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# Detecting the Rank of a Symmetric Tensor

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***Abstract*—This paper deals with the problem of Canonical Polyadic (CP) decomposition of a given tensor. Standard algorithms to perform this decomposition generally require the knowledge of the rank of the sought tensor decomposition. Yet, determining the rank of a given tensor is generally hard. In this paper, we propose a method to find the rank of a symmetric tensor. We reformulate the CP decomposition problem into a truncated moment problem and we derive a sufficient condition to certify the rank of the tensor from the rank of some moment matrices associated with it. For tensors with rank not exceeding a prescribed value, this sufficient condition is also necessary. Finally, we propose to combine our rank detection procedure with existing algorithms. Experimental results show the validity of our results and provide an illustration of its practical use. Our method provides the correct rank even in the presence a moderate level of noise.**

## I. Introduction

Tensors are useful mathematical tools in a wide range of scientific areas. For example, the diffusion kurtosis tensor is used in medical imaging, the elasticity tensor in physics, and tensors play also important roles in image authenticity validation, in crystal study, or in quantum physics [1]. At the origin of the wide spread of tensors, one can find decomposition possibilities, which have become fundamental operations in today's science and engineering. More precisely, tensor factorizations aim at decomposing an intricate or large tensor into a sum of smaller and simpler ones. Several decompositions have been proposed for different applications such as Canonical Polyadic (CP) or Tucker [2], [3] decompositions.

Among these decomposition problems, we are focusing here on the CP decomposition. The current methods for performing this decomposition include mainly optimization techniques such as Alternating Least Squares (ALS), Non-linear Least Squares (NLS) [4], and non-linear unconstrained optimization (OPT) [4], [5]. A few algebraic methods have also been proposed such as the generalized eigenvalue method (GEVD) [6] or methods based on the decomposition of a homogeneous polynomial into a sum of given powers of linear forms [7]. However, in order to perform the decomposition, all the mentioned methods require to know explicitly the tensor rank. Yet, finding the true rank is difficult, as it is known to be an NP-hard problem [8]. Furthermore, any error in the sought rank can yield dramatic consequences, as the set of tensors of given rank does not form a closed set. Common rank estimation methods are based on optimization problems using the nuclear norm as a surrogate for the rank [9], Bayesian models [10], or on matrix unfoldings such as balanced matricization [3].

In this paper, we deal with the CP decomposition of a symmetric tensor, which has for instance application in blind identification of under-determined mixtures [7]. Applications of symmetric tensors to machine learning can also be found in [11] and applications to other areas in [1]. For symmetric tensors, one can introduce the notion of symmetric rank, which is also NP-hard to determine [8] but has the benefit to be computable by some existing algebraic methods [12]. Although, the rank and the symmetric rank may be different [13], they are in many cases equal.

We propose here a moment based method offering theoretical guarantees to determine the symmetric rank in the CP decomposition of a symmetric tensor. Our method has the advantage to provide a necessary and sufficient condition to obtain the true rank whereas for instance, rank-revealing matrix unfoldings [3] give only a sufficient condition. The link between CP decompositions and moment problems has been mentioned [14], but not explored further. Our method shows also similarities with the algebraic methods proposed in [7], and [15]. Nonetheless, we present the method from a completely different perspective that may provide further insight.

Our paper is organized as follows: Section II introduces the CP decomposition and rank for a symmetric tensor. Section III states our main result allowing us to calculate the rank value. The tools used in our method are presented in Section IV, as well as some elements of proof. Section V shows simulation results and Section VI concludes our work.

We use the following notation: $\lfloor \cdot \rfloor$ is the greatest integer smaller than its argument and $\binom{n}{p}$ is the binomial coefficient "among $n$ choose $p$". Upper case calligraphic letters denote tensors ($\mathcal{T}$) and fraktur letter ($\mathfrak{T}$) their values after re-indexing (see Section III-A). Bold upper case letters ($\mathbf{M}$) denote matrices, bold lower case letters ($\mathbf{v}$) denote vectors and simple lower case letters ($s$) denote scalars. For a multi-index $\boldsymbol{\alpha}$ of length $n+1$, we define its absolute value $|\boldsymbol{\alpha}| = \alpha_0 + \cdots + \alpha_n$.

## II. Problem statement

### A. CP decomposition

Consider in the following a tensor of order $d \in \mathbb{N}, d > 2$ on $\mathbb{R}^{n+1}$, which is denoted by $\mathcal{T} \in \mathbb{R}^{n+1} \otimes \cdots \otimes \mathbb{R}^{n+1}$ ($d$ times).

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In this paper, we deal with the case of symmetric tensors, by which we mean that the tensor entries $(\mathcal{T}_{i_1,\ldots,i_d})_{0\le i_1,\ldots,i_d\le n}$ are unchanged by any permutation of the indices. A tensor is said to be symmetric rank-1 if it can be expressed as

$$\mathbf{v}^{\otimes d} = \underbrace{\mathbf{v}\otimes\cdots\otimes\mathbf{v}}_{d \text{ times}}$$

for a vector $\mathbf{v}$ of $\mathbb{R}^{n+1}$, meaning that its elements are given by $\left[\mathbf{v}^{\otimes d}\right]_{i_1,\ldots,i_d} = v_{i_1}\ldots v_{i_d}$. Given any symmetric tensor $\mathcal{T}$, we are concerned with the problem of decomposing it as a sum of rank-1 tensors, that is we want to write it

$$\mathcal{T} = \sum_{r=1}^{R} \mathbf{v}(r)^{\otimes d},$$

or equivalently

$$\mathcal{T}_{i_1,\ldots,i_d} = \sum_{r=1}^{R} v_{i_1}(r)\ldots v_{i_d}(r). \tag{1}$$

The decomposition (1) is called a CP decomposition of $\mathcal{T}$. The symmetric rank[1], denoted by $\operatorname{rank}_S \mathcal{T}$ is the minimum number of terms in any representation of $\mathcal{T}$ as above.

### B. Indeterminacies and dehomogenization

Notice that there are ambiguities in defining the vectors of decomposition (1). Indeed, the order of the vectors $(\mathbf{v}(r))_{r\in[\![1,R]\!]}$ in the sum is arbitrary. For even values of $d$, there is also a sign ambiguity. Going further, by normalizing each $\mathbf{v}(r)$ with its $p^{\text{th}}$ coordinate, the decomposition (1) can be expressed in the equivalent form

$$\mathcal{T} = \sum_{r=1}^{R} \lambda_r \left(\frac{\mathbf{v}(r)}{v_p(r)}\right)^{\otimes d}, \tag{2}$$

where we have set $\lambda_r = v_p(r)^d$. The coordinate index $p \in [\![0,n]\!]$ used for the above normalization is the same for all $r$. With no loss of generality, we take $p = 0$ in the following. Corresponding to this choice, we assume:

**Assumption 1.** $(\forall r \in [\![1,R]\!]) \quad v_0(r) \neq 0.$

For all $r \in [\![0,R]\!]$, we also set $\lambda_r = v_0(r)^d$ and then define $\mathbf{u}(r) = (v_1(r)/v_0(r),\ldots,v_n(r)/v_0(r))$, which is obtained by dividing $\mathbf{v}(r)$ by its first component and then dropping the first coordinate. Such a procedure is known as dehomogenization. Remark that if Assumption 1 does not hold, but if $v_p(r)\neq 0$ for a given coordinate $p$ and for all $r \in [\![1,R]\!]$, dehomogenization can be performed with respect to the $p^{\text{th}}$ coordinate. All the results in the following then still hold, but should be adapted by performing a permutation in the coordinates.

## III. Tensor rank detection

In this section, we give a method to detect the rank of a tensor. Our main result links the tensor rank to the ranks of particular matrices and offers a necessary and sufficient condition in contrast to other methods like matrix unfoldings. We therefore describe first the moment matrices associated to a symmetric tensor.

[1] In the following rank will systematically mean symmetric rank.

### A. Re-indexing of the tensor elements

Due to the symmetry assumption on the tensor $\mathcal{T}$, the order of the indices in $\mathbf{i} = (i_1,\ldots,i_d)$ is of no importance for the value of $\mathcal{T}_{i_1,\ldots,i_d}$ and we can index the elements by indicating the number of times each index value appears in $\mathbf{i}$. More precisely, to any $d$-tuple $\mathbf{i} = (i_1,\ldots,i_d)$, we associate a $(n+1)$-tuple $\boldsymbol{\alpha}(\mathbf{i})$ such that

$$\boldsymbol{\alpha}(\mathbf{i}) = (\alpha_0(\mathbf{i}),\ldots,\alpha_n(\mathbf{i})),$$

where for each $k \in [\![0,n]\!]$, $\alpha_k(\mathbf{i})$ is the number of times the index value $k$ appears in $\mathbf{i}$. Note that $\boldsymbol{\alpha}$ is such that $|\boldsymbol{\alpha}(\mathbf{i})| = d$. Any tensor entry $\mathcal{T}_{i_1,\ldots,i_d}$ depends on $\mathbf{i}$ only through $\boldsymbol{\alpha}(\mathbf{i})$. We therefore define the tensor values as follows

$$\mathcal{T}_{\mathbf{i}} = \mathfrak{T}_{\boldsymbol{\alpha}(\mathbf{i})},$$

where $\mathfrak{T}_{\boldsymbol{\alpha}}$ is indexed by $(n+1)$-tuples $\boldsymbol{\alpha} = (\alpha_0,\ldots,\alpha_n)$ satisfying $|\boldsymbol{\alpha}| = d$.

*Example:* Let us take a tensor $\mathcal{T}$ of order 4 in $\mathbb{R}^3$ ($d = 4$ and $n = 2$). The natural description of any symmetric tensor is by its coefficients $\mathcal{T}_{i_1i_2i_3i_4}$ with $0 \le i_1,i_2,i_3,i_4 \le 2$ and the latter coefficients are unchanged by any permutation of $\{i_1,i_2,i_3,i_4\}$. We can equivalently describe the same tensor with the indices $\alpha_0,\alpha_1,\alpha_2$ counting the number of occurrences of $0,1$ and $2$ respectively. For example, with $\alpha_0 = 2, \alpha_1 = 1, \alpha_2 = 1$ we have

$$\begin{aligned}\mathfrak{T}_{211} \longleftrightarrow \mathcal{T}_{0012} &= \mathcal{T}_{0021} = \mathcal{T}_{0102} = \mathcal{T}_{0120} = \mathcal{T}_{1002} = \mathcal{T}_{1020} \\ &= \mathcal{T}_{1200} = \mathcal{T}_{2001} = \mathcal{T}_{2010} = \mathcal{T}_{2001}.\end{aligned}$$

Note that $\alpha_0+\alpha_1+\alpha_2 = 4 = d$, as already mentioned.

### B. Moment matrix

Let us set $k = \lfloor \frac{d}{2} \rfloor$ and arrange the elements $\mathfrak{T}_{\boldsymbol{\alpha}}$ of the initial tensor $\mathcal{T}$ into a matrix $\mathbf{M}_k$ indexed with the multi-indices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with respect to the lexicographic order, i.e. $\boldsymbol{\alpha}$ is before $\boldsymbol{\beta}$ if the leftmost non-zero entry of $\boldsymbol{\alpha}-\boldsymbol{\beta}$ is positive

$$\left(\forall(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \left(\mathbb{N}^{n+1}\right)^2, |\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = k\right) \quad (\mathbf{M}_k)_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \mathfrak{T}_{\boldsymbol{\alpha}+\boldsymbol{\beta}},$$

The matrix $\mathbf{M}_k$ has size $N = \binom{n+k}{k}$ and is called the moment matrix of order $k$. For any integer $l$ such that $k-l \ge 1$, we define the moment matrix $\mathbf{M}_{k-l}$ of order $k-l$ as the leading principal submatrix of $\mathbf{M}_k$ of size $\binom{n+k-l}{k-l}$, i.e. the matrix composed of the $\binom{n+k-l}{k-l}$ first rows and columns of $\mathbf{M}_k$.

*Example:* Let us proceed with our example from Section III-A. We build the following moment matrices from the tensor $\mathcal{T}$ with respect to the lexicographic ordering:

$$\mathbf{M}_2 = \begin{bmatrix} & & & \mathfrak{T}_{220} & \mathfrak{T}_{211} & \mathfrak{T}_{202} \\ & \mathbf{M}_1 & & \mathfrak{T}_{130} & \mathfrak{T}_{121} & \mathfrak{T}_{112} \\ & & & \mathfrak{T}_{031} & \mathfrak{T}_{022} & \mathfrak{T}_{103} \\ \mathfrak{T}_{220} & \mathfrak{T}_{130} & \mathfrak{T}_{031} & \mathfrak{T}_{040} & \mathfrak{T}_{031} & \mathfrak{T}_{022} \\ \mathfrak{T}_{211} & \mathfrak{T}_{121} & \mathfrak{T}_{022} & \mathfrak{T}_{031} & \mathfrak{T}_{022} & \mathfrak{T}_{013} \\ \mathfrak{T}_{202} & \mathfrak{T}_{112} & \mathfrak{T}_{103} & \mathfrak{T}_{022} & \mathfrak{T}_{013} & \mathfrak{T}_{004} \end{bmatrix}$$

with

$$\mathbf{M}_1 = \begin{bmatrix} \mathfrak{T}_{400} & \mathfrak{T}_{310} & \mathfrak{T}_{301} \\ \mathfrak{T}_{310} & \mathfrak{T}_{220} & \mathfrak{T}_{211} \\ \mathfrak{T}_{301} & \mathfrak{T}_{211} & \mathfrak{T}_{202} \end{bmatrix}.$$

### C. Main result

We now give a sufficient condition on the moment matrices $\mathbf{M}_k$ and $\mathbf{M}_{k-1}$ which certifies the rank of the corresponding symmetric tensor $\mathcal{T}$.

**Theorem 1.** *The tensor $\mathcal{T}$ has rank $R$ if its moment matrices of order $k$ and $k-1$ both have rank equal to $R$*

$$(\operatorname{rank} \mathbf{M}_k = \operatorname{rank} \mathbf{M}_{k-1} = R) \quad \Longrightarrow \quad (\operatorname{rank}_S \mathcal{T} = R) \ .$$

Theorem 1 offers a conceptually simple tool to get the rank of a tensor. Indeed, we can first build the associated moment matrix $\mathbf{M}_k$, then extract the principal submatrix $\mathbf{M}_{k-1}$ and finally check whether their ranks are equal. The conditions in which this theorem is applicable are discussed in more details below and an extension is given.

### D. Extended result

First, the order of the moment matrix is non-negative and the moment matrix $\mathbf{M}_{k-1}$ is thus defined for $k \geq 2$ only. This means that our method can only be applied to symmetric tensors of order at least $d = 4$.

Furthermore, Theorem 1 cannot be used to detect ranks exceeding the size of the moment matrix $\mathbf{M}_{k-1}$. It can thus only be used to detect rank values smaller than $\binom{n+k-1}{k-1}$. The tensor rank can be greater since, to our knowledge, the lowest rank upper-bound (see [7]) is $\binom{n+d}{d}$. The rank value restriction of our method is hence $R \leq \binom{n+k-1}{k-1}$. Note that for a tensor of order 4, it reduces to $R \leq n+1$. In large tensor data, and in many applications where a low rank representation of the tensor is looked for, our result may however provide interesting guarantees.

When the tensor rank is known in advance to have smaller rank than the size of $\mathbf{M}_{k-1}$, the following reciprocal of Theorem 1 can be proved:

**Theorem 2.** *Suppose Assumption 1 holds. If $R \leq \binom{n+k-1}{k-1}$, we have the following equivalence*

$$(\operatorname{rank}_S \mathcal{T} = R) \quad \Longleftrightarrow \quad (\operatorname{rank} \mathbf{M}_k = \operatorname{rank} \mathbf{M}_{k-1} = R) \ .$$

## IV. CP decomposition: a moment problem

We now introduce the necessary notions for a proof of Theorems 1 and 2, leading to an interpretation of the CP decomposition (1) as an integral with respect to a measure supported on $R$ points.

### A. CP decomposition as a measure integration

Following the re-indexing in III-A, and with the dehomogenization performed in Section II-B, the rank $R$ decomposition in (1) also reads

$$\begin{aligned}\mathfrak{T}_{\alpha_0,\ldots,\alpha_n} &= \sum_{r=1}^{R} v_0(r)^{\alpha_0} \ldots v_n(r)^{\alpha_n} \\ &= \sum_{r=1}^{R} \lambda_r u_1(r)^{\alpha_1} \ldots u_n(r)^{\alpha_n} \ . \end{aligned} \tag{3}$$

Now, we write (3) in an equivalent integral form

$$\mathfrak{T}_{\alpha_0,\ldots,\alpha_n} = \int x_1^{\alpha_1} \ldots x_n^{\alpha_n} \mu(\mathrm{d}\mathbf{x}) = \int \mathbf{x}^{\boldsymbol{\gamma}} \mu(\mathrm{d}\mathbf{x}) \ , \tag{4}$$

where $\boldsymbol{\gamma} = (\alpha_1, \ldots, \alpha_n)$ and $\mu$ is the discrete measure

$$\mu = \sum_{r=1}^{R} \lambda_r \delta_{\mathbf{u}(r)} \ , \tag{5}$$

which is defined on $n$ variables and supported on the points $(\mathbf{u}(r))_{r\in[\![1,R]\!]}$. Such a measure, which is concentrated on a finite set of $R$ points, is referred to as an $R$-atomic measure. Finding the vectors $(\mathbf{u}(r))_{r\in[\![1,R]\!]}$ and the coefficients $(\lambda_r)_{r\in[\![1,R]\!]}$ of the CP decomposition is hence equivalent to determine the $R$-atomic measure $\mu$ in (5).

The right hand side of (4) is called the moment of order $\boldsymbol{\gamma}$ of the measure $\mu$ and its degree is $|\boldsymbol{\gamma}|$. The moment matrix $\mathbf{M}_k$ defined in Section III-B contains moments of $\mu$ up to degree $2k$. Therefore, (4) shows that finding a CP decomposition of a symmetric tensor $\mathcal{T}$ as in (1) is equivalent to the estimation of a discrete measure $\mu$ from its moments of degree up to $2k$. This is a truncated moment problem which is often encountered in mathematics, engineering, and physics and has been widely investigated [16], [17]. In the next section, we use information on the moments of $\mu$ contained in the moment matrix $\mathbf{M}_k$ to retrieve the number of points $R$ on which the measure $\mu$ is defined and to solve our original CP decomposition problem.

*Example:* In the example provided in Section III-A, the elements of $\mathcal{T}$ are moments up to degree 4 of a 2-atomic measure $\mu$ on $\mathbf{x} = (x_1, x_2)$. The moment monomials corresponding to the moment matrix $\mathbf{M}_1$ are presented in Table I along with their corresponding tensor elements.

TABLE I
TENSORS ELEMENTS AND RELATED MOMENTS

| Tensor elements | | Moment monomials $\mathbf{x}^{\boldsymbol{\gamma}}$ |
|---|---|---|
| $\mathcal{T}_{0000}$ | $\mathfrak{T}_{400}$ | $1$ |
| $\mathcal{T}_{0001}$ | $\mathfrak{T}_{310}$ | $x_1$ |
| $\mathcal{T}_{0002}$ | $\mathfrak{T}_{301}$ | $x_2$ |
| $\mathcal{T}_{0011}$ | $\mathfrak{T}_{220}$ | $x_1^2$ |
| $\mathcal{T}_{0012}$ | $\mathfrak{T}_{211}$ | $x_1 x_2$ |
| $\mathcal{T}_{0022}$ | $\mathfrak{T}_{202}$ | $x_2^2$ |

### B. Solving the truncated moment problem

A necessary condition for a matrix to be a moment matrix is positive definiteness. In our context, $\mathcal{T}$ admits a CP decomposition; thus there exist an unknown $R$ and vectors $(\mathbf{v}(r))_{r\in[\![1,R]\!]}$ satisfying (1). A corresponding measure $\mu$ can thus always be defined as in (5). Then, we have

$$\begin{aligned}(\forall \mathbf{a} \in \mathbb{R}^N) \quad \mathbf{a}^\top \mathbf{M}_k \mathbf{a} &= \sum_{|\boldsymbol{\gamma}|\leq k, |\boldsymbol{\delta}|\leq k} a_{\boldsymbol{\gamma}} a_{\boldsymbol{\delta}} \mathfrak{T}_{\boldsymbol{\gamma}+\boldsymbol{\delta}} \\ &= \sum_{|\boldsymbol{\gamma}|\leq k, |\boldsymbol{\delta}|\leq k} a_{\boldsymbol{\gamma}} a_{\boldsymbol{\delta}} \int \mathbf{x}^{\boldsymbol{\gamma}+\boldsymbol{\delta}} \mu(\mathrm{d}\mathbf{x}) \\ &= \int p_{\mathbf{a}}(\mathbf{x})^2 \mu(\mathrm{d}\mathbf{x}) \geq 0 \end{aligned}$$

where $p_{\mathbf{a}}$ is the polynomial $p_{\mathbf{a}} = \sum_{|\boldsymbol{\gamma}| \leq k} a_{\boldsymbol{\gamma}} \mathbf{x}^{\boldsymbol{\gamma}}$. It follows that the moment matrix $\mathbf{M}_k$ that we defined for a tensor is always positive semi-definite.

Since $\mathbf{M}_{k-1}$ is positive semi-definite and under assumption of Theorem 1 that $\mathbf{M}_k$ and $\mathbf{M}_{k-1}$ have same rank $R$, [17, Theorem 7.10] applies. Accordingly, the moments contained in $\mathbf{M}_{k-1}$ have a unique representing $R$-atomic measure. Since the points supporting the measure are the generating vectors of the CP decomposition, we obtain that $R$ is also the rank of the tensor $\mathcal{T}$.

To prove the equivalence in Theorem 2, we can also apply [17, Theorem 7.10]. Nevertheless Assumption 1 must hold. Indeed, dehomogenization is always possible when $\mathbf{M}_k$ and $\mathbf{M}_{k-1}$ have same rank. Conversely, if $\mathcal{T}$ has rank $R$ but Assumption 1 does not hold, then one can prove that $\operatorname{rank} \mathbf{M}_{k-1} = \operatorname{rank} \mathbf{M}_k - 1$.

## V. Numerical results

For numerical illustration, we generate randomly rank-$R$ tensors by drawing their CP decomposition vectors $(\mathbf{v}(r))_{r \in [\![1,R]\!]}$ according to a uniform distribution on $[0,1]^{n+1}$.

### A. Importance of rank detection

Most of the standard algorithms to perform CP decomposition require as input, the rank of the sought CP decomposition. Here, we use the algorithms called ALS, NLS, OPT, and GEVD from Tensorlab 3.0 [18]. To show the benefit of our method, we compare the relative error between a tensor of known rank and the CP decomposition returned by algorithms ALS, NLS, OPT, and GEVD for various input ranks. The relative error between tensors $\mathcal{T}$ and $\hat{\mathcal{T}}$ is used to assess the quality of the CP decomposition and it is defined as

$$\text{relative error} = \frac{\left\| \mathcal{T} - \hat{\mathcal{T}} \right\|_F}{\|\mathcal{T}\|_F},$$

where $\|.\|_F$ is the Frobenius norm.

Figure 1 shows the average relative error on 100 random tensors of rank 5, order 4 and dimension 30 for the different input ranks. We can notice the sensitivity of the algorithms to the input rank. More specifically, the algebraic method GEVD shows a high improvement potential when the true rank is known. Similarly NLS shows a particularly good performance for the real rank value. This shows the importance of our method which can thus be used to find directly the rank of a given tensor before feeding it into a CP decomposition algorithm.

### B. Finding the rank of a symmetric tensor

Here, we look at the numerical rank of moment matrices to verify that Theorem 1 applies well in practice. We compute the rank of the tensor through the ranks of moment matrices $\mathbf{M}_k$ and $\mathbf{M}_{k-1}$ and if both are equal, then this delivers the rank of the tensor.

On four different examples with different tensor ranks, Figure 2 plots the first twelve singulars values of the moment matrices, normalized by the largest singular value $\sigma_1$. We observe a significant drop after the same singular value for both moment matrices. We can conclude that they have the same numerical rank and infer that this rank is also the rank of the tensor.

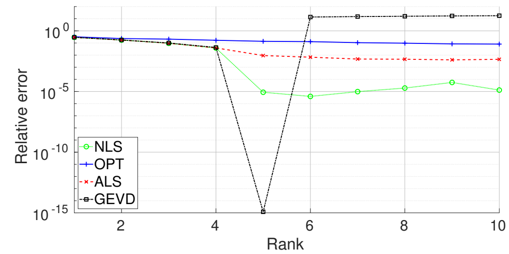



Fig. 1. Relative error of the CP decomposition

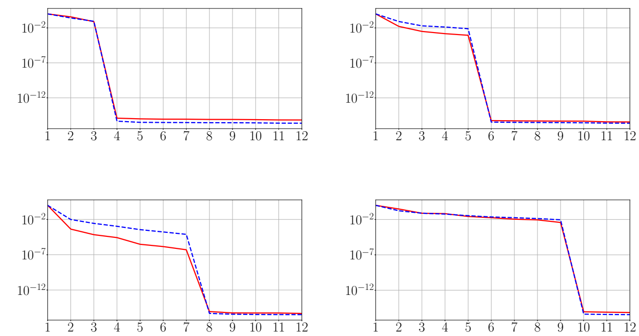
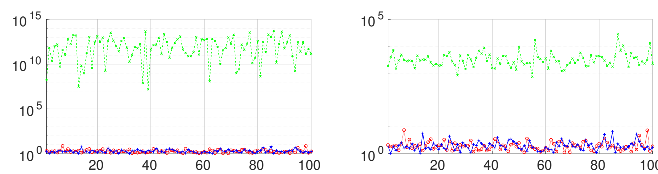


Fig. 2. First singular values of $\mathbf{M}_3$ (red) and $\mathbf{M}_2$ (blue) (Top-left: $R = 3$, Top-right: $R = 5$, Btm-left: $R = 7$, Btm-right: $R = 9$)



(a) No noise (b) Noisy

Fig. 3. Singular value ratios gap of $\mathbf{M}_2$ for 100 tests (In red: $\sigma_5/\sigma_6$, in blue: $\sigma_6/\sigma_7$, and in green: $\sigma_7/\sigma_8$)

We generate 100 random rank-7 tensors of dimension 20 and of order 6. Figure 3 shows the ratio of the successive singular values (sorted in decreasing order) $\sigma_5/\sigma_6$, $\sigma_6/\sigma_7$ and, $\sigma_7/\sigma_8$ of $\mathbf{M}_2$, without noise and in a presence of an additive zero-mean Gaussian noise of variance $10^{-4}$. In both case, we observe a gap in the ratios that indicate a rank of 7 for the moment matrix. We observe a similar gap for $\mathbf{M}_3$ which shows that our method detects the rank of the tensor correctly.

In conclusion, for noiseless or moderate level of noise scenarios, we observe that $\mathbf{M}_k$ and $\mathbf{M}_{k-1}$ always have same rank under conditions of Section III-D. We therefore can determine the rank of the corresponding tensor and confirm that Theorem 1 works well in numerical applications.

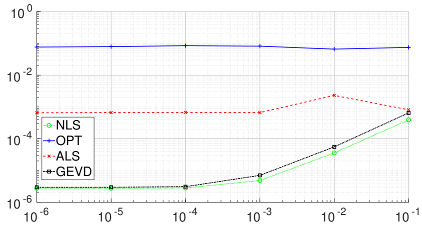



Fig. 4. Average relative error depending on noise variance ($R = 3$)

### C. Application to cumulant-based source separation

We consider the cumulant tensor of a random vector $\mathbf{y}$ such that

$$\mathbf{y} = \mathbf{A}\mathbf{s} + \mathbf{w}$$

where $\mathbf{A} \in \mathbb{R}^{n\times R}$ is an unknown matrix, $\mathbf{w}$ is a white Gaussian noise with zero-mean, and $\mathbf{s}$ is a vector of $R$ independent random variables. In our experiments, we draw the elements of $\mathbf{A}$ according to a uniform distribution on $[0, 1]$ and the elements of $\mathbf{s}$ take value -1 or 1 with equal probability.

Our goal is to retrieve the number of sources $R$ from several samples of the observation vector $\mathbf{y}$. It is known [19] that the tensor of cumulant follows a CP decomposition

$$\mathrm{Cum}(y_i, y_j, y_k, y_l) = \sum_{r=1}^{R} A_{ir}A_{jr}A_{kr}A_{lr}\,\mathrm{Cum}(s_r)\,.$$

We use the empirical estimation of moments to compute the estimated cumulant tensor, which is a noise-corrupted version of a low-rank tensor. We then apply our method to detect the low rank model and thereby recover the number of sources. Table II shows the percentage of cases where the number of sources is correctly detected over 200 runs. For each of them, 100,000 samples of vectors $\mathbf{y}$ of size 20 are generated, for various variance values of the noise $\mathbf{w}$. We detect the rank numerically in the moment matrices by a gap of $10^3$ in the ratio of successive singular values. We then feed the detected ranks into NLS, ALS, OPT and GEVD algorithms and look at the CP decomposition they returned. Figure 4 plots the average relative errors for the four algorithms in the case of three sources.

TABLE II
PERCENTAGE OF SUCCESSFUL DETECTION OF THE NUMBER OF SOURCES

| | Number of sources R | | |
|---|---|---|---|
| Variance of the noise | 3 | 4 | 5 |
| 0 | 100% | 97% | 69% |
| $1 \cdot 10^{-6}$ | 100% | 97% | 69% |
| $1 \cdot 10^{-4}$ | 100% | 97% | 69% |
| $1 \cdot 10^{-2}$ | 100% | 97% | 69% |
| $1 \cdot 10^{-1}$ | 100% | 95% | 57% |

We note that even with a reasonable level of noise, the moment matrices have still the same rank that corresponds to the rank of the tensor. The method shows satisfactory results for low rank tensor corrupted with noise. Nevertheless, we observe that the higher the rank, the higher the sensitivity to the estimation noise and the higher the number of samples must be. Figure 4 also shows that algebraic methods such as GEVD are more sensitive to the noise despite their good performance in the noiseless case.

## VI. Conclusion

We have proposed a method to find the symmetric rank of a symmetric tensor. By interpreting the problem of the CP decomposition of a symmetric tensor as a moment problem, we can use tools related to truncated moment problems and obtain a necessary condition to deduce the tensor rank. Finally our simulations show first, the importance of rank detection and then the successful application of our result in practical situations.

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