Recursive LCMVEs with Non-Stationary Constraints and Partially Coherent Signal Sources

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***Abstract*—In estimating an unknown parameter vector in a linear regression model, it is common to use linearly constrained minimum variance estimators (LCMVEs). For a long time, LCMVEs were studied in the context of stationary constraints under both stationary or non-stationary environments. Recently, a new family of non-stationary constraints leading to Kalman-like recursive LCMVEs has been introduced for fully coherent signal (FCS) sources. A noteworthy feature of this family is to allow the possibility of, at each new observation, incorporating new constraints. This article extends these results to the case of partially coherent signal (PCS) sources. Indeed, without ad hoc modifications of the Kalman-like recursion, estimation of the amplitudes of PCS sources exhibit a performance breakdown even for a slight loss of coherence. Last but not least, it is shown that PCS sources introduce a lower limit in the achievable performance in the large sample regime.**

I. INTRODUCTION

In the signal processing literature, particularly that dealing with parameter estimation, one of the most studied estimation problems is the identification of the components of an N-dimensional complex observation vector ($\mathbf{y}$), being a linear superposition of P individual complex signals ($\mathbf{x}$) and complex noisy data ($\mathbf{v}$). That is, $\mathbf{y} = \mathbf{H}\mathbf{x}+\mathbf{v}$, also known as linear regression, where $\mathbf{H} \in \mathbb{C}^{N\times P}$ and $\mathbf{v} \in \mathbb{C}^{N}$. The importance of this problem stems from the fact that a wide range of problems in communications, array processing, and many other areas can be recast in this form [1], [2]. In many practical problems: a) $\mathbf{v}$ is zero mean; b) $\mathbf{x}$ and $\mathbf{v}$ are uncorrelated; c) both $\mathbf{H}$ and the noise covariance matrix $\mathbf{C_v}$ are either known or specified according to predefined parametric models. In this case, the weighted least squares estimator of $\mathbf{x}$ given $\mathbf{y}$ [1] is[1]

$$\begin{aligned}\widehat{\mathbf{x}}^b &= \arg\min_{\mathbf{x}} \left\{ (\mathbf{y}-\mathbf{H}\mathbf{x})^H \mathbf{C}_\mathbf{v}^{-1} (\mathbf{y}-\mathbf{H}\mathbf{x}) \right\} \\ &= \left(\mathbf{H}^H\mathbf{C}_\mathbf{v}^{-1}\mathbf{H}\right)^{-1}\mathbf{C}_\mathbf{v}^{-1}\mathbf{H}^H\mathbf{y},\end{aligned} \tag{1a}$$

which coincides with the minimum variance distortionless response estimator (MVDRE) [1]–[3], $\widehat{\mathbf{x}}^b = (\mathbf{W}^b)^H\mathbf{y}$, where

$$\begin{aligned}\mathbf{W}^b &= \arg\min_{\mathbf{W}} \left\{\mathbf{W}^H\mathbf{C}_\mathbf{v}\mathbf{W}\right\} \ \text{ s.t. } \mathbf{W}^H\mathbf{H} = \mathbf{I} \\ &= \mathbf{C}_\mathbf{v}^{-1}\mathbf{H}\left(\mathbf{H}^H\mathbf{C}_\mathbf{v}^{-1}\mathbf{H}\right)^{-1},\end{aligned} \tag{1b}$$

and with the maximum-likelihood estimator [1], if $\mathbf{x}$ is deterministic and $\mathbf{v}$ is Gaussian. However, it is well known that the MVDRE achievable performance strongly depends on the accurate knowledge on the parametric model of the observations, that is, $\mathbf{H}$ and $\mathbf{C_v}$ [1, § 6.7]. Linearly constrained minimum variance estimators (LCMVEs) [3]–[5], where additional linear constraints are imposed, have been developed to make MVDREs more robust [1, § 6.7] [6],

$$\begin{aligned}\mathbf{W}^b &= \arg\min_{\mathbf{W}} \left\{\mathbf{W}^H\mathbf{C}_\mathbf{v}\mathbf{W}\right\} \text{s.t.} \mathbf{W}^H \left[\mathbf{H}\ \boldsymbol{\Omega}\right] = \left[\mathbf{I}\ \boldsymbol{\Upsilon}\right] \\ &= \mathbf{C}_\mathbf{v}^{-1}\left[\mathbf{H}\ \boldsymbol{\Omega}\right]\left(\left[\mathbf{H}\ \boldsymbol{\Omega}\right]^H\mathbf{C}_\mathbf{v}^{-1}\left[\mathbf{H}\ \boldsymbol{\Omega}\right]\right)^{-1}\left[\mathbf{I}\ \boldsymbol{\Upsilon}\right]^H,\end{aligned} \tag{2}$$

where $\boldsymbol{\Omega}$ and $\boldsymbol{\Upsilon}$ are known matrices of the appropriate dimensions. However, since additional degrees of freedom are used by LCMVEs (2) in order to satisfy these constraints, this implies an increase of the achievable minimum mean squared error (MSE). Moreover, $\mathbf{C_v}$ may be unknown and must be learned by an adaptive technique. Remarkably, if $\mathbf{x}$ and $\mathbf{v}$ are uncorrelated, $\mathbf{C_v}$ can be replaced by $\mathbf{C_y}$ in (1a), (1b), (2), which means that either $\mathbf{C_v}$ can be learned from auxiliary data containing noise only, if available, or $\mathbf{C_y}$ can be used instead and learned from the observations. Therefore, when several observations are available, $\mathbf{y}_l \in \mathbb{C}^{N_l}$, $1 \leq l \leq k$, recursive adaptive implementations of the LCMVE are available resorting to constrained stochastic gradient [3], constrained recursive least squares [7], [8] and constrained Kalman-type [9] algorithms. The equivalence between the LCMVE and the generalized side lobe canceller [4], [5] allows to resort also on standard stochastic gradient or recursive least squares [2]. However, these recursive algorithms allow to sequentially update the LCMVE (2) in non-stationary environments (i.e. when the observation model changes over time),

$$\mathbf{y}_l = \mathbf{H}_l\mathbf{x}_l + \mathbf{v}_l, \quad 1 \leq l \leq k, \tag{3}$$

only for a given set of linear constraints $\mathbf{W}_l^H \left[\mathbf{H}\ \boldsymbol{\Omega}\right] = \left[\mathbf{I}\ \boldsymbol{\Upsilon}\right]$ [2], [3], [7]–[9]. This leads to the set of recursive LCMVEs for stationary constraints.

On another note, in presence of FCS sources, i.e. $\mathbf{x}_l = \mathbf{x}$, one can concatenate the available observations (3) to obtain an augmented observation model[2] of size $\mathcal{N}_k$, $\mathcal{N}_k = \sum_{l=1}^{k} N_l$,

$$\overline{\mathbf{y}}_k = \overline{\mathbf{H}}_k\mathbf{x} + \overline{\mathbf{v}}_k, \quad \overline{\mathbf{y}}_k, \overline{\mathbf{v}}_k \in \mathbb{C}^{\mathcal{N}_k}, \quad \overline{\mathbf{H}}_k \in \mathbb{C}^{\mathcal{N}_k\times P}. \tag{4a}$$

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[1]The superscript ^{b} is used to remind the reader that the value under consideration is the "best" one according to a given criterion.

[2]Throughout the present communication, the vector resulting from the vertical concatenation of k vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$ is denoted $\overline{\mathbf{a}}_k$. The matrix resulting from the vertical concatenation of k matrices $\mathbf{A}_1, \ldots, \mathbf{A}_k$ of same column number is denoted $\overline{\mathbf{A}}_k$. $[\mathbf{A}\ \mathbf{B}]$ and $\left[\begin{smallmatrix}\mathbf{A}\\ \mathbf{B}\end{smallmatrix}\right]$ denotes respectively the matrix resulting from the horizontal and the vertical concatenation of $\mathbf{A}$ and $\mathbf{B}$. $E\left[\cdot\right]$ denotes the expectation operator.

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Then, provided that the additive noise sequence $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is temporally uncorrelated, authors in [11] have recently introduced the family of recursive LCMVE with non-stationary constraints associated to (4a), that is $\widehat{\mathbf{x}}_k^b = (\overline{\mathbf{W}}_k^b)^H \overline{\mathbf{y}}_k$ where

$$\overline{\mathbf{W}}_k^b = \arg\min_{\overline{\mathbf{W}}_k} \left\{ \overline{\mathbf{W}}_k^H \mathbf{C}_{\overline{\mathbf{v}}_k} \overline{\mathbf{W}}_k \right\} \text{ s.t. } \overline{\mathbf{W}}_k^H \overline{\boldsymbol{\Lambda}}_k = \boldsymbol{\Gamma}_k, \quad (4b)$$

$\overline{\boldsymbol{\Lambda}}_k = \left[\overline{\mathbf{H}}_k \ \overline{\boldsymbol{\Omega}}_k\right]$, $\boldsymbol{\Gamma}_k = [\mathbf{I} \ \boldsymbol{\Upsilon}_k]$, which can be computed according to a Kalman-like recursion [10, §1],

$$\widehat{\mathbf{x}}_k^b = \widehat{\mathbf{x}}_{k-1}^b + \mathbf{W}_k^{bH} \left(\mathbf{y}_k - \mathbf{H}_k \widehat{\mathbf{x}}_{k-1}^b \right). \quad (4c)$$

A particularly noteworthy feature of the recursive formulation introduced in [11] is that it is a fully adaptive sequential estimator, allowing for the incorporation of additional constraints at each step. The addition can be, for instance, triggered by a preprocessing of each new observation or via external information on the environment.

The primary goal of this communication is to extend the results derived in [11] to the case of non FCS sources, i.e. when $\mathbf{x}_k \neq \mathbf{x}_1$. Indeed, in real-life applications different experimental factors may prevent from observing FCS sources (see Section III). In that sense, we address the case of partially coherent signal (PCS) sources where their amplitudes undergo a partial random walk between observations,

$$\mathbf{x}_1 = \mathbf{x}, \ \ \mathbf{x}_l = \mathbf{F}_{l-1}\mathbf{x}_{l-1} + \mathbf{w}_{l-1}, \ \ \mathbf{x}_1, \mathbf{x}_l, \mathbf{w}_{l-1} \in \mathbb{C}^P, \quad (5a)$$

and the random noise sequence $\{\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}\}$ is zero-mean, temporally white and uncorrelated with both $\mathbf{x}_1$ and the measurement noise sequence $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$,

$$\mathbf{C}_{\mathbf{w}_l, \mathbf{w}_k} = \mathbf{C}_{\mathbf{w}_k} \delta_k^l, \ \mathbf{C}_{\mathbf{x}_1, \mathbf{w}_k} = \mathbf{0}, \ \mathbf{C}_{\mathbf{w}_l, \mathbf{v}_k} = \mathbf{0}. \quad (5b)$$

The main merits of the amplitude fluctuation model (5a) are both its simplicity and its capability to model most cases of PCS amplitudes, including the situation where $\mathbf{C}_{\mathbf{x}_l}$ is invariant, i.e. $\mathbf{C}_{\mathbf{x}_l} = \mathbf{C}_{\mathbf{x}_1}$, with a tunable correlation matrix $\mathbf{C}_{\mathbf{x}_l, \mathbf{x}_{l-1}}$ between observations, which is obtained by setting $\mathbf{F}_{l-1} = \mathbf{C}_{\mathbf{x}_l, \mathbf{x}_{l-1}} \mathbf{C}_{\mathbf{x}_1}^{-1}$ and $\mathbf{C}_{\mathbf{w}_{l-1}} = \mathbf{C}_{\mathbf{x}_1} - \mathbf{F}_{l-1} \mathbf{C}_{\mathbf{x}_1} \mathbf{F}_{l-1}^H$.

The significance of the derivation of a recursive LCMVE for PCS sources is twofold: 1) if the parameters $\left(\mathbf{F}_{l-1}, \mathbf{C}_{\mathbf{w}_{l-1}}\right)$ of the fluctuation model (5a) are known, it allows for a recursive computation of the optimal estimate $\widehat{\mathbf{x}}_k^b$ of the amplitude $\mathbf{x}_k$ of the sources and its covariance error matrix; and 2) if the parameters $\left(\mathbf{F}_{l-1}, \mathbf{C}_{\mathbf{w}_{l-1}}\right)$ are unknown, it allows to perform a parametric study of the robustness of the LCMVE for FCS sources (4a) against partial coherency. This can be done by comparing its performance assessed via Monte-Carlo simulations and the best performance achievable for each likely (or possible) value of the parameters $\left(\mathbf{F}_{l-1}, \mathbf{C}_{\mathbf{w}_{l-1}}\right)$.

II. Recursive LCMVEs for PCS Sources

In this section, we consider a completely different approach to the one previously taken in [11]. Indeed, we provide a general definition – see (8) – of a distortionless linear filter/estimator of the amplitude $\mathbf{x}_k$ of the sources at each time index k, which encompasses the usual definition used in [11, (6)] expressed by the non-stationary constraints $\overline{\mathbf{W}}^H \overline{\mathbf{H}}_k = \mathbf{I}$. The combination of this general definition and an insightful breakdown of the variance of the output equivalent noise (10a-b) allows to prove that the LCMVE with non-stationary constraints and PCS sources can also be computed according to a Kalman-like recursion (4c).

A. Equivalent observation model

Since the partial random walk (5a) of the individual signals $\mathbf{x}_1$ can be recast as, $2 \leq l \leq k$,

$$\mathbf{x}_l = \mathbf{B}_{l,1}\mathbf{x}_1 + \mathbf{G}_l \overline{\mathbf{w}}_{l-1}, \ \mathbf{G}_l \overline{\mathbf{w}}_{l-1} = \sum_{i=1}^{l-1} \mathbf{B}_{l,i+1} \mathbf{w}_i,$$

$$\mathbf{G}_l \in \mathbb{C}^{P \times (l-1)P}, \ \mathbf{B}_{l,i} = \left| \begin{array}{cl} \mathbf{F}_{l-1}\mathbf{F}_{l-2}...\mathbf{F}_i, & l > i \\ \mathbf{I} & , l = i \\ \mathbf{0} & , l < i \end{array} \right. ,$$

the observation model (3) becomes

$$\mathbf{y}_l = \mathbf{A}_l \mathbf{x}_1 + \mathbf{n}_l, \ \mathbf{A}_l = \mathbf{H}_l \mathbf{B}_{l,1}, \ \left| \begin{array}{l} \mathbf{n}_1 = \mathbf{v}_1 \\ \mathbf{n}_{l \geq 2} = \mathbf{v}_l + \mathbf{H}_l \mathbf{G}_l \overline{\mathbf{w}}_{l-1} \end{array} \right.$$

leading to the updated augmented observation model

$$\overline{\mathbf{y}}_k = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_k \end{pmatrix} = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_k \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{n}_2 \\ \vdots \\ \mathbf{n}_k \end{bmatrix} = \overline{\mathbf{A}}_k \mathbf{x}_1 + \overline{\mathbf{n}}_k.$$

B. Problem statement

Let $\overline{\mathbf{W}}_k = \left[\begin{smallmatrix} \overline{\mathbf{D}}_{k-1} \\ \mathbf{W}_k \end{smallmatrix}\right]$, $\overline{\mathbf{D}}_{k-1} \in \mathbb{C}^{\mathcal{N}_{k-1} \times P}$, $\mathbf{W}_k \in \mathbb{C}^{N_k \times P}$. As:

$$\overline{\mathbf{W}}_k^H \overline{\mathbf{y}}_k = \left(\left(\overline{\mathbf{W}}_k^H \overline{\mathbf{A}}_k \right) \mathbf{x}_1 + \mathbf{G}_k \overline{\mathbf{w}}_{k-1} \right) + \overline{\mathbf{W}}_k^H \overline{\mathbf{n}}_k - \mathbf{G}_k \overline{\mathbf{w}}_{k-1}, \quad (7)$$

a linear filter $\overline{\mathbf{W}}_k$ is distortionless iff

$$\overline{\mathbf{W}}_k^H \overline{\mathbf{A}}_k = \mathbf{B}_{k,1} \Leftrightarrow \overline{\mathbf{W}}_k^H \overline{\mathbf{y}}_k = \mathbf{x}_k + \overline{\mathbf{W}}_k^H \overline{\mathbf{n}}_k - \mathbf{G}_k \overline{\mathbf{w}}_{k-1}. \quad (8)$$

The aim is to look for the family of linear constraints

$$\overline{\mathbf{W}}_k^H \overline{\boldsymbol{\Lambda}}_k = \boldsymbol{\Gamma}_k, \ \overline{\boldsymbol{\Lambda}}_k = \left[\overline{\mathbf{A}}_k \ \overline{\boldsymbol{\Omega}}_k \right], \ \boldsymbol{\Gamma}_k = \left[\mathbf{B}_{k,1} \ \boldsymbol{\Upsilon}_k \right], \quad (9a)$$

yielding a LCMVE, $\widehat{\mathbf{x}}_k^b = \overline{\mathbf{W}}_k^{bH} \overline{\mathbf{y}}_k$, where

$$\overline{\mathbf{W}}_k^b = \arg\min_{\overline{\mathbf{W}}_k} \left\{ \mathbf{P}_k \left(\overline{\mathbf{W}}_k \right) \right\} \text{ s.t. } \overline{\mathbf{W}}_k^H \overline{\boldsymbol{\Lambda}}_k = \boldsymbol{\Gamma}_k, \quad (9b)$$

$$\mathbf{P}_k \left(\overline{\mathbf{W}}_k \right) = E\left[\widehat{\mathbf{r}}_k \widehat{\mathbf{r}}_k^H \right], \ \widehat{\mathbf{r}}_k = \overline{\mathbf{W}}_k^H \overline{\mathbf{n}}_k - \mathbf{G}_k \overline{\mathbf{w}}_{k-1},$$

where $\mathbf{P}_k \left(\overline{\mathbf{W}}_k \right)$ is the variance of the output equivalent noise, which can be computed recursively as (4c), or equivalently as,

$$\widehat{\mathbf{x}}_k^b = \left(\mathbf{I} - \mathbf{W}_k^{bH} \mathbf{H}_k \right) \widehat{\mathbf{x}}_{k-1}^b + \mathbf{W}_k^{bH} \mathbf{y}_k. \quad (9c)$$

Indeed, since $\overline{\mathbf{W}}_k^b$ in (9b) is analogous to a linearly constrained Wiener filter [2, §2.5], its "batch" form is given by [2, §2]

$$\overline{\mathbf{W}}_k^b = \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\boldsymbol{\Lambda}}_k \left(\overline{\boldsymbol{\Lambda}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\boldsymbol{\Lambda}}_k \right)^{-1} \boldsymbol{\Gamma}_k^H + \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \left(\mathbf{I} - \overline{\boldsymbol{\Lambda}}_k \left(\overline{\boldsymbol{\Lambda}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \overline{\boldsymbol{\Lambda}}_k \right)^{-1} \overline{\boldsymbol{\Lambda}}_k^H \mathbf{C}_{\overline{\mathbf{n}}_k}^{-1} \right) \mathbf{C}_{\overline{\mathbf{n}}_k, \mathbf{G}_k \overline{\mathbf{w}}_{k-1}}, \quad (9d)$$

which reduces to $\overline{\mathbf{W}}_k^b = \mathbf{C}_{\overline{\mathbf{v}}_k}^{-1}\overline{\boldsymbol{\Lambda}}_k\left(\overline{\boldsymbol{\Lambda}}_k^H\mathbf{C}_{\overline{\mathbf{v}}_k}^{-1}\overline{\boldsymbol{\Lambda}}_k\right)^{-1}\boldsymbol{\Gamma}_k^H$, $\overline{\boldsymbol{\Lambda}}_k = \left[\overline{\mathbf{H}}_k\ \overline{\boldsymbol{\Omega}}_k\right]$, $\boldsymbol{\Gamma}_k = [\mathbf{I}\ \boldsymbol{\Upsilon}_k]$, when $\overline{\mathbf{w}}_{k-1} = \mathbf{0}$ [11, (7b)]. Nevertheless, the "batch" form (9d) is hardly likely to be computable as the size of $\overline{\mathbf{n}}_k$ increases.

C. Recursive LCMVEs

Let $\mathbf{P}_k\left(\overline{\mathbf{W}}_k\right) = \mathbf{P}_k\left(\overline{\mathbf{D}}_{k-1}, \mathbf{W}_k\right)$. Then a key point to solve the problem at hand is to notice that under the assumptions in (5b) we have that

$$\mathbf{P}_k\left(\overline{\mathbf{D}}_{k-1}, \mathbf{W}_k\right) = \mathbf{Q}_{k-1}\left(\overline{\mathbf{D}}_{k-1}, \mathbf{W}_k\right) + \left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)\mathbf{C}_{\mathbf{w}_{k-1}}\left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)^H + \mathbf{W}_k^H\mathbf{C}_{\mathbf{v}_k}\mathbf{W}_k, \quad (10a)$$

where

$$\mathbf{Q}_{k-1}\left(\overline{\mathbf{D}}_{k-1}, \mathbf{W}_k\right) = E\left[\widehat{\mathbf{q}}_{k-1}\widehat{\mathbf{q}}_{k-1}^H\right], \quad \widehat{\mathbf{q}}_{k-1} = \overline{\mathbf{D}}_{k-1}^H\overline{\mathbf{n}}_{k-1} - \left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)\mathbf{F}_{k-1}\mathbf{G}_{k-1}\overline{\mathbf{w}}_{k-2} \quad (10b)$$

which reduces to [11, (9)]

$$\mathbf{P}_k\left(\overline{\mathbf{D}}_{k-1}, \mathbf{W}_k\right) = \mathbf{Q}_{k-1}\left(\overline{\mathbf{D}}_{k-1}, \mathbf{W}_k\right) + \mathbf{W}_k^H\mathbf{C}_{\mathbf{v}_k}\mathbf{W}_k, \quad \mathbf{Q}_{k-1}\left(\overline{\mathbf{D}}_{k-1}, \mathbf{W}_k\right) = \overline{\mathbf{D}}_{k-1}^H\mathbf{C}_{\overline{\mathbf{v}}_{k-1}}\overline{\mathbf{D}}_{k-1},$$

when $\overline{\mathbf{w}}_{k-1} = \mathbf{0}$.

• First step

If we recast $\overline{\boldsymbol{\Lambda}}_k = \left[\overline{\mathbf{A}}_k\ \overline{\boldsymbol{\Omega}}_k\right]$ as $\overline{\boldsymbol{\Lambda}}_k = \left[\begin{smallmatrix}\overline{\boldsymbol{\Phi}}_{k-1}\\ \boldsymbol{\Phi}_k\end{smallmatrix}\right]$ where $\overline{\boldsymbol{\Phi}}_{k-1} = \left[\overline{\mathbf{A}}_{k-1}\ \overline{\boldsymbol{\Omega}}_{k-1}\right]$ and $\boldsymbol{\Phi}_k = [\mathbf{A}_k\ \boldsymbol{\Omega}_k]$, then an equivalent form of (9a) is given by

$$\overline{\mathbf{W}}_k^H\overline{\boldsymbol{\Lambda}}_k = \boldsymbol{\Gamma}_k \ \Leftrightarrow\ \overline{\mathbf{D}}_{k-1}^H\overline{\boldsymbol{\Phi}}_{k-1} = \boldsymbol{\Gamma}_k - \mathbf{W}_k^H\boldsymbol{\Phi}_k. \quad (11)$$

Therefore, according to (10a),

$$\overline{\mathbf{D}}_{k-1}^b = \arg\min_{\overline{\mathbf{D}}_{k-1}}\left\{\mathbf{Q}_{k-1}\left(\overline{\mathbf{D}}_{k-1}, \mathbf{W}_k\right)\right\} \quad \text{s.t. } \overline{\mathbf{D}}_{k-1}^H\overline{\boldsymbol{\Phi}}_{k-1} = \boldsymbol{\Gamma}_k - \mathbf{W}_k^H\boldsymbol{\Phi}_k, \quad (12a)$$

that is, provided that $\overline{\boldsymbol{\Phi}}_{k-1}$ and $\mathbf{C}_{\overline{\mathbf{v}}_{k-1}}$ are full rank, (9d):

$$\overline{\mathbf{D}}_{k-1}^b = \mathbf{C}_{\overline{\mathbf{n}}_{k-1}}^{-1}\overline{\boldsymbol{\Phi}}_{k-1}\left(\overline{\boldsymbol{\Phi}}_{k-1}^H\mathbf{C}_{\overline{\mathbf{n}}_{k-1}}^{-1}\overline{\boldsymbol{\Phi}}_{k-1}\right)^{-1}\left(\boldsymbol{\Gamma}_k - \mathbf{W}_k^H\boldsymbol{\Phi}_k\right)^H + \mathbf{C}_{\overline{\mathbf{n}}_{k-1}}^{-1}\left(\mathbf{I} - \overline{\boldsymbol{\Phi}}_{k-1}\left(\overline{\boldsymbol{\Phi}}_{k-1}^H\mathbf{C}_{\overline{\mathbf{n}}_{k-1}}^{-1}\overline{\boldsymbol{\Phi}}_{k-1}\right)^{-1}\overline{\boldsymbol{\Phi}}_{k-1}^H\mathbf{C}_{\overline{\mathbf{n}}_{k-1}}^{-1}\right) \times \mathbf{C}_{\overline{\mathbf{n}}_{k-1},\mathbf{G}_{k-1}\overline{\mathbf{w}}_{k-2}}\mathbf{F}_{k-1}^H\left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)^H. \quad (12b)$$

It is remarkable that (12b) can be recast as

$$\overline{\mathbf{D}}_{k-1}^b = \overline{\mathbf{W}}_{k-1}^b\mathbf{F}_{k-1}^H\left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)^H + \mathbf{C}_{\overline{\mathbf{n}}_{k-1}}^{-1}\overline{\boldsymbol{\Phi}}_{k-1}\left(\overline{\boldsymbol{\Phi}}_{k-1}^H\mathbf{C}_{\overline{\mathbf{n}}_{k-1}}^{-1}\overline{\boldsymbol{\Phi}}_{k-1}\right)^{-1}\boldsymbol{\Theta}_{k-1}^H, \quad (12c)$$

where $\boldsymbol{\Theta}_{k-1} = \boldsymbol{\Gamma}_k - \mathbf{W}_k^H\boldsymbol{\Phi}_k - \left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)\mathbf{F}_{k-1}\boldsymbol{\Gamma}_{k-1}$ and

$$\overline{\mathbf{W}}_{k-1}^b = \arg\min_{\overline{\mathbf{W}}_{k-1}}\left\{\mathbf{P}_{k-1}\left(\overline{\mathbf{W}}_{k-1}\right)\right\} \quad \text{s.t. } \overline{\mathbf{W}}_{k-1}^H\overline{\boldsymbol{\Phi}}_{k-1} = \boldsymbol{\Gamma}_{k-1}. \quad (12d)$$

Thus, the LCMVE (9b) follows a Kalman-like recursion (9c) with separable solutions for $\overline{\mathbf{D}}_{k-1}$ and $\mathbf{W}_k$ iff, $\forall\mathbf{W}_k$

$$\overline{\mathbf{D}}_{k-1}^b = \overline{\mathbf{W}}_{k-1}^b\mathbf{F}_{k-1}^H\left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)^H \Leftrightarrow$$
$$\boldsymbol{\Theta}_{k-1} = \mathbf{0} = \boldsymbol{\Gamma}_k - \mathbf{W}_k^H\boldsymbol{\Phi}_k - \left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)\mathbf{F}_{k-1}\boldsymbol{\Gamma}_{k-1},$$

that is iff, $\forall\mathbf{W}_k$

$$\boldsymbol{\Gamma}_k - \mathbf{W}_k^H\boldsymbol{\Phi}_k = \left[\mathbf{B}_{k,1} - \mathbf{W}_k^H\mathbf{A}_k\ \boldsymbol{\Upsilon}_k - \mathbf{W}_k^H\boldsymbol{\Omega}_k\right] = \left(\mathbf{F}_{k-1} - \mathbf{W}_k^H\mathbf{H}_k\mathbf{F}_{k-1}\right)\boldsymbol{\Gamma}_{k-1},$$

which requires that $\boldsymbol{\Gamma}_{k-1} = [\mathbf{B}_{k-1,1}\ \boldsymbol{\Upsilon}_{k-1}]$, $\boldsymbol{\Upsilon}_k = \mathbf{F}_{k-1}\boldsymbol{\Upsilon}_{k-1}$, and $\boldsymbol{\Omega}_k = \mathbf{H}_k\mathbf{F}_{k-1}\boldsymbol{\Upsilon}_{k-1}$. Ergo, the LCMVE (9b) follows a Kalman-like recursion (9c) iff (12a),

$$\overline{\mathbf{D}}_{k-1}^H\overline{\boldsymbol{\Phi}}_{k-1} = \boldsymbol{\Gamma}_k - \mathbf{W}_k^H\boldsymbol{\Phi}_k = \left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)\mathbf{F}_{k-1}\left[\mathbf{B}_{k-1,1}\ \boldsymbol{\Upsilon}_{k-1}\right],$$

or equivalently, iff

$$\mathcal{C}_k^1 : \overline{\mathbf{W}}_k^H\begin{bmatrix}\overline{\mathbf{A}}_{k-1} & \overline{\boldsymbol{\Omega}}_{k-1}\\ \mathbf{A}_k & \mathbf{H}_k\mathbf{F}_{k-1}\boldsymbol{\Upsilon}_{k-1}\end{bmatrix} = \left[\mathbf{B}_{k,1}\ \mathbf{F}_{k-1}\boldsymbol{\Upsilon}_{k-1}\right], \quad (13)$$

and (12d) amounts to the following

$$\overline{\mathbf{W}}_{k-1}^b = \arg\min_{\overline{\mathbf{W}}_{k-1}}\left\{\mathbf{P}_{k-1}\left(\overline{\mathbf{W}}_{k-1}\right)\right\} \quad \text{s.t. } \overline{\mathbf{W}}_{k-1}^H\left[\overline{\mathbf{A}}_{k-1}\ \overline{\boldsymbol{\Omega}}_{k-1}\right] = \left[\mathbf{B}_{k-1,1}\ \boldsymbol{\Upsilon}_{k-1}\right], \quad (14)$$

which means that $\widehat{\mathbf{x}}_{k-1}^b = \overline{\mathbf{W}}_{k-1}^{bH}\overline{\mathbf{y}}_{k-1}$ is an LCMVE as well. The specific form of (13) reflects the fact that the linear constraints at time $k-1$ in (14) propagates at time k via $\mathcal{C}_k^1$ in (13). Interestingly, additional linear constraints on $\mathbf{W}_k$ can be introduced on-line as shown in a second step.

• Second step

Let us notice that $\mathbf{P}_k\left(\overline{\mathbf{D}}_{k-1}^b, \mathbf{W}_k\right)$ (10a) can be reformulated as,

$$\mathbf{P}_k\left(\overline{\mathbf{D}}_{k-1}^b, \mathbf{W}_k\right) = \left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)\mathbf{F}_{k-1}\mathbf{P}_{k-1}^b\mathbf{F}_{k-1}^H\left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)^H + \left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)\mathbf{C}_{\mathbf{w}_{k-1}}\left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)^H + \mathbf{W}_k^H\mathbf{C}_{\mathbf{v}_k}\mathbf{W}_k,$$

where $\mathbf{P}_k^b = \mathbf{P}_k\left(\overline{\mathbf{W}}_k^b\right)$, or equivalently,

$$\mathbf{P}_k\left(\overline{\mathbf{D}}_{k-1}^b, \mathbf{W}_k\right) = \mathbf{W}_k^H\mathbf{C}_{\mathbf{v}_k}\mathbf{W}_k + \left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)\mathbf{U}_{k-1}^b\left(\mathbf{I} - \mathbf{W}_k^H\mathbf{H}_k\right)^H, \quad (15)$$

where $\mathbf{U}_{k-1}^b = \mathbf{F}_{k-1}\mathbf{P}_{k-1}^b\mathbf{F}_{k-1}^H + \mathbf{C}_{\mathbf{w}_{k-1}}$, which is analogue to [11, (16)], provided that one replaces $\mathbf{P}_{k-1}^b$ with $\mathbf{U}_{k-1}^b$.

Therefore two cases are possible:

1) no additional linear constraints on $\mathbf{W}_k$ are introduced. In that case, as shown at the first step, the LCMVE only propagates at time k the existing linear constraints at time $k-1$ (14) via $\mathcal{C}_k^1$ (13).

Then the solution of $\mathbf{W}_k^b = \arg\min_{\mathbf{W}_k} \left\{ \mathbf{P}_k \left(\overline{\mathbf{D}}_{k-1}^b, \mathbf{W}_k \right) \right\}$, is well known and given by [10, §1],

$$\widehat{\mathbf{x}}_k^b = \widehat{\mathbf{x}}_{k-1}^b + \mathbf{W}_k^{bH} \left(\mathbf{y}_k - \mathbf{H}_k \widehat{\mathbf{x}}_{k-1}^b \right), \quad (16a)$$
$$\mathbf{U}_{k-1}^b = \mathbf{F}_{k-1} \mathbf{P}_{k-1}^b \mathbf{F}_{k-1}^H + \mathbf{C}_{\mathbf{w}_{k-1}}, \quad (16b)$$
$$\mathbf{S}_k = \mathbf{H}_k \mathbf{U}_{k-1}^b \mathbf{H}_k^H + \mathbf{C}_{\mathbf{v}_k}, \quad (16c)$$
$$\mathbf{W}_k^b = \mathbf{S}_k^{-1} \mathbf{H}_k \mathbf{U}_{k-1}^b, \quad (16d)$$
$$\mathbf{P}_k^b = \left(\mathbf{I} - \mathbf{W}_k^{bH} \mathbf{H}_k \right) \mathbf{U}_{k-1}^b. \quad (16e)$$

2) additional linear constraints on $\mathbf{W}_k$, i.e. $\mathbf{W}_k^H \mathbf{\Delta}_k = \mathbf{T}_k$, are introduced on-line and (13) must be updated to take them into account, leading to,

$$\mathcal{C}_k^2 : \overline{\mathbf{W}}_k^H \begin{bmatrix} \overline{\mathbf{A}}_{k-1} & \overline{\mathbf{\Omega}}_{k-1} & \mathbf{0} \\ \mathbf{A}_k & \mathbf{H}_k \mathbf{F}_{k-1} \mathbf{\Upsilon}_{k-1} & \mathbf{\Delta}_k \end{bmatrix} = \left[\mathbf{B}_{k,1} \ \mathbf{F}_{k-1} \mathbf{\Upsilon}_{k-1} \ \mathbf{T}_k \right]. \quad (17)$$

$\mathbf{W}_k^b = \arg\min_{\mathbf{W}_k} \left\{ \mathbf{P}_k \left(\overline{\mathbf{D}}_{k-1}^b, \mathbf{W}_k \right) \right\}$ s.t. $\mathbf{W}_k^H \mathbf{\Delta}_k = \mathbf{T}_k$, is then analogous to a linearly constrained Wiener filter [2, (2.113)]. Thus (9b) follows a Kalman-like recursion given by

$$\widehat{\mathbf{x}}_k^b = \widehat{\mathbf{x}}_{k-1}^b + \mathbf{W}_k^{bH} \left(\mathbf{y}_k - \mathbf{H}_k \widehat{\mathbf{x}}_{k-1}^b \right), \quad (18a)$$
$$\mathbf{U}_{k-1}^b = \mathbf{F}_{k-1} \mathbf{P}_{k-1}^b \mathbf{F}_{k-1}^H + \mathbf{C}_{\mathbf{w}_{k-1}}, \quad (18b)$$
$$\mathbf{S}_k = \mathbf{H}_k \mathbf{U}_{k-1}^b \mathbf{H}_k^H + \mathbf{C}_{\mathbf{v}_k}, \quad (18c)$$
$$\mathbb{W}_k = \mathbf{S}_k^{-1} \mathbf{H}_k \mathbf{U}_{k-1}^b, \quad \mathbb{T}_k = \mathbf{T}_k - \mathbb{W}_k^H \mathbf{\Delta}_k, \quad (18d)$$
$$\mathbf{W}_k^b = \mathbb{W}_k + \mathbf{S}_k^{-1} \mathbf{\Delta}_k \left(\mathbf{\Delta}_k^H \mathbf{S}_k^{-1} \mathbf{\Delta}_k \right)^{-1} \mathbb{T}_k^H, \quad (18e)$$
$$\mathbf{P}_k^b = \left(\mathbf{I} - \mathbb{W}_k^H \mathbf{H}_k \right) \mathbf{U}_{k-1}^b + \mathbb{T}_k \left(\mathbf{\Delta}_k^H \mathbf{S}_k^{-1} \mathbf{\Delta}_k \right)^{-1} \mathbb{T}_k^H. \quad (18f)$$

In both cases,

$$\mathbf{P}_{k-1}^b = \min_{\overline{\mathbf{W}}_{k-1}} \left\{ \mathbf{P}_{k-1} \left(\overline{\mathbf{W}}_{k-1} \right) \right\} \text{ s.t. } \overline{\mathbf{W}}_{k-1}^H \left[\overline{\mathbf{A}}_{k-1} \ \overline{\mathbf{\Omega}}_{k-1} \right] = \left[\mathbf{B}_{k-1,1} \ \mathbf{\Upsilon}_{k-1} \right],$$

which means that the same rationale can be applied at time $k-1$ and so forth until time $k = 2$.

Note that if $\mathbf{C}_{\mathbf{w}_{k-1}} = \mathbf{0}$, then (16a-16e) reduce to [11, (17a)-(17c)], and (18a-18f) reduce to [11, (19a)-(19d)].

• Summary

The linear constraints (9a) allowing the LCMVE to follow a Kalman-like recursion (4c)(9c) in presence of PCS sources (5a), are built as follows:

▷ at time $k = 1$, a set of linear constraints of the form

$$\mathbf{W}_1^H \mathbf{\Lambda}_1 = \mathbf{\Gamma}_1, \ \left\{ \mathbf{\Lambda}_1 = \mathbf{H}_1, \ \mathbf{\Gamma}_1 = \mathbf{I} \right\} \text{ or } \left\{ \mathbf{\Lambda}_1 = \left[\mathbf{H}_1 \ \mathbf{\Omega}_1 \right], \ \mathbf{\Gamma}_1 = \left[\mathbf{I} \ \mathbf{\Upsilon}_1 \right] \right\}, \quad (19a)$$

must be set, leading to

$$\widehat{\mathbf{x}}_1^b = \mathbf{W}_1^{bH} \mathbf{y}_1, \ \mathbf{W}_1^b = \mathbf{C}_{\mathbf{v}_1}^{-1} \mathbf{\Lambda}_1 \left(\mathbf{\Lambda}_1^H \mathbf{C}_{\mathbf{v}_1}^{-1} \mathbf{\Lambda}_1 \right)^{-1} \mathbf{\Gamma}_1^H, \quad \mathbf{P}_1^b = \mathbf{\Gamma}_1 \left(\mathbf{\Lambda}_1^H \mathbf{C}_{\mathbf{v}_1}^{-1} \mathbf{\Lambda}_1 \right)^{-1} \mathbf{\Gamma}_1^H, \quad (19b)$$

▷ at time $k \geq 2$:

a) either no additional linear constraints on $\mathbf{W}_k$ are introduced and $\widehat{\mathbf{x}}_k^b$ must be computed according to (16a-16e) in order to propagate the existing set of linear constraints,

b) or additional linear constraints on $\mathbf{W}_k$, i.e. $\mathbf{W}_k^H \mathbf{\Delta}_k = \mathbf{T}_k$, are introduced on-line and $\widehat{\mathbf{x}}_k^b$ must be computed according to (18a-18f) in order to propagate the updated set of linear constraints.

III. On the impact of PCS sources on recursive LCMVEs performance

Let us consider a uniform linear array with $N = 50$ sensors equally spaced at $\widehat{d} = \lambda/2$ (half-wavelength) and an impinging signal source x_1 with broadside angle $\alpha = 10^\circ$, embedded in a spatially and temporally white noise: $\mathbf{y}_k = \mathbf{h}_k \left(\widehat{d}, \alpha \right) x_1 + \mathbf{v}_k$, $\mathbf{h}_k^T (d, \alpha) = \left(1, \ldots, e^{j 2\pi (N-1) \frac{d}{\lambda} \sin(\alpha)} \right)$, $\mathbf{C}_{\mathbf{v}_l, \mathbf{v}_k} = \mathbf{I} \delta_k^l$. The signal source x_1 is random, Gaussian complex circular with unit variance ($C_{x_1} = 1$), and is assumed to be fully coherent ($x_k = x_1$). However, fluctuation of the propagation medium are sometime unavoidable during the whole observation time interval, which prevents from observing a perfectly coherent signal source. Indeed, the random fluctuation of the propagation medium induces a random fluctuation of the signal amplitude. If the propagation medium fluctuations are small, then the mean power received from the signal source remains unchanged [12], which can be modeled via (5a) as:

$$x_k = f_{k-1} x_{k-1} + w_{k-1}, \ C_{x_k} = C_{x_1}, \ \left| f_{k-1} \right|^2 \leq 1, \quad \rho_{x_{k-1}, x_k} = C_{x_k, x_{k-1}} / C_{x_{k-1}} = f_{k-1}, \quad (20)$$

where f_{k-1} is the correlation coefficient between x_{k-1} and x_k which fully characterizes the loss of coherence between observation $k-1$ and k.

First, we want to investigate the impact of a slight loss of coherence of the signal source on the performance of the recursive MVDRE computed under the hypothesis of an FCS source [11, (17a)-(17c)]. To this end, we compute the MSE in the estimation of x_k, both for a FCS source (reference case) denoted "MVDRE (FCS)", and for a PCS source, denoted by "MVDRE Mismatched to PCS". Secondly, we also want to assess the benefit of the proposed extension to [11], that is, the ability to resort to a recursive MVDRE taking into account (20), with f_{k-1} and $C_{w_{k-1}}$ known. This method is denoted as "MVDRE Matched to PCS" in the results.

The results are summarized in Fig. 1, where the empirical MSEs (denoted "...(Sim)...") are assessed with 10^4 Monte-Carlo trials, whereas the analytic MSEs are assessed according to (16a-16e). Three cases of very small loss of coherence are considered $\left(\sigma_{w_l}^2 = \sigma_w^2 \in \left\{ 10^{-6}, 10^{-5}, 10^{-4} \right\} \right)$. Fig. 1 clearly exemplifies the impact of a slight loss of coherence of the signal source on the MVDRE performance in the large sample regime, which introduces a severe performance breakdown when the loss of coherence is not taken into account. Thanks to the results derived in this article, we can also evaluate which is the minimum achievable MSE when the amplitude fluctuation

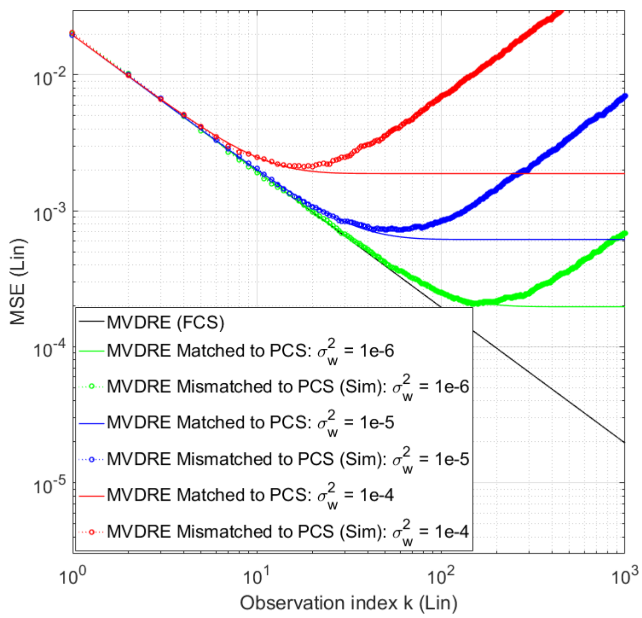

Fig. 1. MSE of the recursive MVDRE of x_k (20) versus k

model is known (20). Fig. 1 clearly shows that, when the signal source amplitude becomes partially coherent, there exists a lower limit in the achievable MSE, and an optimal number of observations that can be combined to estimate the amplitude with a nearly minimum achievable MSE.

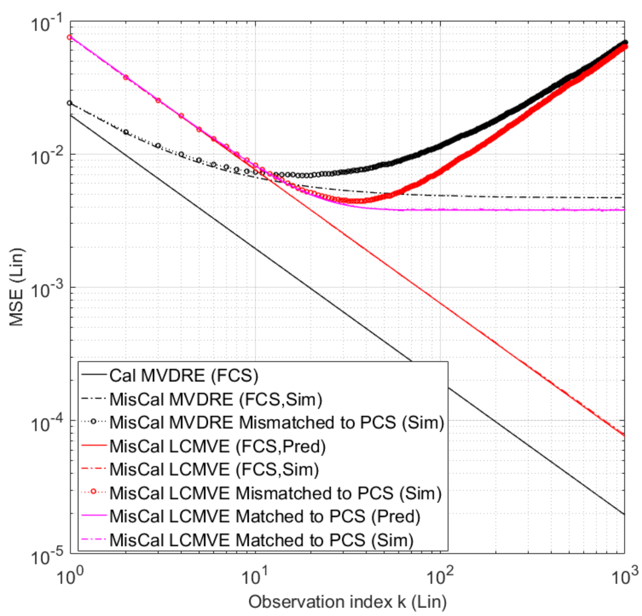

Fig. 2. MSE of the recursive LCMVE of x_k (20) versus k, $\sigma_w^2 = 10^{-4}$

Let us assume now that, due to a calibration error, or array deformation (e.g., thermal effects, aging, etc.), the actual inter-sensor distance is $d = 0.9975\widehat{d}$, i.e. $\widehat{d} - d = \lambda/400$. Thus, we are in the presence of a parametric modelling error in measurement vectors $\mathbf{h}_k(d, \alpha)$ which leads to the computation of a recursive MVDRE that does not match the true observations. The effect of such kind of "miscalibration" on the MVDRE is shown in Fig. 2 where we compare the performance of MVDREs based on recursions [11, (17a)-(17c)] computed with the true value d ("Cal MVDRE (FCS)") and with the assumed value $\widehat{d}$ ("MisCal MVDRE (FCS,SIM)") for an FCS source. It is usual to add derivative constraints in order to mitigate the effect on $\mathbf{h}_k(d, \alpha)$ of a small change in the system parameter d [1, §6.7.1], leading to a recursive LCMVE [11, (19a)-(19d)] where at each iteration the constraint $\partial \mathbf{h}_k\left(\widehat{d}, \alpha\right)/\partial d = 0$ is taken into account ("MisCal LCMVE (FCS,SIM)").

In this context also, we assess the impact of a PCS source on recursive LCMVE performance by considering the amplitude fluctuation model (20) where $\sigma_w^2 = 10^{-4}$. For this purpose, we compare the performance of the recursive LCMVE computed under the hypothesis of FCS source [11, (19a)-(19d)], denoted by "MisCal LCMVE Mismatched to PCS (SIM)" in Fig. 2, and the proposed extension (18a-18f), that is the ability to resort to a recursive LCMVE taking into account (20), denoted by "MisCal LCMVE Matched to PCS (SIM)" in Fig. 2. Again, even a slight loss of coherence introduces a severe LCMVE performance breakdown when the loss of coherence is ignored, breakdown which can be mitigated when the amplitude fluctuation model (20) is taken into account thanks to the proposed results. Last but not least, in case of a "small" miscalibration effect, the analytic LCMVE recursion (18a-18f) provides a tight prediction of the actual behaviour of the LCMVE, both in presence of an FCS source ("MisCal LCMVE (FCS,Pred)") and of a PCS source ("MisCal LCMVE Matched to PCS (Pred)").

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