A Continuum Model for Route Optimization in Large-Scale Inhomogeneous Multi-Hop Wireless Networks

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***Abstract*—Multi-hop route optimization in large-scale inhomogeneous networks typically requires the use of constrained optimization tools, yielding processing complexity that scales like $O(N^3)$, N being the number of relays employed. Here, we propose an alternative approach to route optimization by considering the limit of infinite relay node density to develop a continuum model, which yields an optimized equivalent continuous relay path. The model is carefully constructed to maintain a constant connection density even though the node density scales without bound. This leads to a formulation for minimizing the end-to-end outage probability that can be solved using methods from the calculus of variations. We demonstrate the effectiveness of this new approach and its potential for reducing processing complexity by considering a network subjected to a point source of interference.**

***Index Terms*—Multi-hop relaying, calculus of variations, outage, continuum modeling.**

I. Introduction

As networks evolve towards 5th generation (5G) wireless systems, network densification [1] is considered to be a key enabler in satisfying the increasing capacity demands on wireless communication networks. The unprecedented growth in Internet of Things (IoT) services [2] driven by a number of technological, social, and economic factors [3], will fuel further densification. While the heterogeneous integration of wireless networks is likely to provide the required levels of connection ubiquity between humans, machines, and devices, it will lead to inhomogeneous spatial interference characteristics.

Multi-hop relaying has long been studied as a way to extend coverage in wireless networks, reduce energy consumption, and improve the overall quality of service (QoS) (see, e.g., [4] and references therein). If two devices wish to communicate via a large number of relays over a network subject to spatial inhomogeneities, the challenge is in the determination of the multi-hop route, or the placement of devices such that this route can be constructed, that optimizes the target performance metric (see, e.g. [5] and references therein). Furthermore, if the network and/or environment are subject to temporal variations, the time required to compute the optimal route or device locations must be minimized. Typically, determining the location of N relay devices in d-dimensional space (subject to a small number of constraints) scales like $O(N^3)$, associated with the cost of matrix inversion in the Lagrangian method of optimisation (see [6], sections 2.11 and 8.1). Although a number of numerical optimization tools exist for this purpose, such tools are computational unwieldy when applied in the context of large, energy-constrained relay networks.

In this paper, we present an alternative optimization approach, which is based upon the idea of treating the multi-hop route as a continuum. We argue that this approach has the potential to reduce the complexity in the node placement/route selection process for dense networks. Perhaps more importantly, however, the proposed model presents a radically different view of wireless networks, which we hope will stimulate further interest in developing new approaches to treating complex wireless networks in the future. Specifically, we offer the following contributions:

1) we develop a continuum model that describes the end-to-end outage probability in a multi-hop relay network;
2) we apply the model to find an analytic solution for the outage-optimal device locations when the system is subject to an inhomogeneous scalar field of interference.
3) the efficacy of the continuum model is verified by considering a discrete sampling of the continuum with a finite number of relay nodes.

The remainder of the paper is organized as follows. In section II, we introduce the continuum model for end-to-end outage probability calculation. We elaborate on the model in section III by applying it to study the outage-optimization problem in a simple system with a point source of interference. Section IV provides thoughts on how to sample the continuum to obtain discrete node positions for use in practical, finite systems. Conclusions are drawn in section V.

II. Continuum Model for End-to-End Outage Probability

Consider the goal of transmitting a message wirelessly from a source node to a destination node via N relay nodes located in $\mathbb{R}^2$. We denote the positions of the source and destination nodes by $\mathbf{p}_0$ and $\mathbf{p}_{N+1}$, respectively, and the position of the ith

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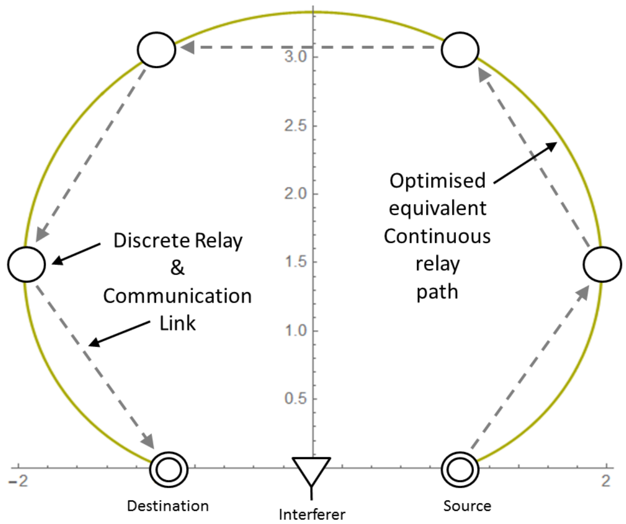

Fig. 1. Multihop wireless network subject to an interferer.

relay node by $\mathbf{p}_i$[1]. For convenience, we collect the positions of the relay nodes in the set $\mathcal{R} = \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N\}$.

Our goal is to develop a formulation of the end-to-end outage probability for the source-destination link that is a functional of the curve in $\mathbb{R}^2$ along which relay devices are placed, i.e., the points in $\mathcal{R}$ should be contained in (or be very close to) the curve. This curve will effectively signify a path of infinite node density. To maintain a practical viewpoint, however, we will retain a *finite connection density* in the model. It is important to note that without this consideration, the end-to-end outage probability would converge to zero as $N \to \infty$. To give a better picture of this approach, Fig. 1 illustrates a continuous path along with discrete node placements for a system subject to a point source of interference. We will return to this example later.

A. General Formulation

To make progress, we consider a model whereby a connection between two arbitrary points $\mathbf{p}_i$ and $\mathbf{p}_j$ in $\mathbb{R}^2$ exists with probability $c(\mathbf{p}_i, \mathbf{p}_j)$, which is independent of all other connections. We then form a multi-hop route, between a source $\mathbf{p}_0$ and a destination $\mathbf{p}_{N+1}$, via the sequence of relays at the points in $\mathcal{R}$. The probability that an end-to-end connection exists between the source and destination can be written as

$$P_c = \prod_{n=0}^{N} c(\mathbf{p}_n, \mathbf{p}_{n+1}). \tag{1}$$

We define the outage probability as the complement of the connection probability, i.e., $P_o = 1 - P_c$. It will be convenient to define the *logarithmic connection probability* (LCP) as

$$\psi_c := \log P_c = \sum_{n=0}^{N} \log c(\mathbf{p}_n, \mathbf{p}_{n+1}). \tag{2}$$

[1]We use $\mathbf{p}_i$ to refer to the ith node, itself, as well as its position in $\mathbb{R}^2$.

We will find it useful to specify node locations in a parametric manner. Define the tagged partition $\mathcal{P}_z = \{z_0, z_1, \dots, z_{n+1}\}$ with $0 = z_0 < z_1 < \dots < z_{n+1} = 1$ along with the sequence $\zeta_0, \dots, \zeta_N$, the elements of which satisfy $\zeta_n \in [z_n, z_{n+1}]$. The path of points that the communication will traverse can be written as the sequence $\mathbf{p}(0), \mathbf{p}(z_1), \dots, \mathbf{p}(z_{n+1})$ where $\mathbf{p}(z) = (x(z), y(z)) \in \mathbb{R}^2$. Now, we can write

$$\psi_c = \sum_n \frac{\log c(\mathbf{p}(z_n), \mathbf{p}(z_{n+1}))}{z_{n+1} - z_n}(z_{n+1} - z_n). \tag{3}$$

Let the mesh of the partition $\mathcal{P}_z$ be denoted by $\delta_z = \max_n(z_{n+1} - z_n)$. Now suppose that as $\delta_z \to 0$, we have

$$\frac{\log c(\mathbf{p}(z_n), \mathbf{p}(z_{n+1}))}{z_{n+1} - z_n} \to -q(\zeta_n) < 0 \tag{4}$$

for some function $q : [0, 1] \to \mathbb{R}_+$. It follows that if the function q is Riemann integrable, then

$$\lim_{\delta_z \to 0} \psi_c = -\int_0^1 q(z)\,\mathrm{d}z. \tag{5}$$

The outage probability in this case can be written as

$$P_o = 1 - \mathrm{e}^{-\int_0^1 q(z)\,\mathrm{d}z}. \tag{6}$$

B. Homogeneous Systems

Note that if the system is homogeneous, then pairwise connectivity would only depend upon the distance between nodes. In this case, it can be shown that the general formulation given above reduces to the outage probability expression

$$P_o = 1 - \mathrm{e}^{-qL} \tag{7}$$

where L is the length of the continuous path of relays connecting the source point $\mathbf{p}_0$ to the destination point $\mathbf{p}_{N+1}$ and q is a positive parameter that is inversely related to the connection density. Due to space restrictions, the details of this discussion are omitted; however, it is worth noting that intuition is satisfied with this result, since one would expect to place nodes along the shortest curve between the source and destination in a homogeneous environment.

C. Inhomogeneous Systems Subject to Rayleigh Fading

In the general case where the network is situated in an environment that is subject to inhomogeneities, q is a function of z, and it must be obtained in order to compute the continuum outage approximation. The specific formulation of q is dependent upon the connection model, which is governed by the channel fading statistics, modulation and coding schemes employed, and various other physical characteristics of the system. In what follows, we develop the model further by assuming each pairwise communication link is affected by Rayleigh fading, and the mean path loss follows an inverse square law[2] with respect to the Euclidean distance between devices. As such, it

[2]The consideration of higher order path loss exponents is deferred to future work.

can be shown that the connection probability for two devices located close to each other can be written as

$$\begin{aligned} &c(\mathbf{p}(z), \mathbf{p}(z+\delta_z)) \\ &= \exp\left(-\frac{1}{r_0^2\mu(\mathbf{p}(z+\delta_z))}\|\mathbf{p}(z+\delta_z)-\mathbf{p}(z)\|^2\right) \\ &= \exp\left(-\frac{(x(z+\delta_z)-x(z))^2+(y(z+\delta_z)-y(z))^2}{r_0^2\mu(x(z+\delta_z),y(z+\delta_z))}\right) \end{aligned} \tag{8}$$

where r_0 is a length scale that defines the connection range [7] and $\mu : \mathbb{R}^2 \to \mathbb{R}_+$ is a function that signifies the link quality at a given point in the plane. For now, we leave μ undefined, but note that it represents a scalar field in $\mathbb{R}^2$. We will provide a specific definition of μ in the next section.

It is necessary at this point to assume the channels for different hops are statistically independent, regardless of the connection distance, i.e., the coherence distance is zero. Expanding the left-hand side of the limit given in (4) for small δ_z yields

$$q(z) = \frac{\dot{x}(z)^2+\dot{y}(z)^2}{r_0^2\mu(x(z),y(z))}\delta_z + O(\delta_z^2) \tag{9}$$

where the dot notation, e.g., $\dot{x}(z)$, represents the first derivative of the function with respect to its argument. With a slight abuse of notation, we note that the expression given above contains the path length differential

$$\delta_l = \sqrt{\dot{x}(z)^2+\dot{y}(z)^2}\delta_z. \tag{10}$$

Consequently, under the condition that $\delta_l/r_0^2 \to \theta \in \mathbb{R}_+$ as $\delta_z \to 0$, the limit becomes

$$q(z) = \theta\frac{\sqrt{\dot{x}(z)^2+\dot{y}(z)^2}}{\mu(x(z),y(z))} \tag{11}$$

and the outage probability can be written as

$$P_o[x,y] = 1-\exp\left(-\theta\int_0^1\frac{\sqrt{\dot{x}(z)^2+\dot{y}(z)^2}}{\mu(x(z),y(z))}\mathrm{d}z\right). \tag{12}$$

Note that the ratio δ_l/r_0^2 is proportional to the connection density, which is defined as the average number of connections per unit of length. Hence, the condition $\delta_l/r_0^2 \to \theta$ indicates that the connection density must be constant in the limit.

This formulation provides an expression of the outage probability as a functional[3] of the parametric path variables x and y. We note that the path that minimizes the outage probability is independent of the scaling factor θ.

III. Optimization in the Presence of an Interferer

We now present an application of the model described in the previous section to optimize the placement of relays when communication takes place in the presence of a single interfering transmission[4] (see Fig. 1). In this particular example, we can arbitrarily center the coordinate system on the interference source. By doing so, we may cast the outage probability expression in polar coordinates to obtain

$$P_o[r] = 1-\exp\left(-\theta\int_{\phi_1}^{\phi_2}\frac{\sqrt{r^2+\dot{r}^2}}{\mu(\phi,r)}\mathrm{d}\phi\right) \tag{13}$$

where the radius r is a function of the angle ϕ, ϕ_1 and ϕ_2 represent the respective angle-ordinates of the source and destination nodes, and the link quality function μ is written in terms of the new polar variables.

A. Link Quality Function Definition

Suppose the interference transmission is directive with gain function given by

$$g(\phi,\phi_o) = 1+\epsilon\cos(\phi-\phi_o), \qquad \epsilon\in[0,1]. \tag{14}$$

This definition gives a simple cardioid pattern in the azimuth variable ϕ, where ϕ_o denotes the orientation angle of the peak antenna gain. The parameter ϵ defines the magnitude of the peak gain, such that $\epsilon = 1$ provides a peak power gain of two and $\epsilon = 0$ defines an isotropic radiation pattern.

Assume the interference power level follows an inverse square law with Euclidean distance from the origin. By computing the signal-to-interference-plus-noise ratio (SINR), we can define the link quality function as[5]

$$\mu(\phi,r) = \left(1+k\frac{g(\phi,\phi_o)}{r(\phi)^2}\right)^{-1} \tag{15}$$

where

$$k = \left(\frac{\lambda}{4\pi}\right)^2\frac{P_I}{\sigma^2}. \tag{16}$$

The parameter P_I is the average interference power level, λ signifies the wavelength, and σ^2 is the noise variance at the receivers. Hereafter, with a slight abuse of terminology, we refer to k as the normalized interference-to-noise ratio (INR)[6].

B. Differential Equation for the Stationary Paths

We now proceed to apply the calculus of variations to compute the stationary paths in polar coordinates, which are described by the function $r(\phi)$. Denoting the integrand in (13) by $L(\phi,r,\dot{r})$, the Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{\partial L}{\partial \dot{r}}\right). \tag{17}$$

Using the link quality function μ defined above, one can rewrite (17) as

$$F(\phi,r,\dot{r}) \coloneqq \ddot{r} = r+\frac{2\dot{r}^2}{r}-\frac{(2gr+\dot{g}\dot{r})(1+\left(\frac{\dot{r}}{r}\right)^2)}{\frac{r^2}{k}+g}. \tag{18}$$

The solutions of (18) are the stationary paths for the outage probability functional given in (13).

[3]The notation $P_o[x,y]$ is used to highlight the fact that the outage probability depends on the functions x and y.

[4]Due to space restriction, we defer analysis of more complex scenarios, e.g. multiple interferers, to future work.

[5]The calculation follows that outlined in [7].

[6]In fact, k is proportional to the INR, where the proportionality constant is related to the effective antenna aperture.

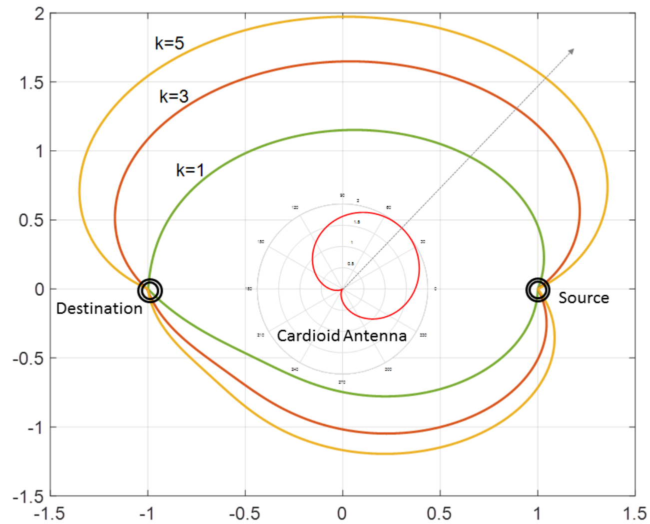

Fig. 2. Optimized routes in the presence of a directional interferer. The directivity is a cardioid pattern with 3dB peak gain at $\pi/4$; $k \in \{1, 3, 5\}$.

Algorithm 1: Successive Route Approximation

initialize S, h, r_d, $r(S+2) \gg r_d$, $\dot{r}_l(1)$, $\dot{r}_u(1)$, ζ;
while $|r(S+2) - r_d|/r_d > \zeta$ **do**
  $\dot{r}(1) = (\dot{r}_l(1) + \dot{r}_u(1))/2$;
  `// Euler's Explicit Method`
  for $i = 1 : S+1$ **do**
    $r(i+1) = r(i) + h\dot{r}(i)$;
    $\dot{r}(i+1) = \dot{r}(i) + hF(\phi(i), r(i), \dot{r}(i))$;
    $\phi(i+1) = \phi(i) + h$;
  end
  if $r(S+2) > r_d$ **then**
    $\dot{r}_u(1) = \dot{r}(1)$;
  else
    $\dot{r}_l(1) = \dot{r}(1)$;
  end
end
return $\{\phi\}, \{r\}$.

C. Numerical Solution

An analytic solution to (18) does not immediately present itself; hence, we employ numerical methods to solve the differential equation here. Specifically, Euler's explicit method lends itself to this example. We consider S discrete samples over the continuum between the source and destination nodes, uniformly distributed over the interval $[\phi_1, \phi_2]$, such that the angular sample spacing is $h = (\phi_2 - \phi_1)/(S+1)$. Let $i \in \{1, 2, \ldots, S+2\}$ denote the index of these samples, where $(\phi(i), r(i))$ represent the coordinates of the ith sample, and $i = 1$ and $i = S+2$ are the source and destination indices, respectively. By this definition, we have that $\phi(1) = \phi_1$ and $r(1)$ is the radial distance of the source from the origin. One may estimate $\dot{r}(1)$ and invoke Euler's explicit method to return the calculated polar coordinates of the path described by (18). We then successively approximate $\dot{r}(1)$, over the interval $[\dot{r}_l(1), \dot{r}_u(1)]$[7], until the calculated radial ordinate of the destination node $r(S+2)$ is within an acceptable fractional deviation ζ of the actual radial distance r_d. Associated pseudo-code is outlined in Algorithm 1. It is clear from the description of the algorithm that its complexity does not scale with the number of nodes N in the network; instead, it scales linearly in the number of samples S, which may be considerably less than N in large-scale practical scenarios.

As an illustration of this approach, numerically optimized routes for $k \in \{1, 3, 5\}$ are shown in Fig. 2 where $S = 1000$. We found that setting $S = 100$ gave satisfactory results. With the direction of the peak gain set to $\phi_o = \pi/4$, we clearly observe longer routes between the source and destination nodes in the half-plane interference is directed. We further observe the optimal routes are repelled from the origin in the direction of peak radiation.

Fig. 2 also depicts one path for each value of INR in the lower half-plane. These are also stationary paths that satisfy (18). Interestingly, we see the stationary paths veering toward the interference source in the lower half-plane as a result of the lower level of interference. It can be argued that the paths shown in the figure are all local minima, since one can imagine perturbations of the paths toward the interference source would increase the INR, while perturbations away from the source would lead to a reduction in the connectivity probability (due to the increased path length). A rigorous treatment of this argument is omitted due to space restrictions.

IV. Sampling the Continuum for Practical Deployments

Having determined a methodology for the creation of an equivalent continuous multi-hop route, We now distribute a finite number of relay nodes onto the stationary path, according to four placement strategies. Here, we list these in increasing order of processing complexity.

- Equi-angle: nodes distributed with constant angular offset
- Equi-spacing: nodes distributed with constant distance along the continuum
- Equi-Euclidean: nodes distributed with constant Euclidean distance
- Optimum-angle: optimal angle ordinates of nodes are determined using the `fmincon` function in Matlab®.

Given a required outage probability (0.05 is considered a realistic target here), we determine the value of θ by rearranging (13). We then calculate the length scale $r_o = \sqrt{L/\theta}$, where $L = \int_{\phi_1}^{\phi_2} \sqrt{r^2 + \dot{r}^2}\, \mathrm{d}\phi$ is the length of the continuum. The multi-hop outage probability is then calculated from

$$P_o = 1 - \prod_{n=0}^{N} \exp\left(-\frac{\|\mathbf{p}(n+1) - \mathbf{p}(n)\|^2}{r_0^2 \mu(\phi(n+1), \|\mathbf{p}(n+1)\|)}\right). \quad (19)$$

[7]The successive approximation is conditional on the variation of $r(S+2)$ is strictly increasing monotonic over the interval $[\dot{r}_l(1), \dot{r}_u(1)]$. Due to space restrictions, we omit further discussion relating to initial approximation of $\dot{r}(1)$.

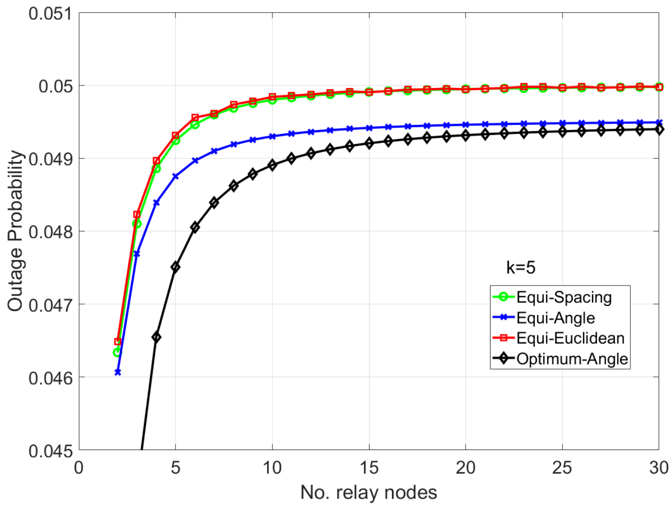

Fig. 3. Outage probability versus the number of relay nodes placed on the continuum according to equi-spacing, equi-angle, equi-Euclidean and optimum angle placement strategies for $k = 5$.

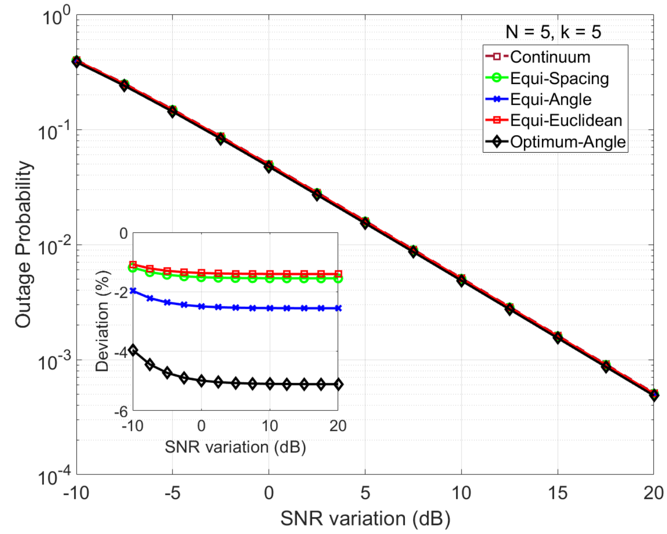

Fig. 4. Outage probability and % deviation in outage probability from the continuous path versus SNR for five relay nodes placed on the continuous path according to equi-spacing, equi-angle, equi-Euclidean and optimum angle placement strategies for $k = 5$.

We vary N over the interval 2 and 30 and calculate the corresponding outage probabilities for each placement strategy for $k = 5$. Results are shown in Fig. 3. With increasing numbers of nodes, the equi-Euclidean relay spacing converges to the equi-spacing case, which represents uniform relay spacing over the continuum and which satisfies the constant connectivity density condition that underpins our continuum model. In this case, we observe a rapid convergence to the $P_o = 0.05$ asymptote. Since the equi-angle and optimum angle scenarios violate the constant connection density condition, they do not exactly converge to the $P_o = 0.05$ asymptote. Notwithstanding this, the simplest form of relay placement yields an acceptably low deviation from the required outage probability. This is further highlighted in Fig. 4, where we vary the effective SNR from the level that yields an outage probability of 0.05 for the continuum. The nonuniform angular distribution yields a lower outage probability due to the interference being nonuniform over the continuum, and reducing the Euclidean distance between adjacent relay nodes subject to higher interference will intuitively yield lower outage probability overall. However, this example illustrates that the suboptimality of the continuum approach is not significant.

V. Conclusion and Discussions

In this paper, we have demonstrated the effectiveness of the continuum model for end-to-end outage probability calculation when applied to multi-hop route optimization in a wireless network subject to an inhomogeneous scalar field of interference. Taking a canonical example of interference originating from a single point in $\mathbb{R}^2$, we have shown that the scale of processing complexity reduces from $O(N^3)$ for a discrete Lagrangian method of constrained optimization (where N is the number of relays) to $O(S)$ in the continuum model, with S samples used to approximate the continuous path. Although analytic forms representing the optimal path do not exist in general, it is possible to employ basic numerical techniques to approximate it. From this approximation, it is straightforward to sample the continuum with a finite number of points (relays), thus yielding an acceptably low deviation in outage probability relative to the optimum.

Several open problems remain. Analytic solutions to particular problems are desirable, and more complicated scenarios and cost functions should be considered. Furthermore, optimal sampling of the continuum is also of interest. We hope this work inspires others to consider continuum modeling in related problems in communications and signal processing.

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