Modelling Mismatch and Noise Statistics Uncertainty in Linear MMSE Estimation

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***Abstract*—Standard filtering techniques, such as Kalman, sigma-point or particle filters, assume a perfect knowledge of the system. This implies that both process and measurement functions, and the system noise statistics, are assumed known and fit the reality. Regarding the noise statistics this involves knowing not only the distributions but also their parameters. In this contribution, we explore the impact of system model mismatch and uncertain noise statistics parameters into linear minimum mean square error estimators for linear discrete state-space models. Illustrative examples are shown to support the discussion.**

***Index Terms*—Wiener filtering, Kalman filtering, model mismatch, system noise uncertainty, robustness.**

I. Introduction: from Wiener to Kalman

We first recall the basics of Wiener filtering and its recursive form for linear discrete state-space (LDSS) models, directly related to the Kalman filter (KF), which in turn is the starting point of the performance analysis provided in this contribution.

If $\mathbf{x}$ and $\mathbf{y}$ are two zero-mean complex random vectors, the Wiener filter (WF) is the linear minimum mean square error (LMMSE) estimator of $\mathbf{x}$ [1], that is, it minimises the error covariance matrix $\mathbf{P}(\mathbf{L}) = \mathbb{E}\left[(\mathbf{L}\mathbf{y}-\mathbf{x})(\mathbf{L}\mathbf{y}-\mathbf{x})^H\right]$. If $\mathbf{C}_\mathbf{x}$, $\mathbf{C}_\mathbf{y}$, $\mathbf{C}_{\mathbf{x},\mathbf{y}}$, $\mathbf{C}_{\mathbf{x}|\mathbf{y}}$ are the covariance matrices of $\mathbf{x}$, $\mathbf{y}$, cross-covariance of $\mathbf{x}$ and $\mathbf{y}$, and conditional covariance of $\mathbf{x}$ given $\mathbf{y}$, respectively, and $\mathbf{C}_\mathbf{y}$ is invertible, then[1]

$$\hat{\mathbf{x}}^b(\mathbf{y}) = \mathbf{L}^b\mathbf{y},\ \mathbf{L}^b = \arg\min_{\mathbf{L}}\{\mathbf{P}(\mathbf{L})\} = \mathbf{C}_{\mathbf{x},\mathbf{y}}\mathbf{C}_\mathbf{y}^{-1}, \tag{1}$$

$$\mathbf{P}(\mathbf{L}^b) = \mathbf{C}_{\mathbf{x}|\mathbf{y}} = \mathbf{C}_\mathbf{x} - \mathbf{C}_{\mathbf{x},\mathbf{y}}\mathbf{C}_\mathbf{y}^{-1}\mathbf{C}_{\mathbf{x},\mathbf{y}}^H. \tag{2}$$

The best affine estimator in the MSE sense, $\hat{\mathbf{x}}(\mathbf{y}) = \mathbf{L}\mathbf{y} + \mathbf{a}$, which minimises the error covariance matrix $\mathbf{P}(\mathbf{L},\mathbf{a}) = \mathbb{E}\left[(\mathbf{L}\mathbf{y}+\mathbf{a}-\mathbf{x})(\mathbf{L}\mathbf{y}+\mathbf{a}-\mathbf{x})^H\right]$, admits a similar WF form

$$\hat{\mathbf{x}}^b(\mathbf{y}) = \mathbf{m}_\mathbf{x} + \mathbf{L}^b(\mathbf{y}-\mathbf{m}_\mathbf{y}), \tag{3}$$

$$\mathbf{a}^b = \mathbf{m}_\mathbf{x} - \mathbf{L}^b\mathbf{m}_\mathbf{y}\ ;\ \mathbf{m}_\mathbf{x} = \mathbb{E}\left[\mathbf{x}\right]\ ;\ \mathbf{m}_\mathbf{y} = \mathbb{E}\left[\mathbf{y}\right] \tag{4}$$

$$\mathbf{L}^b = \arg\min_{\mathbf{L}}\{\mathbf{P}(\mathbf{L},\mathbf{a}^b)\} = \mathbf{C}_{\mathbf{x},\mathbf{y}}\mathbf{C}_\mathbf{y}^{-1}\ ;\ \mathbf{P}(\mathbf{L}^b,\mathbf{a}^b) = \mathbf{C}_{\mathbf{x}|\mathbf{y}}$$

The LDSS models of interest in this article are given by,

$$\mathbf{x}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{w}_{k-1}\ ;\ \mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k, \tag{5}$$

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[1]The superscript $(\cdot)^b$ stands for the best solution according to a given estimation criterion previously defined, for instance in the MSE sense.

where $\mathbf{x}_k \in \mathbb{C}^{n_x}$, $\mathbf{y}_k \in \mathbb{C}^{n_y}$ (for $k \geq 1$) are the state vector to be inferred and a measurement vector related to the states; $\mathbf{F}_{k-1}$ and $\mathbf{H}_k$ are the system model (process and measurement) matrices; the process, $\mathbf{w}_k$, and measurement noise, $\mathbf{v}_k$, have mean vectors and covariance matrices respectively given by $\mathbf{m}_{\mathbf{w}_k}$, $\mathbf{m}_{\mathbf{v}_k}$, $\mathbf{C}_{\mathbf{w}_k}$ and $\mathbf{C}_{\mathbf{v}_k}$; in this case the WF estimate of $\mathbf{x}_k$ using measurements up to time k, $\mathbf{y}_{1:k}$, is

$$\hat{\mathbf{x}}_{k|k}^b = \mathbf{m}_{\mathbf{x}_k} + \mathbb{L}_k^b(\bar{\mathbf{y}}_k - \mathbf{m}_{\bar{\mathbf{y}}_k}),\ \mathbb{L}_k^b = \mathbf{C}_{\mathbf{x}_k,\bar{\mathbf{y}}_k}\mathbf{C}_{\bar{\mathbf{y}}_k}^{-1} \tag{6}$$

with $\bar{\mathbf{y}}_k^\top = \left[\mathbf{y}_1^\top,\ldots,\mathbf{y}_k^\top\right]$. Notice that in practice this is not a useful solution as each estimate depends on the complete set of observations. If the following minimum set of uncorrelation conditions hold

$$\forall\ k \geq 2 : \mathbf{C}_{\mathbf{w}_{k-1},\bar{\mathbf{y}}_{k-1}} = \mathbf{0} \text{ and } \mathbf{C}_{\mathbf{v}_k,\bar{\mathbf{y}}_{k-1}} = \mathbf{0}, \tag{7}$$

both state and measurement predictions are given by

$$\hat{\mathbf{x}}_{k|k-1}^b = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1}^b + \mathbf{m}_{\mathbf{w}_{k-1}}, \tag{8}$$

$$\hat{\mathbf{y}}_{k|k-1}^b = \mathbf{H}_k\hat{\mathbf{x}}_{k|k-1}^b + \mathbf{m}_{\mathbf{v}_k}, \tag{9}$$

which leads to a convenient recursive predictor/corrector general form (for $k \geq 2$) [2]

$$\begin{aligned}\hat{\mathbf{x}}_{k|k}^b &= \hat{\mathbf{x}}_{k|k-1}^b + \mathbf{L}_k^b(\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}^b)\\ &= (\mathbf{I} - \mathbf{L}_k^b\mathbf{H}_k)\mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1}^b + \mathbf{L}_k^b\mathbf{y}_k\\ &\quad + (\mathbf{I} - \mathbf{L}_k^b\mathbf{H}_k)\mathbf{m}_{\mathbf{w}_{k-1}} - \mathbf{L}_k^b\mathbf{m}_{\mathbf{v}_k}.\end{aligned} \tag{10}$$

The optimal gain and estimation error covariance are computed in a general manner using the following recursion

$$\begin{aligned}\mathbf{P}_{k|k-1}^b &= \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}^b\mathbf{F}_{k-1}^H + \mathbf{C}_{\mathbf{w}_{k-1}}\\ &+ \mathbf{F}_{k-1}\mathbf{C}_{\mathbf{w}_{k-1},\mathbf{x}_{k-1}}^H + \mathbf{C}_{\mathbf{w}_{k-1},\mathbf{x}_{k-1}}\mathbf{F}_{k-1}^H\end{aligned} \tag{11}$$

$$\begin{aligned}\mathbf{S}_{k|k-1}^b &= \mathbf{H}_k\mathbf{P}_{k|k-1}^b\mathbf{H}_k^H + \mathbf{C}_{\mathbf{v}_k}\\ &+ \mathbf{H}_k\mathbf{C}_{\mathbf{v}_k,\mathbf{x}_k}^H + \mathbf{C}_{\mathbf{v}_k,\mathbf{x}_k}\mathbf{H}_k^H\end{aligned} \tag{12}$$

$$\mathbf{L}_k^b = (\mathbf{P}_{k|k-1}^b\mathbf{H}_k^H + \mathbf{C}_{\mathbf{v}_k,\mathbf{x}_k}^H)(\mathbf{S}_{k|k-1}^b)^{-1}, \tag{13}$$

$$\mathbf{P}_{k|k}^b = (\mathbf{I} - \mathbf{L}_k^b\mathbf{H}_k)\mathbf{P}_{k|k-1}^b - \mathbf{L}_k^b\mathbf{C}_{\mathbf{v}_k,\mathbf{x}_k}, \tag{14}$$

which are also valid for $k = 1$ if $\mathbf{P}_{0|0}^b = \mathbf{C}_{\mathbf{x}_0}$ and $\hat{\mathbf{x}}_{0|0}^b = \mathbb{E}\{\mathbf{x}_0\}$. If we consider zero-mean process and measurement noise (i.e., $\mathbf{m}_{\mathbf{w}_{k-1}} = \mathbf{0}$ and $\mathbf{m}_{\mathbf{v}_k} = \mathbf{0}$) and the usual KF uncorrelation conditions,

$$\left\{\begin{array}{l}\mathbf{C}_{\mathbf{x}_0,\mathbf{w}_k} = \mathbf{0}, \mathbf{C}_{\mathbf{x}_0,\mathbf{v}_k} = \mathbf{0}, \mathbf{C}_{\mathbf{w}_l,\mathbf{w}_k} = \mathbf{C}_{\mathbf{w}_k}\delta_k^l\\ \mathbf{C}_{\mathbf{v}_l,\mathbf{v}_k} = \mathbf{C}_{\mathbf{v}_k}\delta_k^l, \mathbf{C}_{\mathbf{w}_l,\mathbf{v}_k} = \mathbf{0}\end{array}\right. \tag{15}$$

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then we recover the standard KF equations [3]. Notice that only the first and second order moments are taken into account, and nothing is said about the distribution of the noise, then, both the general formulation above and the standard KF solution provide the LMMSE estimator even if the system noises are not Gaussian distributed. Finally, from the recursions above we can see that the KF is a recursive form of the WF.

In the previous derivations, besides the different uncorrelation conditions, the main assumptions are [3]:

- Known system matrices $\mathbf{F}_k$ and $\mathbf{H}_k$.
- Known 1st and 2nd order process and measurement noise statistics, $\mathbf{m}_{\mathbf{w}_k}$, $\mathbf{m}_{\mathbf{v}_k}$, $\mathbf{C}_{\mathbf{w}_k}$, $\mathbf{C}_{\mathbf{v}_k}$.
- Perfect initialisation, $\mathbf{P}^b_{0|0} = \mathbf{C}_{\mathbf{x}_0}$ and $\hat{\mathbf{x}}^b_{0|0} = \mathbb{E}\{\mathbf{x}_0\}$.

The performance of LMMSE estimators for LDSS models strongly depends on this knowledge [4], [5]. The main goal of this contribution is to analyse the impact on the final estimates of both modelling mismatch (i.e., error on the system matrices) and system noise uncertainty (i.e., lack of knowledge on the noise mean and covariance). Notice that we do not pretend to provide a robust filtering solution to mitigate such errors (see for instance [6]), but rather analyse the possible performance degradation if we do not take them into account. Mathematically, this can be translated to

$$\text{Mismatched LDSS} \left\{ \begin{array}{l} \tilde{\mathbf{x}}_k = \mathbf{F}_{k-1}\tilde{\mathbf{x}}_{k-1} + \tilde{\mathbf{w}}_{k-1} \\ \mathbf{y}_k = \mathbf{H}_k\tilde{\mathbf{x}}_k + \tilde{\mathbf{v}}_k \end{array} \right. \tag{16}$$

$$\text{True LDSS} \left\{ \begin{array}{l} \mathbf{x}_k = (\mathbf{F}_{k-1} + d\mathbf{F}_{k-1})\,\mathbf{x}_{k-1} + \mathbf{w}_{k-1} \\ \mathbf{y}_k = (\mathbf{H}_k + d\mathbf{H}_k)\,\mathbf{x}_k + \mathbf{v}_k \end{array} \right. \tag{17}$$

where $\tilde{\mathbf{m}}_{\mathbf{w}_{k-1}} \neq \mathbf{m}_{\mathbf{w}_{k-1}}$, $\tilde{\mathbf{C}}_{\mathbf{w}_{k-1}} \neq \mathbf{C}_{\mathbf{w}_{k-1}}$, $\tilde{\mathbf{m}}_{\mathbf{v}_k} \neq \mathbf{m}_{\mathbf{v}_k}$ and $\tilde{\mathbf{C}}_{\mathbf{v}_k} \neq \mathbf{C}_{\mathbf{v}_k}$, which are unknown (up to a certain extent) system noise 1st and 2nd order moments.

II. Standard KF under Uncertain Noise Statistics

The standard KF solution is considered, that is, assuming zero-mean noises and that both uncorrelation conditions (7) and (15) are satisfied. First, the impact on the estimation performance of system noise mismatch at time k is analysed, assuming that no mismatch was present at the previous time step $k-1$ ($\hat{\mathbf{x}}_{k-1|k-1} = \hat{\mathbf{x}}^b_{k-1|k-1}$ is an unbiased estimator of $\mathbf{x}_{k-1}$). The filter assumes the LDSS model (16) but the true one is given by (17). In this case, at $k-1$, there is no system model mismatch, i.e., $d\mathbf{F}_{k-1} = d\mathbf{H}_k = \mathbf{0}$. The noise statistics parameters are unknown to a certain extent: while $\{\tilde{\mathbf{m}}_{\mathbf{w}_{k-1}}, \tilde{\mathbf{m}}_{\mathbf{v}_k}\} = \mathbf{0}$ the true model is $\{\mathbf{m}_{\mathbf{w}_{k-1}}, \mathbf{m}_{\mathbf{v}_k}\} \neq \mathbf{0}$, $\mathbf{C}_{\mathbf{w}_{k-1}} = \tilde{\mathbf{C}}_{\mathbf{w}_{k-1}} + d\mathbf{C}_{\mathbf{w}_{k-1}}$, and both noise covariances $\mathbf{C}_{\mathbf{v}_k} = \tilde{\mathbf{C}}_{\mathbf{v}_k} + d\mathbf{C}_{\mathbf{v}_k}$. At time k, considering (16),

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1}, \tag{18}$$

$$\hat{\mathbf{y}}_{k|k-1} = \mathbf{H}_k\hat{\mathbf{x}}_{k|k-1} = \mathbf{H}_k\mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1}, \tag{19}$$

$$\begin{aligned} \hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{L}_k\left(\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}\right) \\ &= \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1} + \mathbf{L}_k\left(\mathbf{y}_k - \mathbf{H}_k\mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1}\right), \end{aligned} \tag{20}$$

and the estimation error ($\mathbf{e}_{k|k} = \hat{\mathbf{x}}_{k|k} - \mathbf{x}_k$) is then

$$\begin{aligned} \mathbf{e}_{k|k} &= \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1} + \mathbf{L}_k\left(\mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k\right) \\ &- \mathbf{L}_k\mathbf{H}_k\mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1} - \mathbf{F}_{k-1}\mathbf{x}_{k-1} - \mathbf{w}_{k-1} \\ &= \mathbf{A}_k(\mathbf{F}_{k-1}\mathbf{e}_{k-1|k-1} - \mathbf{w}_{k-1}) + \mathbf{L}_k\mathbf{v}_k, \end{aligned} \tag{21}$$

with $\mathbf{A}_i = (\mathbf{I} - \mathbf{L}_i\mathbf{H}_i)$. This is the error expected from a standard KF, but in this case $\mathbf{w}_{k-1}$ and $\mathbf{v}_k$ are not properly characterised (i.e., the mismatched model assumes $\tilde{\mathbf{w}}_{k-1}$ and $\tilde{\mathbf{v}}_k$), directly impacting into the estimation performance.

A. Estimator Bias under Noise Mismatch

The estimator bias is computed from

$$\begin{aligned} \mathbb{E}\{\mathbf{e}_{k|k}\} &= \mathbf{A}_k\mathbf{F}_{k-1}\mathbb{E}\{\hat{\mathbf{x}}_{k-1|k-1} - \mathbf{x}_{k-1}\} \\ &- \mathbf{A}_k\mathbf{m}_{\mathbf{w}_{k-1}} + \mathbf{L}_k\mathbf{m}_{\mathbf{v}_k} \\ &= \mathbf{L}_k\mathbf{m}_{\mathbf{v}_k} - \mathbf{A}_k\mathbf{m}_{\mathbf{w}_{k-1}}, \end{aligned} \tag{22}$$

where the previous estimate was assumed unbiased, i.e., $\mathbb{E}\{\hat{\mathbf{x}}_{k-1|k-1} - \mathbf{x}_{k-1}\} = \mathbf{0}$. Then, unknown noise mean vectors introduce a recursive bias into the estimation.

- From k to $k+l$:

From the previous expression at k, the general bias expression at $k+l$, taking into account that in the following steps the term $\mathbb{E}\{\mathbf{e}_{k+l-1|k+l-1}\}$ is no longer zero, is given by

$$\mathbb{E}\{\mathbf{e}_{k+l|k+l}\} = \sum_{i=k}^{k+l}\prod_{j=i}^{k+l-1}\mathbf{A}_{j+1}\mathbf{F}_j\left(\mathbf{L}_i\mathbf{m}_{\mathbf{v}_i} - \mathbf{A}_i\mathbf{m}_{\mathbf{w}_{i-1}}\right),$$

where the weighting term $\prod_{j=i}^{k+l-1}(\mathbf{I} - \mathbf{L}_{j+1}\mathbf{H}_{j+1})\mathbf{F}_j$ plays an important role. In the time-invariant scalar case, under steady-state conditions ($L_{j+1} = L_\infty$) and iff $(1 - L_\infty H)F < 1$, then

$$\prod_{j=1}^{N}(1 - L_{j+1}H)F = \left((1 - L_\infty H)F\right)^N \underset{N\to\infty}{\longrightarrow} 0, \tag{23}$$

which leads to a constant bias in steady-state conditions. Finally, the bias can be expressed in a convenient form as

$$\text{Bias}_k = \mathbf{A}_k\mathbf{F}_{k-1}\text{Bias}_{k-1} + \left(\mathbf{L}_k\mathbf{m}_{\mathbf{v}_k} - \mathbf{A}_k\mathbf{m}_{\mathbf{w}_{k-1}}\right) \tag{24}$$

B. Error Covariance under Noise Mismatch

In the sequel, the impact on the MSE is analysed. First, the noise mismatch implies that

$$\begin{aligned} \mathbb{E}\{\mathbf{w}_{k-1}\mathbf{w}_{k-1}^H\} &= \tilde{\mathbf{C}}_{\mathbf{w}_{k-1}} + d\mathbf{C}_{\mathbf{w}_{k-1}} + \mathbf{m}_{\mathbf{w}_{k-1}}\mathbf{m}_{\mathbf{w}_{k-1}}^H, \\ \mathbb{E}\{\mathbf{v}_k\mathbf{v}_k^H\} &= \tilde{\mathbf{C}}_{\mathbf{v}_k} + d\mathbf{C}_{\mathbf{v}_k} + \mathbf{m}_{\mathbf{v}_k}\mathbf{m}_{\mathbf{v}_k}^H. \end{aligned}$$

The estimation error covariance is obtained as follows

$$\begin{aligned} &\mathbf{P}^b_{k-1|k-1} = \mathbb{E}\{\mathbf{e}_{k-1|k-1}(\mathbf{e}_{k-1|k-1})^H\}, \\ &\mathbf{P}_{k|k-1} = \mathbf{F}_{k-1}\mathbf{P}^b_{k-1|k-1}\mathbf{F}_{k-1}^H + \mathbb{E}\{\mathbf{w}_{k-1}\mathbf{w}_{k-1}^H\} = \\ &\underbrace{\mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k-1}^H + \tilde{\mathbf{C}}_{\mathbf{w}_{k-1}}}_{\mathbf{P}^b_{k|k-1}} + \underbrace{d\mathbf{C}_{\mathbf{w}_{k-1}} + \mathbf{m}_{\mathbf{w}_{k-1}}\mathbf{m}_{\mathbf{w}_{k-1}}^H}_{\text{noise statistics mismatch}} \end{aligned}$$

$$\begin{aligned} \mathbf{P}_{k|k} &= \mathbf{A}_k\mathbf{P}_{k|k-1}\mathbf{A}_k^H + \mathbf{L}_k\mathbb{E}\{\mathbf{v}_k\mathbf{v}_k^H\}\mathbf{L}_k^H \\ &= \underbrace{\mathbf{A}_k\mathbf{P}^b_{k|k-1}\mathbf{A}_k^H + \mathbf{L}_k\tilde{\mathbf{C}}_{\mathbf{v}_k}\mathbf{L}_k^H}_{\mathbf{P}^b_{k|k}} + \mathbf{P}_{e,k}, \end{aligned} \tag{25}$$

where the recursive estimation error covariance term induced by the process and measurement noise mismatch is

$$\begin{aligned}\mathbf{P}_{e,k} &= \mathbf{A}_k(d\mathbf{C}_{\mathbf{w}_{k-1}} + \mathbf{m}_{\mathbf{w}_{k-1}}\mathbf{m}_{\mathbf{w}_{k-1}}^H)\mathbf{A}_k^H \\ &+ \mathbf{L}_k(d\mathbf{C}_{\mathbf{v}_k} + \mathbf{m}_{\mathbf{v}_k}\mathbf{m}_{\mathbf{v}_k}^H)\mathbf{L}_k^H \end{aligned} \tag{26}$$

Again, $\mathbf{L}_k$ was computed to minimise the expected trace of $\mathbf{P}_{k|k}^b$, using $\mathbf{P}_{k|k-1}^b$ (i.e., $\tilde{\mathbf{C}}_{\mathbf{w}_{k-1}}$) and $\tilde{\mathbf{C}}_{\mathbf{v}_k}$ will be suboptimal when model assumptions differ.

- From k to $k+l$:

It is easy to show that $\mathbf{P}_{k+l|k+l} = \mathbf{P}_{k+l|k+l}^b + \mathbf{P}_{e,k+l}$, with the corresponding error term at $k+l$ given by

$$\begin{aligned}\mathbf{P}_{e,k+l} &= \sum_{i=k}^{k+l} \prod_{j=i}^{k+l-1} \mathbf{A}_{j+1}\mathbf{F}_j(\mathbf{A}_i d\mathbf{C}_{\mathbf{w}_{i-1}}\mathbf{A}_i^H \\ &+ \mathbf{A}_i\mathbf{m}_{\mathbf{w}_{i-1}}\mathbf{m}_{\mathbf{w}_{i-1}}^H\mathbf{A}_i^H + \mathbf{L}_i d\mathbf{C}_{\mathbf{v}_i}\mathbf{L}_i^H \\ &+ \mathbf{L}_i\mathbf{m}_{\mathbf{v}_i}\mathbf{m}_{\mathbf{v}_i}^H\mathbf{L}_i^H) \prod_{j=i}^{k+l-1} \mathbf{F}_j^H\mathbf{A}_{j+1}^H. \end{aligned} \tag{27}$$

As already stated for the bias computation, under certain conditions, both terms $\prod_{j=i}^{k+l-1}\mathbf{A}_{j+1}\mathbf{F}_j$ and $\prod_{j=i}^{k+l-1}\mathbf{F}_j^H\mathbf{A}_{j+1}^H$ tend to zero in the large sample regime, and then the steady-state estimation error covariance $\mathbf{P}_\infty$ reaches a constant value. It is convenient to express the error term computation in a recursive manner

$$\begin{aligned}\mathbf{P}_{e,k} &= \mathbf{A}_k\mathbf{F}_{k-1}\mathbf{P}_{e,k-1}\mathbf{F}_{k-1}^H\mathbf{A}_k^H \\ &+ \mathbf{A}_k(d\mathbf{C}_{\mathbf{w}_{k-1}} + \mathbf{m}_{\mathbf{w}_{k-1}}\mathbf{m}_{\mathbf{w}_{k-1}}^H)\mathbf{A}_k^H \\ &+ \mathbf{L}_k(d\mathbf{C}_{\mathbf{v}_k} + \mathbf{m}_{\mathbf{v}_k}\mathbf{m}_{\mathbf{v}_k}^H)\mathbf{L}_k^H. \end{aligned} \tag{28}$$

III. General Recursive LMMSE Estimation under Uncertain Noise Statistics

If we consider the general predictor/corrector LMMSE estimator form in (10)-(14), where only the minimum set of uncorrelation conditions (7) are satisfied (i.e., the uncorrelation between the states and both system noises in (15) are in general not satisfied), the estimation error for the mismatched LDSS model (16) is again

$$\begin{aligned}\mathbf{e}_{k|k} &= \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1} + \mathbf{L}_k\left(\mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k\right) \\ &- \mathbf{L}_k\mathbf{H}_k\mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1} - \mathbf{F}_{k-1}\mathbf{x}_{k-1} - \mathbf{w}_{k-1} \\ &= \mathbf{A}_k(\mathbf{F}_{k-1}\mathbf{e}_{k-1|k-1} - \mathbf{w}_{k-1}) + \mathbf{L}_k\mathbf{v}_k. \end{aligned} \tag{29}$$

Then, the recursive estimator bias expression in the general case at time k is the same given in (24). But is the mismatched MSE the same as in the standard mismatched KF? If the cross-covariances $\mathbf{C}_{\mathbf{w}_{k-1},\mathbf{x}_{k-1}}$ and $\mathbf{C}_{\mathbf{v}_k,\mathbf{x}_k}$ are assumed to be known, the estimation error covariance is obtained as follows

$$\begin{aligned}&\mathbf{P}_{k-1|k-1}^b = \mathbb{E}\{\mathbf{e}_{k-1|k-1}(\mathbf{e}_{k-1|k-1})^H\}, \\ &\mathbf{P}_{k|k-1} = \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}^b\mathbf{F}_{k-1}^H + \mathbb{E}\{\mathbf{w}_{k-1}\mathbf{w}_{k-1}^H\} \\ &+ \mathbf{F}_{k-1}\mathbf{C}_{\mathbf{w}_{k-1},\mathbf{x}_{k-1}}^H + \mathbf{C}_{\mathbf{w}_{k-1},\mathbf{x}_{k-1}}\mathbf{F}_{k-1}^H \\ &= \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k-1}^H + \tilde{\mathbf{C}}_{\mathbf{w}_{k-1}} + 2\mathbf{F}_{k-1}\mathbf{C}_{\mathbf{w}_{k-1},\mathbf{x}_{k-1}}^H \\ &+ \underbrace{d\mathbf{C}_{\mathbf{w}_{k-1}} + \mathbf{m}_{\mathbf{w}_{k-1}}\mathbf{m}_{\mathbf{w}_{k-1}}^H}_{\text{noise statistics mismatch}}, \end{aligned} \tag{30}$$

where the first term corresponds to the prediction error covariance without mismatch $\mathbf{P}_{k|k-1}^b$, and the updated covariance is

$$\begin{aligned}\mathbf{P}_{k|k} &= \mathbf{A}_k\mathbf{P}_{k|k-1}\mathbf{A}_k^H + \mathbf{L}_k\mathbb{E}\{\mathbf{v}_k\mathbf{v}_k^H\}\mathbf{L}_k^H - 2\mathbf{A}_k\mathbf{C}_{\mathbf{x}_k,\mathbf{x}_v}\mathbf{L}_k^H \\ &= \mathbf{A}_k\mathbf{P}_{k|k-1}^b\mathbf{A}_k^H + \mathbf{L}_k\tilde{\mathbf{C}}_{\mathbf{v}_k}\mathbf{L}_k^H - 2\mathbf{A}_k\mathbf{C}_{\mathbf{x}_k,\mathbf{x}_v}\mathbf{L}_k^H + \mathbf{P}_{e,k}, \end{aligned}$$

where again the left-hand term corresponds to $\mathbf{P}_{k|k}^b$, which is the Joseph stabilised form of the estimation error covariance matrix update, valid for any gain $\mathbf{L}_k$, and the error term is

$$\begin{aligned}\mathbf{P}_{e,k} &= \mathbf{A}_k(d\mathbf{C}_{\mathbf{w}_{k-1}} + \mathbf{m}_{\mathbf{w}_{k-1}}\mathbf{m}_{\mathbf{w}_{k-1}}^H)\mathbf{A}_k^H \\ &+ \mathbf{L}_k(d\mathbf{C}_{\mathbf{v}_k} + \mathbf{m}_{\mathbf{v}_k}\mathbf{m}_{\mathbf{v}_k}^H)\mathbf{L}_k^H, \end{aligned} \tag{31}$$

then we have the same covariance error update as in (28).

- What if we have uncertainties on the cross-correlations?

$$\mathbf{C}_{\mathbf{w}_{k-1},\mathbf{x}_{k-1}} = \tilde{\mathbf{C}}_{\mathbf{w}_{k-1},\mathbf{x}_{k-1}} + d\mathbf{W}_{k-1},$$
$$\mathbf{C}_{\mathbf{v}_k,\mathbf{x}_k} = \tilde{\mathbf{C}}_{\mathbf{v}_k,\mathbf{x}_k} + d\mathbf{V}_k.$$

In this case, the covariance error term is

$$\begin{aligned}\mathbf{P}_{e,k} &= \mathbf{A}_k(d\mathbf{C}_{\mathbf{w}_{k-1}} + \mathbf{m}_{\mathbf{w}_{k-1}}\mathbf{m}_{\mathbf{w}_{k-1}}^H)\mathbf{A}_k^H \\ &+ \mathbf{L}_k(d\mathbf{C}_{\mathbf{v}_k} + \mathbf{m}_{\mathbf{v}_k}\mathbf{m}_{\mathbf{v}_k}^H)\mathbf{L}_k^H - 2\mathbf{A}_k d\mathbf{V}_k\mathbf{L}_k^H \\ &+ 2\mathbf{A}_k\mathbf{F}_{k-1}d\mathbf{W}_{k-1}^H\mathbf{A}_k^H, \end{aligned} \tag{32}$$

and the recursive error term computation at time k, assuming that a possible mismatch is present at any time $< k$,

$$\begin{aligned}\mathbf{P}_{e,k} &= \mathbf{A}_k\mathbf{F}_{k-1}\mathbf{P}_{e,k-1}\mathbf{F}_{k-1}^H\mathbf{A}_k^H \\ &+ \mathbf{A}_k(d\mathbf{C}_{\mathbf{w}_{k-1}} + \mathbf{m}_{\mathbf{w}_{k-1}}\mathbf{m}_{\mathbf{w}_{k-1}}^H)\mathbf{A}_k^H \\ &+ \mathbf{L}_k(d\mathbf{C}_{\mathbf{v}_k} + \mathbf{m}_{\mathbf{v}_k}\mathbf{m}_{\mathbf{v}_k}^H)\mathbf{L}_k^H - 2\mathbf{A}_k d\mathbf{V}_k\mathbf{L}_k^H \\ &+ 2\mathbf{A}_k\mathbf{F}_{k-1}d\mathbf{W}_{k-1}^H\mathbf{A}_k^H, \end{aligned} \tag{33}$$

which gives us a general recursive equation to assess the impact on the MSE of any possible uncertainty in the system model noise characterisation.

IV. Standard KF under Model Mismatch

We consider again the standard KF solution, that is, zero-mean noises and both uncorrelation conditions (7) and (15) are satisfied. We analyse first the impact on the estimation performance of a model mismatch at time k, assuming that no mismatch was present at the previous time step $k-1$. The filter assumes the LDSS model (16) but the true one is given by (17), whereas the true noise covariance matrices are known (the noise mismatch is analysed in Sections II and III). At time k, the estimation error is

$$\mathbf{e}_{k|k} = \underbrace{(\mathbf{I} - \mathbf{L}_k\mathbf{H}_k)(\mathbf{F}_{k-1}\mathbf{e}_{k-1|k-1} - \mathbf{w}_{k-1}) + \mathbf{L}_k\mathbf{v}_k}_{\mathbf{e}_{k|k}^b \text{ without mismatch}} + \boldsymbol{\epsilon}_k,$$

where $\mathbf{e}_{k|k}^b$ is the standard KF estimation error expression without model mismatch, and the extra error term is

$$\begin{aligned}\boldsymbol{\epsilon}_k &= \mathbf{L}_k d\mathbf{H}_k(\mathbf{F}_{k-1} + d\mathbf{F}_{k-1})\mathbf{x}_{k-1} \\ &- (\mathbf{I} - \mathbf{L}_k\mathbf{H}_k)d\mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{L}_k d\mathbf{H}_k\mathbf{w}_{k-1}, \end{aligned} \tag{34}$$

which is zero if $d\mathbf{F}_{k-1}$ and $d\mathbf{H}_k$ are null. For convenience we define $\mathbf{D}_k = \mathbf{L}_k d\mathbf{H}_k(\mathbf{F}_{k-1} + d\mathbf{F}_{k-1}) - (\mathbf{I} - \mathbf{L}_k\mathbf{H}_k)d\mathbf{F}_{k-1}$.

A. Estimator Bias under Model Mismatch

From the previous error expression we can compute

$$\mathbb{E}\{\mathbf{e}_{k|k}\} = \mathbf{A}_k\mathbf{F}_{k-1}\mathbb{E}\{\mathbf{e}_{k-1|k-1}\} + \mathbb{E}\{\boldsymbol{\epsilon}_k\} \\ = \mathbf{D}_k\mathbb{E}\{\mathbf{x}_{k-1}\} = \mathbf{D}_k\mathbf{m}_{\mathbf{x}_{k-1}}, \quad (35)$$

again we have a recursive bias term due to model mismatch.

- From k to $k+l$:

$$\mathbb{E}\{\mathbf{e}_{k+l|k+l}\} = \sum_{i=k}^{k+l}\prod_{j=i}^{k+l-1}\mathbf{A}_{j+1}\mathbf{F}_j\mathbf{D}_i\mathbf{m}_{\mathbf{x}_{i-1}}. \quad (36)$$

where the estimate at $k-1$ was considered to be unbiased. The bias at time k can be computed in a recursive form as

$$\text{Bias}_k = \mathbf{A}_k\mathbf{F}_{k-1}\text{Bias}_{k-1} + \mathbf{D}_k\mathbf{m}_{\mathbf{x}_{k-1}}, \quad (37)$$

which depends on the probably unknown mean value $\mathbf{m}_{\mathbf{x}_{k-1}}$.

B. Error Covariance under Model Mismatch

The corresponding estimation error covariance is given by

$$\mathbf{P}_{k|k} = \mathbb{E}\{\mathbf{e}_{k|k}\mathbf{e}_{k|k}^H\} = \mathbf{P}_{k|k}^b \\ \underbrace{+\,\mathbb{E}\{\mathbf{e}_{k|k}^b\boldsymbol{\epsilon}_k^H\} + \mathbb{E}\{\boldsymbol{\epsilon}_k(\mathbf{e}_{k|k}^b)^H\} + \mathbb{E}\{\boldsymbol{\epsilon}_k\boldsymbol{\epsilon}_k^H\}}_{\text{extra terms due to model mismatch}}, \quad (38)$$

$$\mathbf{P}_{k|k}^b = \mathbf{A}_k\mathbf{P}_{k|k-1}^b\mathbf{A}_k^H + \mathbf{L}_k\mathbf{C}_{\mathbf{v}_k}\mathbf{L}_k^H, \\ \mathbf{P}_{k|k-1}^b = \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}^b\mathbf{F}_{k-1}^H + \mathbf{C}_{\mathbf{w}_{k-1}},$$

The extra covariance terms due to modelling mismatch are

$$\mathbb{E}\{\boldsymbol{\epsilon}_k\boldsymbol{\epsilon}_k^H\} = \mathbf{D}_k\mathbb{E}\{\mathbf{x}_{k-1}\mathbf{x}_{k-1}^H\}\mathbf{D}_k^H + \mathbf{L}_k d\mathbf{H}_k\mathbf{C}_{\mathbf{w}_{k-1}}d\mathbf{H}_k^H\mathbf{L}_k^H \\ = \mathbf{D}_k(\mathbf{C}_{k-1} + \mathbf{m}_{\mathbf{x}_{k-1}}\mathbf{m}_{\mathbf{x}_{k-1}}^H)\mathbf{D}_k^H + \mathbf{L}_k d\mathbf{H}_k\mathbf{C}_{\mathbf{w}_{k-1}}d\mathbf{H}_k^H\mathbf{L}_k^H, \quad (39)$$

$$\mathbb{E}\{\mathbf{e}_{k|k}^b\boldsymbol{\epsilon}_k^H\} = \mathbf{A}_k\mathbf{F}_{k-1}\mathbb{E}\{\mathbf{e}_{k-1|k-1}\mathbf{x}_{k-1}^H\}\mathbf{D}_k^H \\ - \mathbf{A}_k\mathbf{C}_{\mathbf{w}_{k-1}}d\mathbf{H}_k^H\mathbf{L}_k^H \\ = \mathbf{A}_k(\mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}^b\mathbf{D}_k^H - \mathbf{C}_{\mathbf{w}_{k-1}}d\mathbf{H}_k^H\mathbf{L}_k^H), \quad (40)$$

where we used the orthogonality condition of the optimal solution, $\mathbb{E}\{(\hat{\mathbf{x}}_{k-1|k-1} - \mathbf{x}_{k-1})\hat{\mathbf{x}}_{k-1|k-1}^H\} = \mathbf{0}$. Then we can rearrange $\mathbf{P}_{k|k} = \mathbf{P}_{k|k}^b + \mathbf{P}_{e,k}$ to get

$$\mathbf{P}_{e,k} = \mathbf{D}_k(\mathbf{C}_{k-1} + \mathbf{m}_{\mathbf{x}_{k-1}}\mathbf{m}_{\mathbf{x}_{k-1}}^H)\mathbf{D}_k^H \\ + 2\mathbf{A}_k\mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}^b\mathbf{D}_k^H \\ + (\mathbf{L}_k d\mathbf{H}_k - 2\mathbf{A}_k)\mathbf{C}_{\mathbf{w}_{k-1}}d\mathbf{H}_k^H\mathbf{L}_k^H, \quad (41)$$

where we have three distinct contributions on the error term. Again, notice that this covariance error term depends on the covariance and mean values of $\mathbf{x}_{k-1}$.

- From k to $k+l$:

In general we have that $\mathbf{P}_{k+l|k+l} = \mathbf{P}_{k+l|k+l}^b + \mathbf{P}_{e,k+l}$, then the goal is to find the recursive expression for $\mathbf{P}_{e,k+l}$, taking into account that the orthogonality condition is no longer verified, $\mathbb{E}\{(\hat{\mathbf{x}}_{k+l|k+l} - \mathbf{x}_{k+l})\hat{\mathbf{x}}_{k+l|k+l}^H\} \neq \mathbf{0}$. After some manipulations, the error term is computed as

$$\mathbf{P}_{e,k+l} = \sum_{i=k}^{k+l}\prod_{j=i}^{k+l-1}\mathbf{A}_{j+1}\mathbf{F}_j(2\boldsymbol{\Gamma}_i + \boldsymbol{\Delta}_i)\prod_{j=i}^{k+l-1}\mathbf{F}_j^H\mathbf{A}_{j+1}^H,$$

where

$$\boldsymbol{\Delta}_i = \mathbb{E}\{\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^H\} = \mathbf{D}_i(\mathbf{C}_{i-1} + \mathbf{m}_{\mathbf{x}_{i-1}}\mathbf{m}_{\mathbf{x}_{i-1}}^H)\mathbf{D}_i^H \\ + \mathbf{L}_i d\mathbf{H}_i\mathbf{C}_{\mathbf{w}_{i-1}}d\mathbf{H}_i^H\mathbf{L}_i^H, \\ \boldsymbol{\Gamma}_i = \mathbf{A}_i\mathbf{F}_{i-1}\mathbb{E}\{\mathbf{e}_{i-1|i-1}\mathbf{x}_{i-1}^H\}\mathbf{D}_i^H - \mathbf{A}_i\mathbf{C}_{\mathbf{w}_{i-1}}d\mathbf{H}_i^H\mathbf{L}_i^H,$$

and the recursive error term computation at time k, assuming that a possible mismatch is present at any time $< k$ is

$$\mathbf{P}_{e,k} = \mathbf{A}_k\mathbf{F}_{k-1}\mathbf{P}_{e,k-1}\mathbf{F}_{k-1}^H\mathbf{A}_k^H + 2\boldsymbol{\Gamma}_k + \boldsymbol{\Delta}_k. \quad (42)$$

The system model mismatch for the general recursive LMMSE estimator is not considered in this contribution, mainly because the analytical expressions derived above are only useful in some specific cases, that is, where both state covariance and mean values are known, and the computation of the term $\mathbb{E}\{\mathbf{e}_{i-1|i-1}\mathbf{x}_{i-1}^H\}$ is available. An illustrative example for model mismatch is shown in next Section V.

V. ILLUSTRATIVE EXAMPLES

A. Case 1: Noise Statistics Mismatch

To illustrate the validity of the derivation provided in Section II, a scalar autoregressive (AR) process estimation example is considered. The mismatched LDSS is given by

$$\tilde{x}_k = \alpha\tilde{x}_{k-1} + \tilde{w}_k,\ \tilde{w}_k \sim \mathcal{N}(0, \tilde{Q}) \\ \tilde{y}_k = \tilde{x}_k + \tilde{v}_k,\ \tilde{v}_k \sim \mathcal{N}(0, \tilde{R})$$

with $F = \alpha = 0.9$, $H = 1$, $\tilde{Q} = 0.5$ and $\tilde{R} = 1$. The simulated bias and covariance are obtained from $1e3$ Monte Carlo runs.

- Estimator bias: the true LDSS model has non-zero mean noises, with $m_w = 3$ and $m_v = 1$. The results obtained are shown in Fig. 1 (top), where the theoretical bias computed from (24) coincides with the simulated one.
- Estimation error covariance (MSE): the true LDSS has zero-mean noises but $Q = \tilde{Q} + dQ$, $R = \tilde{R} + dR$, $dQ = -0.2\tilde{Q}$ and $dR = -0.8\tilde{R}$, which implies that the mismatched LDSS considers overestimated system noise covariances. The results obtained are shown in Fig. 1 (bottom), where the theoretical MSE computed from $\mathbf{P}_{k|k} = \mathbf{P}_{k|k}^b + \mathbf{P}_{e,k}$ and $\mathbf{P}_{e,k}$ in (28), coincides with the simulated one.

In the case of noise statistics uncertainty, if a rough knowledge of the system noise first and second order moments is available, the expressions provided in Section II and III enable to predict the maximum performance degradation with respect to the optimal LMMSE estimator.

B. Case 2: System Model Mismatch

A simple array processing example is considered to illustrate the system model mismatch case. A fully coherent random Gaussian complex circular source, i.e., $x_k = x_{k-1}$, with unit variance, $C_{x_k} = 1$, is received with a uniform linear array of N equally-spaced sensors.

The mismatched observation model is

$$\mathbf{y}_k = \mathbf{H}(\hat{d}, \alpha)x_k + \mathbf{v}_k, \quad (43)$$

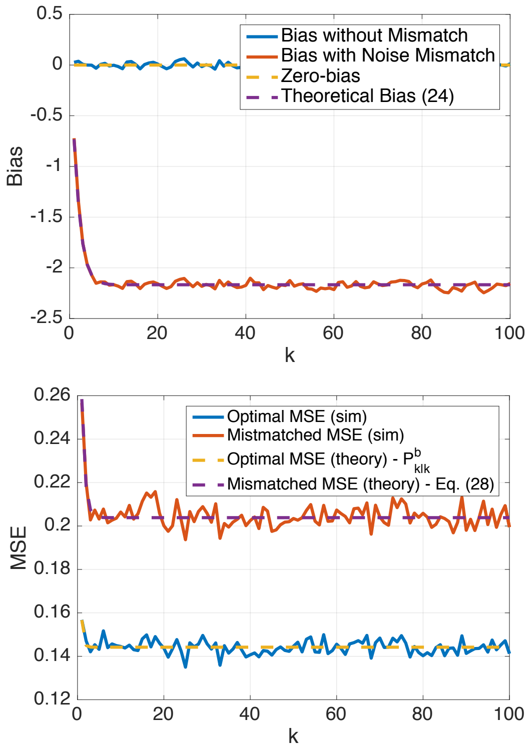

Fig. 1. Simulated and theoretical estimation bias (top) and MSE (bottom), with and without noise statistics mismatch.

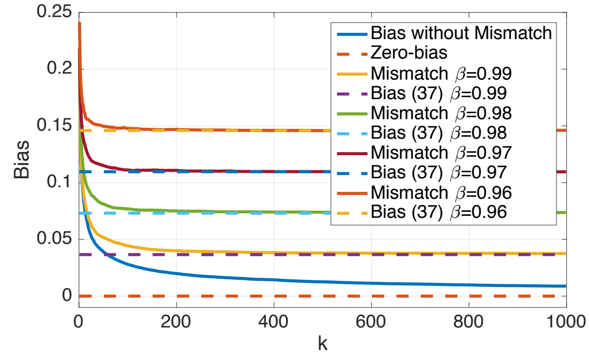

Fig. 2. Simulated and theoretical bias with and without model mismatch, for $N = 10$ sensors, broadside angle $\alpha = 15°$ and different values of β.

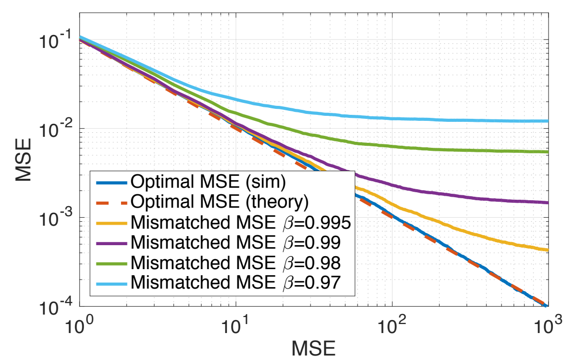

Fig. 3. Simulated MSE with and without model mismatch, considering $N = 10$ sensors, broadside angle $\alpha = 15°$ and different values of β.

where $\hat{d} = \lambda/2$ is the sensor spacing, α is the broadside angle of the impinging signal x_k, $\mathbf{v}_k$ is spatially and temporally white noise, $C_{\mathbf{v}_k} = \mathbf{I}$, and

$$\mathbf{H}(d, \alpha) = \left[1, e^{j2\pi \frac{d}{\lambda} sin(\alpha)}, \ldots, e^{j2\pi (N-1) \frac{d}{\lambda} sin(\alpha)}\right]^{\top}. \quad (44)$$

Due to a possible calibration error or array sensors aging, there exists a small inter-sensor spacing error, then the true LDSS is given by (43) but with $d = \beta \hat{d}$, $\beta < 1$. Notice that the impact on the estimation depends on the number of sensors N, the impinging angle α and the mismatch in d. In the following results, the simulated bias and covariance are obtained from $1e3$ Monte Carlo runs.

Both simulated and theoretical estimator bias computed from (37), for different values of the inter-sensor spacing mismatch β, are shown in Fig. 2, where $N = 10$ sensors and a broadside angle $\alpha = 15°$ were considered. Notice that while the optimal estimator bias tends to zero when more samples are considered, the mismatched estimator tends to a non-zero bias. It is remarkable that a very small model mismatch has a nonmarginal impact on the estimator performance.

Notice that in this example the term $\mathbb{E}\{\mathbf{e}_{i-1|i-1}\mathbf{x}_{i-1}^H\}$ is not available, then the analytical mismatched MSE cannot be computed. Anyway, for completeness, Fig. 3 shows the impact of model mismatch into the MSE. Again, a very small error severely degrades the estimator performance, and while the optimal MSE tends to zero, the mismatched MSE tends to a constant value in the large sample regime.

VI. Conclusions

In this contribution, we considered linear discrete state-space models with a possible model or system noise statistics mismatch. The impact on linear minimum mean square error estimators' performance due to such model/noise mismatch were theoretically analysed and analytical expressions on the bias and mean square error performance loss were provided. These equations can be useful in assessing the estimation performance degradation in uncertain (to a certain extent) environments, as discussed through illustrative examples.

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