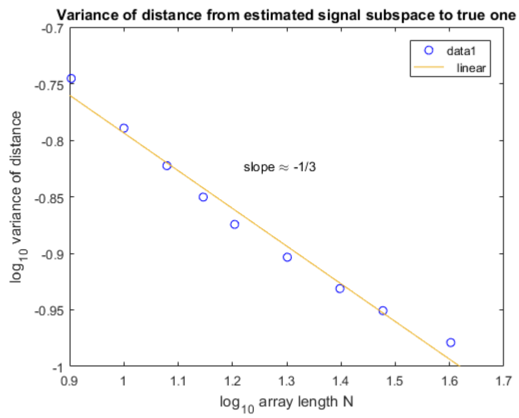


Fig. 2. Shows the AoA estimation accuracy (RMSE) for ESPRIT (i) applied to the full array data, (ii) using the described interpolation method to estimate full signal subspace, (iii) setting the missing sensor data to zero, and (iv) the average performance using the two separate subarrays (left and right). Here the missing sensor was sensor $N_m = 5$ and the ULA has $N = 11$ elements.

Discussion

We observed that the AoA estimation errors are roughly the same for each subarray as would be expected because each sub-array has the same number of elements in this example. In this case, when we combine the two estimated signal subspace terms for the missing sensor (one from each subarray) with equal weighting as in eqn. (5). If the sub-arrays have different numbers of elements we should take into account the different variances in these signal subspace estimates for each sub-array when combining them. So rather than a simple arithmetic average of the two subspace estimates as in (5), we should weight them with respect to their variances. In order to introduce this refinement to the algorithm, we need some idea as to how the variance of the signal subspace estimate for a ULA augmented with a single additional element, varies with the number of array elements. There's nothing obvious about this in the literature and may be a matter for future study. However, we can use simulations to obtain an empirical estimate of this relationship. The task is made more difficult in that the signal subspace estimates will generally be specified by different bases (i.e. eigenvectors). As a measure of the accuracy of the signal subspace estimates, we use the projective distance between the estimated subspace and the true one (the span of the columns of A). Figure 3 shows the behaviour of this distance for the estimated signal subspace for a N element ULA with one additional "virtual" element appended to the array. The same signal scenario as above is used with SNR fixed at -9 dB for each signal. The linear fit to the data (on a log-log axis scale) gives a slope of approximately -1/3. So an empirical relationship might be conjectured as variance $\propto N^{-1/3}$.



(a)

Fig. 3. The estimated variance of the distance $d(\hat{E}_s, A)$ between the true signal subspace (columns of A) and its estimate $\hat{E}_s$ for a one-element augmented ULA of N elements. Here there are two incident signals at $\theta = -20°, 30°$ with common SNR = -9 dB. There are $T = 100$ snapshots.

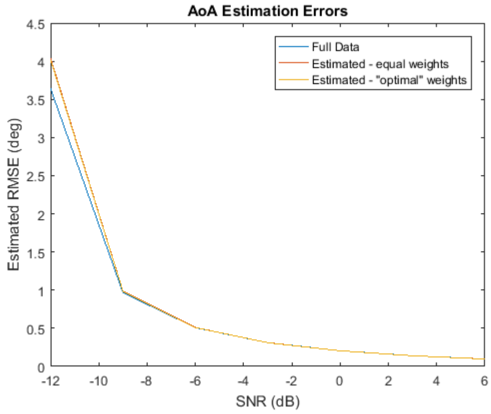
If we have a convex combination $Y = \alpha X_1 + (1 - \alpha) X_2$ of random variables X_1 and X_2, the value of parameter α which minimises the variance of Y is given by

$$\alpha = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} ,$$

where σ_1^2, σ_2^2 are the variances of X_1 and X_2 respectively. So for the problem at hand, where we have two estimates (eqn. (5)) $\hat{E}_L(N_m - 1, :)$ determined from the left subarray having N_m elements, and $\hat{E}_R(0, :)$ determined from the right subarray having $N - N_m - 1$ elements. Accordingly, we should construct the estimate for the missing element signal subspace according to

$$\hat{e}_m^T = \frac{(N - N_m - 1)^{-1/3}\, \hat{E}_L(N_m - 1, :) + N_m^{-1/3}\, \hat{E}_R(0, :)}{N_m^{-1/3} + (N - N_m - 1)^{-1/3}}. \quad (6)$$

We assessed the performance of the weighting scheme in (6) with the $N = 16$ element array and subarrays of size 5 and 10. Thus the weighting terms in (6) are 0.44 and 0.56, quite close to the equal weighting case. Indeed, as shown on figure 4, there is virtually no difference observed between the two methods. The weights would be expected to be significantly different if there was a large difference in the number of elements in each subarray, but in this case, the performance gain in using the interpolation scheme proposed in this paper would be expected to be small compared to just using the larger of the subarrays.



(a)

Fig. 4. Estimated MSE for AoA estimation using (i) full array, (ii) interpolation with equal weight (eqn. (5)), and (iii) interpolation with wights defined in eqn. (6) as SNR is varied. Here there are two incident signals at $\theta = -20^\circ, 30^\circ$ with equal SNR. There are $T = 100$ snapshots.

VI. Conclusion and Ongoing Work

We have considered the problem of AoA estimation for a ULA with missing sensors. In particular, we have considered the case where there is a single missing sensor, although the method is easily generalised. The idea is to use ULA ESPRIT to estimate the signal subspaces for the array without including the missing sensor. We then solve the ESPRIT invariance equations for each contiguous subarray (a ULA) and then use these solutions to "propagate" the signal subspaces to the missing sensor. These subspace estimates are combined and then an estimate for the signal subspace for the complete ULA is obtained. We can then apply ULA ESPRIT for the complete array to obtain AoA estimates.

Numerical simulations are used to validate the approach. In the example considered, AoA estimation accuracy is very close to the case where there is no missing sensor data whilst using both subarrays together, or ignoring the missing sensor data yields significantly worse estimates. The issue of combining the signal subspace estimates obtained from subarrays having different numbers of sensors was addressed. Optimally combining the estimates requires knowledge of how the variance of these estimates behaves as the number of subarray elements varies. For the example considered, simulations suggest that this variance decreases as $N^{1/3}$ (for N elements) which is somewhat surprising given that, at least for the one source signal case, we know the variance of the AoA estimate decreases as N^3 [13]. This result is supported by anecdotal "evidence" that good AoA estimates can still be obtained even if the subspace approximation is relatively poor.

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