Waveform Optimization for FDA Radar

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***Abstract*—This paper addresses the problem of transmit signal design for target localization in a frequency diverse array (FDA) radar. For this purpose, we derive the Cramér-Rao bound (CRB) for target localization in FDA radar. The derived CRB is optimized with respect to the transmit signal parameters. It is commonly assumed that in radar systems the direction-of-arrival (DOA) estimation accuracy and resolution are determined by the transmit-receive array aperture. In FDA radar, a coupling between range and DOA estimation is generated. Using the derived CRB, we show that in FDA radar one is able to improve the DOA estimation accuracy and resolution by increasing the transmit signal bandwidth. The target localization performance is analyzed theoretically and via simulations, and it is shown that using the proposed approach for transmit signal optimization, results in superior target localization performance compared to conventional methods.**

***Index Terms*—Frequency diverse array (FDA), Cramér-Rao bound (CRB), waveform optimization, MIMO radar**

I. Introduction

Waveform or beamforming optimization for active radar is an emerging topic in array signal processing. In order to improve the localization or detection performance, one may optimize a criterion such as statistical lower bounds on localization performance, output signal-to-noise ratio (SNR), probability of detection, or information theoretic measures, with respect to (w.r.t.) the transmit waveform or the transmit signal parameters.

Frequency diverse array (FDA) radar, proposed in [1], is a new technology in array signal processing that attracts a lot of attention [2]. In contrast to a standard phased-array radar, which transmits scaled versions of a single waveform, the FDA radar can be interpreted as a special case of general multiple-input multiple-output (MIMO) radar [3], [4], where the multiple transmit signals obey to a small carrier frequency increment across the transmit array elements. This configuration offered by the FDA radar, allows superior capabilities compared to conventional methods, e.g., [5]–[7]. In addition, Chen *et al.* [8] restudy the works of time-invariant beampattern design by FDA radar, which can focus the transmit energy to a desired position. The conclusion was that it is impossible to design such a beampattern, since the propagation process of the transmit signals should be considered.

Although the general MIMO radar with optimized transmit signal can theoretically achieve better performance than FDA radar, waveform optimization for target localization in MIMO radar is complicated due to its many degrees of freedom. In [9], optimal beamform design for MIMO radar was derived according to minimization of the Cramér-Rao bound (CRB) for target direction-of-arrival (DOA) estimation. However, this method does not take into account the lack of knowledge of the target range. In [10], the frequency increments in FDA radar were optimized w.r.t. the ambiguity function using the simulated annealing algorithm. In addition, several works on waveform optimization for target localization and tracking in cognitive FDA radar have been carried out, (see e.g., [11], [12]). In [11], a cognitive target tracking scheme via angle range-Doppler estimation with transmit subaperturing FDA radar, was proposed. For improved tracking performance, the transmit weight matrix is adaptively designed at each step based on SNR or Bayesian CRB criterion using historical observations. However, most of the research works on FDA radar focus on its contribution in cases where the target range information from the propagation delay is neglected. Furthermore, many works consider the FDA-MIMO radar, that is, FDA radar where the transmit signals are orthogonal.

In this paper, we consider the problem of FDA radar waveform optimization for target localization using the CRB. First, the standard FDA radar model is generalized by considering the propagation delay information, and by allowing general frequency increment value, which does not restrict the transmit signals to be neither orthogonal or to create the decoupling property [5]. We then optimize the FDA transmit signal parameters for target localization based on the CRB, derived for the above-mentioned general model. The main advantage of the proposed method is that it allows superior localization performance compared to conventional methods, under both scenarios: small and large transmit signal bandwidth. In order to demonstrate the benefit of the proposed method, numerical examples are provided.

II. Problem Formulation and CRB

A. Signal and Data Model

Consider the standard FDA radar, also known as linear FDA (LFDA) radar, that is, a mono-static radar consisting of two co-located uniform linear arrays (ULAs) of M_R receivers and M_T transmitters, where d is the space between adjacent elements, and the arrays center is chosen as the reference point. The transmit bandpass signal by the mth transmitter element is $w_m s_0(t)e^{j2\pi f_m t}$, where $s_0(t)$ is the basic waveform which is complex baseband signal with pulse width T_p, bandwidth B, and unit energy, i.e. $\int_{T_p} |s_0(t)|^2 dt = 1$. In addition, $f_m = f_c+(m-1)\Delta f$, where f_c and Δf are the carrier frequency and the frequency increment, respectively. We assume that $|\Delta f| \cdot$

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$(M_T - 1) \ll f_c$ and consider interference and clutter free environment. Similarly to [4], after down conversion of the received signal and in the presence of L_T stationary point targets located in the far-field, the FDA radar data model at time t can be written as

$$\mathbf{y}(t) = \sum_{l=1}^{L_T} \alpha_l \mathbf{A}(\theta_l)\mathbf{s}(t - 2r_l/c) + \mathbf{v}(t),\ t \in [0, T_o] \quad (1)$$

where T_o is the observation time, $\mathbf{A}(\theta) = \mathbf{a}_R(\theta)\mathbf{a}_T^T(\theta)$, and $\mathbf{a}_R(\cdot)$ and $\mathbf{a}_T(\cdot)$ are the receive and transmit steering vectors, respectively. The weighted FDA transmit signal vector is given by

$$\mathbf{s}(t) = \left[s_1(t), \ldots, s_{M_T}(t)\right]^T = \mathbf{D_w}\tilde{\mathbf{s}}(t) \quad (2)$$

where $\tilde{\mathbf{s}}(t) = s_0(t) \cdot \left[1, e^{j2\pi\Delta f t}, \ldots, e^{j2\pi\Delta f(M_T-1)t}\right]^T$ is the FDA signal vector, $\mathbf{w} = [w_1, \ldots, w_{M_T}]^T \in \mathbb{C}^{M_T}$ is the weight vector, and $\mathbf{D_w} \triangleq \text{diag}(\mathbf{w})$ is diagonal matrix whose diagonal elements are given by the elements of $\mathbf{w}$. In addition, $\mathbf{v}(t)$ is additive circularly symmetric complex Gaussian noise, which is assumed to be temporally and spatially white, i.e. $E\left[\mathbf{v}(t)\mathbf{v}^H(t-\tilde{\tau})\right] = \sigma_v^2 \mathbf{I}_{M_R}\delta(\tilde{\tau})$, where $\delta(\cdot)$ is the Dirac delta function, and $\mathbf{I}_{M_R}$ is the identity matrix of size M_R. The unknown deterministic parameters are $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_{L_T}]^T$, $\mathbf{r} = [r_1, \ldots, r_{L_T}]^T$, $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_{L_T}]^T$, where α_l, r_l, and θ_l are the complex amplitude, range, and DOA of the *l*th target, respectively.

B. Cramér-Rao Bound for Target Localization

In this subsection, the CRB [13] for target localization in FDA radar is derived. The CRB is used as a criterion for waveform optimization, as well as FDA performance analysis. In particular, it allows to demonstrate that one is able to improve the DOA estimation accuracy by increasing the transmit signal bandwidth. In [4], the Fisher information matrix (FIM) for the general MIMO radar model is derived. As stated above, the FDA is a particular case of MIMO radar, where the transmit signals are in the form of (2), thus we use expressions from [4] in order to derive the CRB for FDA radar. The FIM for estimating $\boldsymbol{\Psi} = [\tilde{\boldsymbol{\alpha}}^T, \boldsymbol{\Phi}^T]^T$ from the data model in (1), under single target ($L_T = 1$) scenario, can be partitioned as

$$\mathbf{J}_{\boldsymbol{\Psi}} = \begin{bmatrix} \mathbf{J}_{\tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\alpha}}} & \mathbf{J}_{\boldsymbol{\Phi}\tilde{\boldsymbol{\alpha}}}^T \\ \mathbf{J}_{\boldsymbol{\Phi}\tilde{\boldsymbol{\alpha}}} & \mathbf{J}_{\boldsymbol{\Phi}\boldsymbol{\Phi}} \end{bmatrix} \quad (3)$$

where $\tilde{\boldsymbol{\alpha}} = [\text{Re}(\alpha), \text{Im}(\alpha)]^T$, and $\boldsymbol{\Phi} = [r, \theta]^T$. In addition, by assuming large enough observation time such that $2r/c + T_p < T_o$, the FIM (3) elements are given by

$$\mathbf{J}_{\tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\alpha}}} = \frac{2M_R}{\sigma_v^2}\mathbf{a}_T^T(\theta)\mathbf{R_s}\mathbf{a}_T^*(\theta)\,\mathbf{I}_2 \quad (4)$$

$$J_{rr} = \left(\frac{2}{c}\right)^2 \cdot \frac{2|\alpha|^2 M_R}{\sigma_v^2}\mathbf{a}_T^T(\theta)\mathbf{R}_{\dot{\mathbf{s}}\dot{\mathbf{s}}}\mathbf{a}_T^*(\theta) \quad (5)$$

$$J_{\theta\theta} = \frac{2|\alpha|^2}{\sigma_v^2}\text{tr}\left(\dot{\mathbf{A}}(\theta)\mathbf{R_s}\dot{\mathbf{A}}^H(\theta)\right) \quad (6)$$

$$J_{\theta r} = -\left(\frac{2}{c}\right)\frac{2|\alpha|^2 M_R}{\sigma_v^2}\text{Re}\left\{\mathbf{a}_T^T(\theta)\mathbf{R}_{\dot{\mathbf{s}}\mathbf{s}}\dot{\mathbf{a}}_T^*(\theta)\right\} \quad (7)$$

$$J_{r\tilde{\boldsymbol{\alpha}}} = -\left(\frac{2}{c}\right)\frac{2M_R}{\sigma_v^2}\text{Re}\left\{\alpha^*\mathbf{a}_T^T(\theta)\mathbf{R}_{\dot{\mathbf{s}}\mathbf{s}}^H\mathbf{a}_T^*(\theta)[1, j]\right\} \quad (8)$$

$$J_{\theta\tilde{\boldsymbol{\alpha}}} = \frac{2M_R}{\sigma_v^2}\text{Re}\left\{\alpha^*\mathbf{a}_T^T(\theta)\mathbf{R_s}\dot{\mathbf{a}}_T^*(\theta)[1, j]\right\} \quad (9)$$

where the FDA matrices $\mathbf{R_s}$, $\mathbf{R}_{\dot{\mathbf{s}}\mathbf{s}}$, and $\mathbf{R}_{\dot{\mathbf{s}}\dot{\mathbf{s}}}$ are defined as

$$\mathbf{R_s} \triangleq \int_{T_p}\mathbf{s}(t)\mathbf{s}^H(t)dt = \mathbf{D_w}\mathbf{F_{ss}}\,\mathbf{D_w}^H = \mathbf{w}\mathbf{w}^H \odot \mathbf{F_{ss}} \quad (10)$$

$$\mathbf{R}_{\dot{\mathbf{s}}\mathbf{s}} \triangleq \int_{T_p}\dot{\mathbf{s}}(t)\mathbf{s}^H(t)dt = \mathbf{D_w}\mathbf{F}_{\dot{\mathbf{s}}\mathbf{s}}\,\mathbf{D_w}^H = \mathbf{w}\mathbf{w}^H \odot \mathbf{F}_{\dot{\mathbf{s}}\mathbf{s}} \quad (11)$$

$$\mathbf{R}_{\dot{\mathbf{s}}\dot{\mathbf{s}}} \triangleq \int_{T_p}\dot{\mathbf{s}}(t)\dot{\mathbf{s}}^H(t)dt = \mathbf{D_w}\mathbf{F}_{\dot{\mathbf{s}}\dot{\mathbf{s}}}\,\mathbf{D_w}^H = \mathbf{w}\mathbf{w}^H \odot \mathbf{F}_{\dot{\mathbf{s}}\dot{\mathbf{s}}} \quad (12)$$

where $\mathbf{F_{ab}} \triangleq \int_{T_p}\tilde{\mathbf{a}}(t)\tilde{\mathbf{b}}^H(t)dt$, $\dot{\mathbf{s}}(t) = \frac{d\mathbf{s}(t)}{dt}$, and $\odot$ denotes the Hadamard product. Therefore, one can notice that the FIM for a single stationary target does not depend on the target range and that the FIM depends on the transmit signal (2) via the FDA matrices $\mathbf{R_s}$, $\mathbf{R}_{\dot{\mathbf{s}}\mathbf{s}}$, and $\mathbf{R}_{\dot{\mathbf{s}}\dot{\mathbf{s}}}$. Note that for cases where $|s_0(t)|^2 = 1/T_p$, $\forall t \in [0, T_p]$, which is satisfied for phase and frequency coding, the (k, q)th element of $\mathbf{F_{ss}}$ is given by

$$[\mathbf{F_{ss}}]_{k,q} = e^{j\pi T_p\Delta f(k-q)}\text{sinc}(\pi T_p\Delta f(k-q)) \quad (13)$$

where $k, q \in \{1, \ldots, M_T\}$, and $\text{sinc}(x) \triangleq \sin(x)/x$.

Note that by substitution of (4)-(12) into (3), the FIM can be rewritten as

$$\mathbf{J}_{\boldsymbol{\Psi}} = \frac{2M_R}{\sigma_v^2}\text{Re}\left\{\mathbf{Q_w}^H(\tilde{\mathbf{H}} \odot \tilde{\mathbf{F}})\mathbf{Q_w}\right\} \quad (14)$$

where $\mathbf{Q_w} \triangleq \mathbf{I}_4 \otimes \mathbf{w}$, and $\otimes$ denotes the Kronecker product. In addition, the matrices $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{F}}$ are given by

$$\tilde{\mathbf{H}} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0}_{M_T} & \alpha^*\mathbf{A}_1 & \alpha^*\mathbf{A}_2 \\ \mathbf{0}_{M_T} & \mathbf{A}_1 & j\alpha^*\mathbf{A}_1 & j\alpha^*\mathbf{A}_2 \\ \alpha^*\mathbf{A}_1 & j\alpha^*\mathbf{A}_1 & |\alpha|^2\mathbf{A}_1 & -|\alpha|^2\mathbf{A}_2 \\ \alpha^*\mathbf{A}_2 & j\alpha^*\mathbf{A}_2 & -|\alpha|^2\mathbf{A}_2 & |\alpha|^2\mathbf{A}_3 \end{bmatrix} \quad (15)$$

$$\tilde{\mathbf{F}} = \begin{bmatrix} \mathbf{F}_{\mathbf{ss}}^T & \mathbf{F}_{\mathbf{ss}}^T & \mathbf{F}_{\dot{\mathbf{s}}\mathbf{s}}^* & \mathbf{F}_{\mathbf{ss}}^T \\ \mathbf{F}_{\mathbf{ss}}^T & \mathbf{F}_{\mathbf{ss}}^T & \mathbf{F}_{\dot{\mathbf{s}}\mathbf{s}}^* & \mathbf{F}_{\mathbf{ss}}^T \\ \mathbf{F}_{\dot{\mathbf{s}}\mathbf{s}}^* & \mathbf{F}_{\dot{\mathbf{s}}\mathbf{s}}^* & \mathbf{F}_{\dot{\mathbf{s}}\dot{\mathbf{s}}}^T & \mathbf{F}_{\dot{\mathbf{s}}\mathbf{s}}^T \\ \mathbf{F}_{\mathbf{ss}}^T & \mathbf{F}_{\mathbf{ss}}^T & \mathbf{F}_{\dot{\mathbf{s}}\mathbf{s}}^T & \mathbf{F}_{\mathbf{ss}}^T \end{bmatrix} \quad (16)$$

where $\mathbf{A}_1 = \mathbf{a}_T^*(\theta)\mathbf{a}_T^T(\theta)$, $\mathbf{A}_2 = \dot{\mathbf{a}}_T^*(\theta)\mathbf{a}_T^T(\theta)$, $\mathbf{A}_3 = \dot{\mathbf{a}}_T^*(\theta)\dot{\mathbf{a}}_T^T(\theta) + \left(||\dot{\mathbf{a}}_R(\theta)||^2/M_R\right)\mathbf{A}_1$, and $\mathbf{0}_{M_T}$ is a square matrix of size M_T whose elements are equal to zero. Therefore, one can notice that the FIM depends on Δf and $\mathbf{w}$ via the matrices $\tilde{\mathbf{F}}$ and $\mathbf{Q_w}$, respectively.

Finally, the CRB on target range and DOA estimation can be expressed as

$$CRB(\boldsymbol{\Phi}) = \left[\mathbf{J}_{\boldsymbol{\Phi\Phi}} - \mathbf{J}_{\boldsymbol{\Phi}\tilde{\boldsymbol{\alpha}}}\mathbf{J}_{\tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\alpha}}}^{-1}\mathbf{J}_{\boldsymbol{\Phi}\tilde{\boldsymbol{\alpha}}}^T\right]^{-1} = \frac{\sigma_v^2}{2|\alpha|^2 M_R(d_1 d_3 - d_2^2)}\begin{bmatrix} (\frac{c}{2})^2 d_1 & (\frac{c}{2})d_2 \\ (\frac{c}{2})d_2 & d_3 \end{bmatrix} \quad (17)$$

where

$$d_1 = \dot{\mathbf{a}}_T^T(\theta)\mathbf{R}_\mathbf{s}\dot{\mathbf{a}}_T^*(\theta) + \frac{||\dot{\mathbf{a}}_R(\theta)||^2}{M_R}\mathbf{a}_T^T(\theta)\mathbf{R}_\mathbf{s}\mathbf{a}_T^*(\theta) - \frac{\left|\mathbf{a}_T^T(\theta)\mathbf{R}_\mathbf{s}\dot{\mathbf{a}}_T^*(\theta)\right|^2}{\mathbf{a}_T^T(\theta)\mathbf{R}_\mathbf{s}\mathbf{a}_T^*(\theta)} \tag{18}$$

$$d_2 = \mathrm{Re}\left\{\mathbf{a}_T^T(\theta)\mathbf{R}_{\dot{\mathbf{s}}\mathbf{s}}\dot{\mathbf{a}}_T^*(\theta) - \frac{\mathbf{a}_T^T(\theta)\mathbf{R}_{\dot{\mathbf{s}}\mathbf{s}}^H\mathbf{a}_T^*(\theta)\left(\mathbf{a}_T^T(\theta)\mathbf{R}_\mathbf{s}\dot{\mathbf{a}}_T^*(\theta)\right)^*}{\mathbf{a}_T^T(\theta)\mathbf{R}_\mathbf{s}\mathbf{a}_T^*(\theta)}\right\} \tag{19}$$

$$d_3 = \mathbf{a}_T^T(\theta)\mathbf{R}_{\dot{\mathbf{s}}\dot{\mathbf{s}}}\mathbf{a}_T^*(\theta) - \frac{\left|\mathbf{a}_T^T(\theta)\mathbf{R}_{\dot{\mathbf{s}}\mathbf{s}}^H\mathbf{a}_T^*(\theta)\right|^2}{\mathbf{a}_T^T(\theta)\mathbf{R}_\mathbf{s}\mathbf{a}_T^*(\theta)} \tag{20}$$

In the case of coherent signals ($\Delta f = 0$), it can be verified that $\mathbf{R}_\mathbf{s} = \mathbf{w}\mathbf{w}^H$ and $\mathbf{R}_{\dot{\mathbf{s}}\mathbf{s}} = j\mathrm{Im}\{\int_{T_p}\dot{s}_0(t)s_0^*(t)dt\}\mathbf{w}\mathbf{w}^H$, and consequently $d_2 = 0$. Therefore, there is no coupling between θ and r, and the CRB on DOA estimation does not depend on the transmit signal bandwidth, B. However, when using $\Delta f \neq 0$, the transmit signals are not fully correlated and thus, d_2 is not necessarily equal to zero, which results in the dependency of the CRB on DOA estimation on the transmit signal bandwidth.

III. FDA Waveform Optimization

As can be seen from (2), the degrees of freedom in the weighted FDA transmit signal are Δf and $\mathbf{w}$. It can be seen that the dependency of the FIM on Δf is complex, thus we adopt a grid search approach to find the optimal Δf. In order to answer the question of what should be the grid domain of Δf, we examine the FDA signal autocorrelation matrix (10). It can be verified that $\mathbf{R}_\mathbf{s}$ depends on Δf via the product $T_p\Delta f$. Without loss of generality we can write, $\Delta f = \gamma/T_p$, $\gamma \in [-c_1, c_1]$, where $c_1 \in \mathbb{R}^+$ is a constant which depends on some constraints such as: total system bandwidth, the assumption $|\Delta f| \cdot (M_T - 1) \ll f_c$, range ambiguity constraints, etc. Usually, in pulse-Doppler radars the pulse width is determined as $10 - 15$ percent of the pulse repetition interval (PRI), where the last is determined according to range-Doppler considerations.

Since we focus on the FDA parameters influence (the frequency increment and weight vector), we consider a fixed pulse width. As mentioned above, if $\Delta f = 0$, then $\mathbf{R}_\mathbf{s} = \mathbf{w}\mathbf{w}^H$, which represents the case of coherent transmit signals. Furthermore, if $\Delta f = \eta/T_p$, where $\eta \in \{\pm 1, \ldots, \pm\lfloor c_1 \rfloor\}$, then the transmit signals are orthogonal and $\mathbf{R}_\mathbf{s} = \mathrm{diag}\left([|w_1|^2, \ldots, |w_{M_T}|^2]^T\right)$ which yields omnidirectional transmission. As can be seen from (10) and (13), the off-diagonal elements of $\mathbf{R}_\mathbf{s}$ approach to zero for $|\Delta f| \gg 1/T_p$. Therefore, we conclude that in terms of target DOA estimation it is enough to require $|\Delta f| \leq 1/T_p$. On the other hand, in terms of target range estimation, $|\Delta f| > 1/T_p$ yields larger total bandwidth, which is given by $B + (M_T - 1)|\Delta f|$.

We are interested to design the weighted FDA transmit signal parameters for target localization in Cartesian coordinates. Therefore, for a given transmit signal bandwidth, the parameters Δf and $\mathbf{w}$ can be optimized based on the CRB matrix, under either a total energy constraint or energy constraint per element. The CRB for target location in Cartesian coordinates is given by

$$CRB(\bar{\mathbf{x}}) = \mathbf{G}(\mathbf{\Phi})CRB(\mathbf{\Phi})\mathbf{G}^T(\mathbf{\Phi}) \tag{21}$$

where

$$\mathbf{G}(\mathbf{\Phi}) = \begin{bmatrix} \sin\theta & r\cos\theta \\ \cos\theta & -r\sin\theta \end{bmatrix} \tag{22}$$

and $\bar{\mathbf{x}} \triangleq [x\ y]^T = \begin{bmatrix} r\sin\theta & r\cos\theta \end{bmatrix}^T$ denotes the target location in a two-dimensional Cartesian coordinate system. The CRB in the vector parameter case is a matrix, thus, several optimization criteria can be adopted, e.g., minimizing the trace, the determinant, or the largest eigenvalue of the CRB matrix, w.r.t. to the frequency increment and the weight vector. We consider the trace optimization criterion because of its physical meaning and its superior performance compared to the other mentioned criteria, as described in [9]. Therefore, the optimization problem can be written as

$$(\overline{\Delta f}, \overline{\mathbf{w}}) = \arg\min_{\Delta f, \mathbf{w}} \mathrm{tr}\left(CRB(\bar{\mathbf{x}})\right) \quad \text{s.t. } ||\mathbf{w}||_2^2 = P \text{ or } \left\{|w_m|^2\right\}_{m=1}^{M_T} = P/M_T \tag{23}$$

From (21) one can notice that $\mathrm{tr}\left(CRB(\bar{\mathbf{x}})\right) = CRB(r) + r^2 CRB(\theta)$. Therefore, (23) can be rewritten as follows

$$(\overline{\Delta f}, \overline{\mathbf{w}}) = \arg\min_{\Delta f, \mathbf{w}} \mathrm{tr}\left(\widetilde{\mathbf{W}}\mathbf{J}_\mathbf{\Psi}^{-1}\right) \quad \text{s.t. } ||\mathbf{w}||_2^2 = P \text{ or } \left\{|w_m|^2\right\}_{m=1}^{M_T} = P/M_T \tag{24}$$

where $\widetilde{\mathbf{W}} = \mathrm{diag}\left([0, 0, 1, r^2]\right)$. In [9], it was shown that the minimization problem (24) can be cast as the following semidefinite programming (SDP)

$$(\overline{\Delta f}, \overline{\mathbf{w}}) = \arg\min_{\Delta f, \mathbf{w}} \min_{\{t_l\}_{l=1}^4} \sum_{l=1}^4 \mu_l t_l \quad \text{s.t. } \begin{bmatrix} \mathbf{J}_\mathbf{\Psi} & \mathbf{e}_i \\ \mathbf{e}_i^T & t_i \end{bmatrix} \succeq \mathbf{0}, \quad i = 1, \ldots, 4, \quad ||\mathbf{w}||_2^2 = P \text{ or } \left\{|w_m|^2\right\}_{m=1}^{M_T} = P/M_T \tag{25}$$

where μ_l is the lth diagonal element of $\widetilde{\mathbf{W}}$, $\{t_l\}_{l=1}^4$ are auxiliary variables, $\mathbf{e}_i$ denotes the ith column of the identity matrix, and the FIM, $\mathbf{J}_\mathbf{\Psi}$, is given in (3) and (14).

IV. Simulation Results

In this section, we present via simulations the influence of the transmit signal bandwidth (BW) on DOA estimation in FDA radar. In addition, we demonstrate that the optimized FDA waveform improves the target localization performance in terms of CRB, compared to coherent and orthogonal transmit signals. Consider a ULAs with half a wavelength spacing ($d = \lambda/2$), $M_R = 2$ receivers, and $M_T = 7$ transmitters. The basic waveform is chosen as linear frequency modulation

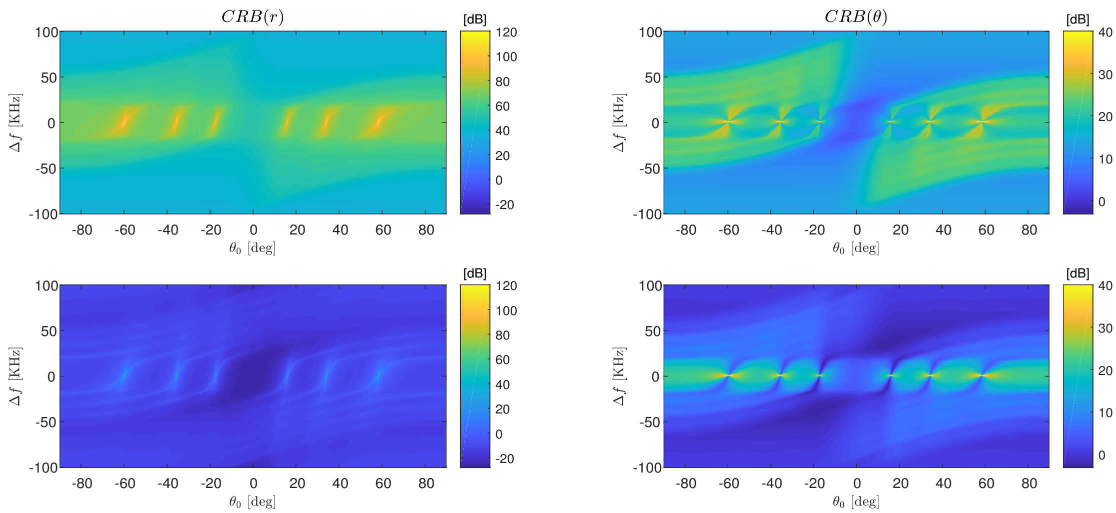

Fig. 1. CRB on target range (left column) and DOA (right column) estimation as function of $(\Delta f, \theta_0)$, where $T_p = 10\,\mu$sec, $M_R = 2$, $M_T = 7$, ASNR $=$ 15 dB. In the first row - small BW ($T_p\tilde{B} = 0.1, B = 110$ KHz), and in the second row - large BW ($T_p\tilde{B} = 10^3, B = 100.1$ MHz).

(LFM) signal, that is, $s_0(t) = \left(1/\sqrt{T_p}\right)\exp\left(j\pi\tilde{B}/T_p t^2\right)$, $t \in [0, T_p]$, where its bandwidth is given by $B = \tilde{B} + 1/T_p$, and the pulse width is fixed to $T_p = 10\,\mu$sec. We define the array signal-to-noise ratio (ASNR) as ASNR $\triangleq |\alpha|^2 M_R P/\sigma_v^2$, which has been set to ASNR $= 15$ dB. In the examples below we assume a single target ($L_T = 1$) in a single Doppler bin, where the unknown target range and DOA are $r = 1$ Km and $\theta = 0°$, respectively.

In Fig. 1, the CRB on target range and DOA estimation (17), is shown as a function of $(\Delta f, \theta_0)$ under both small and large BW scenarios, where the weight vector is chosen as $\mathbf{w} = \sqrt{P/M_T}\mathbf{a}_T^*(\theta_0)$. It can be noticed that in the small BW case, the CRB on range estimation decreases as $|\Delta f|$ increases. This is due to the fact that the total BW of the transmit signal is increased. However, in large BW scenario, the contribution of Δf is negligible because $|\Delta f| \cdot (M_T - 1) \ll B$, while transmission of steered beam to the target direction with coherent signals ($\Delta f = 0, \theta_0 = \theta = 0°$) yields the highest gain, thus the CRB on range is lowest for those parameters. In regard to the CRB on DOA estimation, one can notice an interesting phenomenon: for most values of $\Delta f \neq 0$ and θ_0, the CRB is lower for the large BW case, while for coherent signals the CRB has not changed.

In order to demonstrate the BW influence on DOA estimation performance, in Fig. 2 we present the CRB on DOA estimation versus the frequency increment. In this figure, for every Δf the weight vector is optimized by two methods:
1) $\mathbf{w}$ was set as in Fig. 1, and the CRB is minimized w.r.t. θ_0.
2) minimization of the CRB w.r.t. general weight vector under energy constraint per element. It can be seen that for each method, if $\Delta f = 0$ the BW does not affect on the CRB, while for $\Delta f \neq 0$ the CRB is improved. This phenomenon can be explained by the coupling between range and DOA in the CRB. Therefore, if the BW increases, more information on

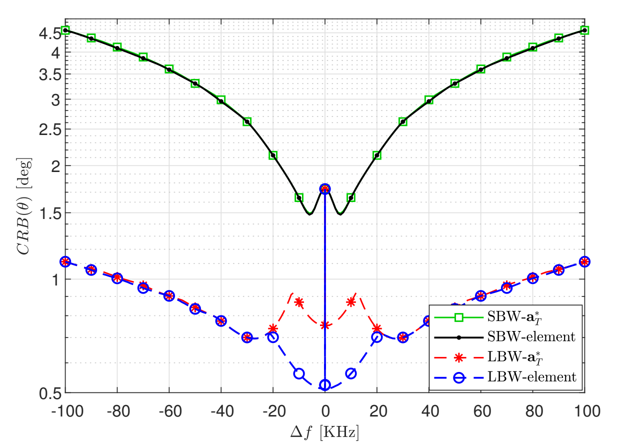

Fig. 2. Optimal CRB on target DOA estimation versus frequency increment, for small BW ($T_p\tilde{B} = 0.1, B = 110$ KHz) and large BW ($T_p\tilde{B} = 10^3, B = 100.1$ MHz), where $T_p = 10\,\mu$sec, $M_R = 2$, $M_T = 7$, ASNR $= 15$ dB.

the target range is available, which affects on the information about θ and α. Note that since the case of known target range is equivalent to the case of unknown range where the BW approaches to infinity, the results of the known range scenario are similar to those in the large BW scenario. Thus, for $\Delta f \neq 0$ there is a trade-off between the gain, and ability to derive more information on the target DOA.

In Fig. 3 the minimized CRB on target localization (23) is shown for the large BW case in order to demonstrate that the optimized FDA waveform improves the target localization performance, compared to coherent and orthogonal (FDA-MIMO) transmit signals. It can be seen that for each of the four proposed optimization methods of the weight vector (norm or element constraint, or choosing the weight vector as proportional to $\mathbf{a}_T^*(\theta_0)$ or $\dot{\mathbf{a}}_T^*(\theta_0)$), the lowest CRB

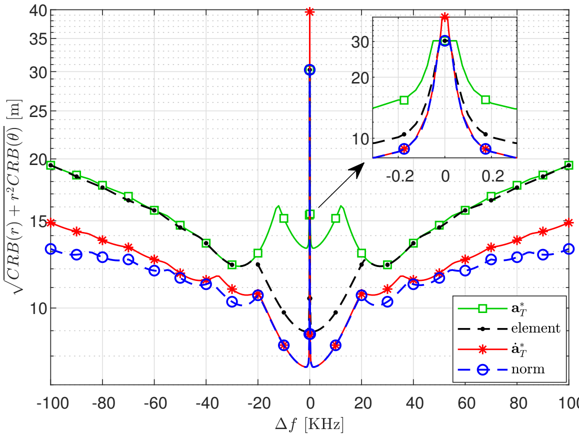

Fig. 3. Optimal CRB on target localization estimation, where $B = 100.1\,\text{MHz}$, $T_p = 10\,\mu\text{sec}$, $M_R = 2$, $M_T = 7$, ASNR $= 15\,\text{dB}$.

is achieved by neither coherent ($\Delta f = 0$) or orthogonal ($\Delta f = \pm 1/T_p = \pm 100$ KHz) signals. The optimal FDA parameters are given by ($\overline{\Delta f} = \pm 0.015/T_p = \pm 1.5$ KHz, $\overline{\mathbf{w}} = \sqrt{P}\dot{\mathbf{a}}_T^*(\theta_{opt})/||\dot{\mathbf{a}}_T^*(\theta_{opt})||$), where $\theta_{opt} = \pm 0.85°$. In Fig. 1 we have seen that in the large BW case, transmission of coherent signals yields the best performance in terms of CRB on range estimation. Nevertheless, the superior DOA estimation performance by FDA radar decreases the CRB on cross-range estimation and thus reduces the CRB on localization performance. Note that since the minimized CRB is symmetric w.r.t. Δf, one-sided grid domain can be utilized.

V. Conclusion

In this paper, we have investigated the problem of waveform design for target localization in FDA radar using the CRB. We have found that minimizing the CRB w.r.t. to both the weight vector and the frequency increment results in superior target localization performance, compared to both coherent and orthogonal transmit signals. Because of the coupling between range and DOA, increasing the transmit signal bandwidth results in superior DOA estimation accuracy, and consequently the localization performance improves.

The analysis and the waveform design method described in this paper, consider standard linear FDA radar. Generalization to arbitrary frequency increments or arbitrary array geometries can be further investigated in future work. This work can also be extended to use the additional degrees of freedom provided by FDA, to cope with interference such as clutter or jammer sources. Thus, adaptive or non-adaptive FDA transmit signal design in the presence of interference is an interesting topic for future research.

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