Analytical Method to Convert Circular Harmonic Expansion Coefficients for Sound Field Synthesis by Using Multipole Loudspeaker Array

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***Abstract*—We propose a method to synthesize the two-dimensional (2D) exterior sound field of a directional sound source using a Cartesian multipole loudspeaker array in which each loudspeaker unit is located on a Cartesian grid. We also propose an analytical method that models the sound field of the desired directional source in order to obtain weighting coefficients for each multipole from the circular harmonic expansion coefficients. The conversion method is derived by comparing the sound field created by a higher-order derivative of the free field Green's function and the corresponding field expressed by circular harmonic expansion coefficients in 2D space. In contrast to an existing analytical conversion method, the proposed method reproduces not only directivity patterns but also phases of the radiated sound from a target sound source, thereby enabling accurate sound field synthesis. We used numerical simulations to show that the proposed method achieved more accurate sound field reproduction than an existing pressure-matching-based method at higher frequency regions.**

***Index Terms*—multipole loudspeaker array, circular harmonics, sound field synthesis, analytical method, Cartesian multipole**

I. Introduction

Spatial audio reproduction is key for providing highly realistic experiences to audiences in theaters, and such systems have recently been introduced for live events [1] [2]. Most of these systems are implemented on the basis of two major sound field expressions: higher order ambisonics (HOA) [3] [4] and wave field synthesis (WFS) [5] [6]. HOA decomposes the sound field at an arbitrary point with spherical harmonic expansion [4], while WFS reproduces an arbitrary sound field by using secondary source distributions on the basis of the first Rayleigh integral [6].

There is another method to express sound fields, namely, multipole superposition. Multipole superposition is a technique to reproduce desired sound fields or directional patterns by superposing weighted "basic multipoles" such as dipoles, quadrupoles, and so on [7]–[11]. It has been indicated that multipole sources span the same space as that described by circular or spherical harmonics [12]. Most applications of multipole sources have aimed at reproducing only directivity patterns of target acoustic sources using loudspeaker arrays [13]–[16]. These applications are based on the least-squares method [14] [15], orthogonal projection into basis functions [13], and analytical conversion from circular harmonics to weighting coefficients [16]. Several methods that reproduce the phase as well as the directivity of an arbitrary-order multipole are based on circular loudspeaker arrays and least-squares-based methods [17]–[19].

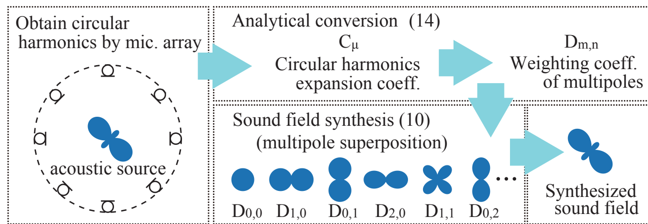

Fig. 1. Overview of proposed method.

In this paper, we propose a method to synthesize the two-dimensional (2D) exterior sound field of a directional sound source on the basis of multipole superposition (Fig. 1) by using a Cartesian multipole loudspeaker array in which loudspeaker units are arranged on a Cartesian grid. The proposed method analytically converts circular harmonic expansion coefficients, which are obtained from the sound field of a target sound source, into weighting coefficients for multipoles. Thus, compared to least-squares-based methods, the proposed method can avoid becoming ill-conditioned when calculating weighting coefficients. The conversion method is derived from comparing the coefficients of two different expressions of the desired sound field: one, the circular harmonic expansion, and two, the weighted sum of partial derivatives of the free field Green's function of the Helmholtz equation. Numerical simulations are performed to compare the accuracy of the sound fields reproduced by the proposed method and an existing pressure-matching-based method.

II. Sound field of a directional source

A. Free field Green's function in 2D space

The sound field created by a point monopole located at the position $\mathbf{r}'$ is modeled by the Green's function $G(\mathbf{r}|\mathbf{r}', k)$ that satisfies the following Helmholtz equation driven by a spatio-temporal Dirac pulse positioned at $\mathbf{r}'$.

$$(\nabla^2 + k^2)G(\mathbf{r}|\mathbf{r}', k) = \delta(\mathbf{r} - \mathbf{r}'), \quad (1)$$

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where $\mathbf{r} = (x, y) = (r, \phi)$ is an arbitrary point in the Cartesian or polar coordinate systems, respectively. $k = \frac{\omega}{c}$ is the wave number. ∇ denotes the spatial gradient. ω and c are the angular frequency and the speed of sound respectively. Assuming that the target field is the free field in 2D space, the Green's function is expressed using the zero-th order Hankel function of the second kind as follows [11].

$$G_{2\text{D}}(\mathbf{r}|\mathbf{r}', k) = \frac{j}{4} H_0^{(2)}(k|\mathbf{r} - \mathbf{r}'|), \quad (2)$$

where $j = \sqrt{-1}$ is the imaginary unit.

B. Partial derivatives of the Green's function

The sound field created by the first-order partial derivative of the Green's function (2) along the x-axis is given as

$$\begin{aligned} \frac{\partial G_{2\text{D}}(\mathbf{r}, k)}{\partial x} &= \frac{j}{4} \left(k \frac{dr}{dx} \right) \left. \frac{dH_0^{(2)}(z)}{dz} \right|_{z=kr} \\ &= \frac{j}{4} \left(k \cos\phi \right) \left. \frac{H_{-1}^{(2)}(z) - H_1^{(2)}(z)}{2} \right|_{z=kr} \\ &= \frac{j}{4} H_0^{(2)}(kr) \left(-jk \cos\phi \right), \end{aligned} \quad (3)$$

where ϕ satisfies $\cos\phi = \frac{x}{r}$. We used the following relationships to obtain the above equation: $H_{-n}^{(2)}(z) = (-1)^n H_n^{(2)}(z)$, and $H_n^{(2)}(z) \approx j^n H_0^{(2)}(z)$ which holds for the large value approximation of the n-th order Hankel function [20]. This means that the partial derivative of the Green's function (2) with respect to x is approximated by multiplying the term $-jk\cos\phi$. Thus, the n-th-order derivative with regard to x can be obtained by multiplying $(-jk\cos\phi)^n$ by $G_{2\text{D}}$. Partial derivatives with regard to y can also be obtained by replacing $\cos\phi$ with $\sin\phi$. Thus, the higher-order derivative of the Green's function is expressed as follows.

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} G_{2\text{D}}(\mathbf{r}, k) = G_{2\text{D}}(\mathbf{r}, k)(-jk)^{m+n} \cos^m\phi \sin^n\phi. \quad (4)$$

C. Sound field expressed by circular harmonics

The sound field created by an arbitrary sound source can also be expressed by the following circular harmonic expansion [11].

$$S(\mathbf{r}, k) = \sum_{\mu=-\infty}^{\infty} C_\mu H_\mu^{(2)}(kr) e^{j\mu\phi}. \quad (5)$$

In actual implementation, the circular harmonics coefficients C_μ are obtained by a Fourier transformation of acoustic signals recorded by a circular microphone array placed around the sound sources [21].

III. Sound field created by multipole superposition

A. Multipole sources

The sound field created by a dipole source along the x-axis with a distance of $2d$ between adjacent monopole sources is given by the following equation:

$$\begin{aligned} S_{\text{dipole}}(\mathbf{r}, k) &= \frac{j}{4} \{ H_0^{(2)}(k|\mathbf{r} - \mathbf{d}|) - H_0^{(2)}(k|\mathbf{r} + \mathbf{d}|) \} \\ &= \frac{j}{4} d \sum_{n=-\infty}^{\infty} \left\{ \frac{J_n(kd)}{d} - (-1)^n \frac{J_n(-kd)}{d} \right\} H_n^{(2)}(kr) e^{jn\phi}. \end{aligned} \quad (6)$$

Using l'Hôpital's rule, the term related to the Bessel's function yields,

$$\lim_{d \to 0} \frac{J_n(kd)}{d} = \lim_{d \to 0} k \left. \frac{dJ_n(z)}{dz} \right|_{z=kd} = k \frac{J_{n-1}(0) - J_{n+1}(0)}{2}. \quad (7)$$

At $x = 0$, the Bessel's function $J_n(x)$ gives 0 for any value of n other than 0. Thus the last equation of (6) yields

$$\begin{aligned} S_{\text{dipole}}(\mathbf{r}, k) &= \frac{j}{4} dk \left\{ H_1^{(2)}(kr) e^{j\phi} - H_{-1}^{(2)}(kr) e^{-j\phi} \right\} \\ &= \frac{j}{4} \{ 2dk H_1^{(2)}(kr) \cos\phi \} \\ &\approx G_{2\text{D}}(kr) \{ 2jdk \cos\phi \}. \end{aligned} \quad (8)$$

To obtain the above equation, we used the following relationships again: $H_{-n}^{(2)}(z) = (-1)^n H_n^{(2)}(z)$, and $H_n^{(2)}(z) \approx j^n H_0^{(2)}(z)$.

This equation means that the first-order multipole (dipole) along the x-axis is approximated by multiplying the term $2jdk\cos\phi$ by $G_{2\text{D}}$. Thus, the n-th-order multipole along the x-axis can be obtained by multiplying $(2jdk\cos\phi)^n$ by $G_{2\text{D}}$. Partial derivatives with regard to y can also be obtained by replacing $\cos\phi$ with $\sin\phi$. Thus, the higher-order multipole is defined as follows:

$$\begin{aligned} S_{m,n}(\mathbf{r}, k) &= G_{2\text{D}}(kr)(2jdk)^{m+n} \cos^m\phi \sin^n\phi \\ &= (-2d)^{m+n} \frac{\partial^{m+n}}{\partial x^m \partial y^n} G_{2\text{D}}(\mathbf{r}, k). \end{aligned} \quad (9)$$

As can be seen from this equation, the sound fields created by higher-order multipoles correspond to higher-order derivatives of the free field Green's function, other than a coefficient $(-2d)^n$.

B. Multipole superposition

The sound field created by a directional source can be expressed by superposition of basic multipoles weighted by coefficients obtained by the source's directivity pattern. The sound field is expressed by the following equation.

$$S(\mathbf{r}, k) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D_{m,n} \frac{\partial^{m+n}}{\partial x^m \partial y^n} G_{2\text{D}}(\mathbf{r}, k), \quad (10)$$

where $D_{m,n}$ is the weighting coefficient for the (m, n)-th-order multipole. It is important to obtain these weighting coefficients for each multipole correctly for accurate sound field synthesis of the target directional sound source using multipole superposition.

Such an acoustic inverse problem sometimes becomes ill-conditioned for obtaining weighting coefficients if least-squares-based methods are employed [11] [22]; in such cases, analytical methods are useful. Methods of analytical conversion among circular harmonics, spherical harmonics, and an angular spectrum have been investigated in the field of WFS [22] [23]; however, there are few analytical methods for obtaining weights of multipole sources from those expansion coefficients [16]. In the following section, we propose an analytical method to convert circular harmonic expansion coefficients to weighting coefficients of multipoles.

IV. Analytical Conversion of Circular Harmonics for Multipole Superposition

A. Derivation of analytical conversion

As can be seen from (9), the (m, n)-th-order multipole creates a directivity pattern expressed by a combination $\cos^m\phi\sin^n\phi$. Thus, we modify the equation (5) to obtain the coefficients of $\cos^m\phi\sin^n\phi$. In the following equation, $H_\mu^{(2)}(kr)$ is denoted by $H_\mu^{(2)}$.

$$S(\mathbf{r},k) = \sum_{\mu=-\infty}^{\infty} C_\mu H_\mu^{(2)} e^{j\mu\phi}$$
$$= C_0 H_0^{(2)} + \sum_{\mu=1}^{\infty} H_\mu^{(2)}\{C_\mu e^{j\mu\phi} + (-1)^\mu C_{-\mu} e^{-j\mu\phi}\}.$$

Using the relationship $H_\mu^{(2)} \approx j^\mu H_0^{(2)}$, we obtain the following equation:

$$S(\mathbf{r},k) = H_0^{(2)}\left[C_0 + \sum_{\mu=1}^{\infty} j^\mu\{C_\mu e^{j\mu\phi} + (-1)^\mu C_{-\mu}e^{-j\mu\phi}\}\right].$$

Applying Euler's equation followed by binomial expansion to the term $e^{j\mu\phi}$ in (5), the above equation yields,

$$S(\mathbf{r},k) = C_0 H_0^{(2)} + \sum_{\mu=1}^{\infty} H_\mu^{(2)}\{C_\mu e^{j\mu\phi} + (-1)^\mu C_{-\mu}e^{-j\mu\phi}\}$$
$$= H_0^{(2)}\left[C_0 + \sum_{\mu=1}^{\infty} j^\mu \sum_{n=0}^{\mu} j^n \binom{\mu}{n}\{C_\mu + (-1)^{\mu-n}C_{-\mu}\}\cdot \cos^{\mu-n}\phi\sin^n\phi\right]. \tag{11}$$

Using (2), (4), and (10), the same sound field is expressed as

$$S(\mathbf{r},k) = \sum_{\mu=0}^{\infty}\sum_{n=0}^{\mu} D_{\mu-n,n}\frac{j}{4}H_0^{(2)}(-jk)^\mu \cos^{\mu-n}\phi\sin^n\phi. \tag{12}$$

Comparing coefficients of $\cos^{\mu-n}\phi\sin^n\phi$, the following equation holds.

$$D_{\mu-n,n}\frac{j}{4}(-k)^\mu = j^n\binom{\mu}{n}\{C_\mu + (-1)^{\mu-n}C_{-\mu}\}.$$

Replacing μ with $m+n$, we obtain weighting coefficients as

$$D_{m,n} = \frac{4j^{n-1}}{(-k)^{m+n}}\binom{m+n}{n}\{C_{m+n} + (-1)^m C_{-m-n}\}. \tag{13}$$

To reproduce the sound field of higher-order derivatives of the Green's function using multipole superposition, the term $(-2d)^{m+n}$ must be multiplied by the partial derivatives. Considering this term, the weighting coefficients for multipole superposition yield

$$D_{m,n} = \frac{4j^{n-1}}{(2dk)^{m+n}}\binom{m+n}{n}\{C_{m+n} + (-1)^m C_{-m-n}\}. \tag{14}$$

B. Relation to existing analytical conversion

An analytical method for obtaining weighting coefficients from circular harmonic coefficients has been previously proposed [16]. This method derived weighting coefficients by comparing the sound field expressed by circular harmonic expansion and Taylor's expansion on a unit circle with a directional source at its center. The weighting coefficients derived in [16] are given as the following equation:

$$D_{m,n}^{(\text{old})} = \frac{j^n}{(j2dk)^{m+n}}\binom{m+n}{n}\cdot \left\{C_{m+n}H_{m+n}^{(2)}(k) + (-1)^n C_{-m-n}H_{-m-n}^{(2)}(k)\right\}$$
$$= \frac{j^n H_0^{(2)}(k)}{(2dk)^{m+n}}\binom{m+n}{n}\{C_{m+n} + (-1)^m C_{-m-n}\}. \tag{15}$$

Compared with the proposed method (14), the existing method is expressed as follows:

$$D_{m,n}^{(\text{old})} = \frac{j}{4}H_0^{(2)}(k)D_{m,n}. \tag{16}$$

This equation shows that the weighting coefficients obtained by the proposed method were weighted by the value of the free field Green's function along a unit circle in the weighting coefficient obtained by [16]. This factor affected the phase of the reproduced sound field, making accurate sound field synthesis difficult.

V. Implementation of Multipole Sources

A. Sound field of an arbitrary-order multipole

The sound field created by the (m, n)-th-order multipole (9) is expressed by the following linear combination of the Hankel functions.

$$S_{m,n}(\mathbf{r},k) = \frac{j}{4}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} g_{\mu,\nu}^{m,n} H_0^{(2)}(k|\mathbf{r} - \mathbf{r}_{\mu,\nu}^{m,n}|), \tag{17}$$

where $\mathbf{r}_{\mu,\nu}^{m,n} = (x_\mu^m, y_\nu^n)$ and $g_{\mu,\nu}^{m,n}$ denote the position of the (μ,ν)-th monopole source in the multipole and the gain of the corresponding monopole source, respectively. Positions and gains for monopole sources in an arbitrary-order multipole are formulated in the following subsections.

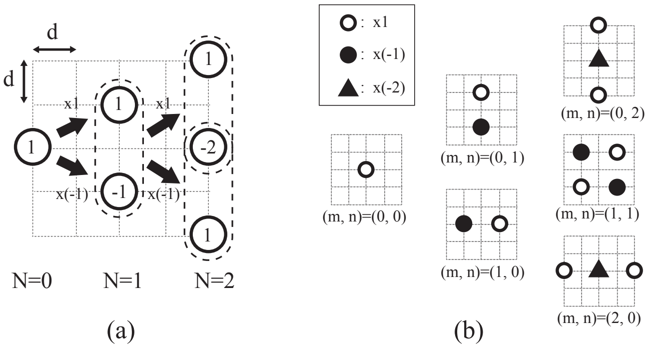

Fig. 2. Positions and gains of monopoles in multipoles up to the second order. (a) 1D space. (b) 2D space.

B. Positions of monopole sources in a multipole

The n-th-order multipole is obtained by arranging two of the (n-1)-th-order multipoles with an interval of $2d$. Because multipoles have some volume, these lower-order multipoles are superposed. As a result, $n+1$ monopole sources form the n-th-order multipole as depicted in Fig. 2 (a). The positions of monopole sources in the (m, n)-th-order multipole are summarized by the following equation:

$$\begin{aligned} x_\mu^m &= x_c + (m - 2\mu)d \qquad (0 \le \mu \le m) \\ y_\nu^n &= y_c + (n - 2\nu)d \qquad (0 \le \nu \le n), \end{aligned} \tag{18}$$

where (x_μ^m, y_ν^n) is the position of the (μ, ν)-th monopole in the (m, n)-th-order multipole. (x_c, y_c) is the center of the multipole.

C. Gains of monopole sources in a multipole

Gains for each monopole source in a multipole are obtained by multiplying $+1$ for the lower-order multipole shifted in a positive direction and -1 for that shifted in a negative direction, followed by superposing these lower-order multipoles as depicted in Fig. 2 (a). The gains of monopole sources in the (m, n)-th-order multipole are summarized by the following equation:

$$g_\zeta^X = \begin{cases} 1 & (\zeta = 0) \\ g_\zeta^{X-1} - g_{\zeta-1}^{X-1} & (0 < \zeta < X) \\ -g_{\zeta-1}^{X-1} & (\zeta = X). \end{cases} \tag{19}$$

The above rule is applied to obtain gains for 2D cases. As a result, the gain for the (μ, ν)-th monopoles in the (m, n)-th-order multipole $g_{\mu,\nu}^{m,n}$ in a 2D case are calculated using the product of the gain along the x-axis g_μ^m and y-axis g_ν^n.

$$g_{\mu,\nu}^{m,n} = g_\mu^m \cdot g_\nu^n. \tag{20}$$

Both g_μ^m and g_ν^n are computed by (19). Gains for each monopole in the 2D cases up to $N = 2$ are illustrated in Fig. 2 (b).

VI. Experimental results

We performed computer simulations to evaluate the proposed method in comparison with an existing least-squares-method based pressure-matching method [24].

$$\mathbf{d}^{\mathrm{PM}} = (\mathbf{G}^{\mathrm{H}}\mathbf{G} + \lambda \mathbf{I})^{-1}\mathbf{G}^{\mathrm{H}}\mathbf{s}, \tag{21}$$

where $\mathbf{d}^{\mathrm{PM}}$ is a vector whose elements are weighting coefficients for each multipole as $\mathbf{d}^{\mathrm{PM}} = [D_{0,0}^{\mathrm{PM}}, \cdots, D_{0,N}^{\mathrm{PM}}]^{\mathrm{T}}$, $\mathbf{s}$ is a vector of sound pressures observed at controlling points placed along a unit circle with a radius of 1 m (180 points with an interval of 2° between adjacent controlling points), and transfer function matrix $\mathbf{G}$ whose columns g_l were defined by (9) at the same controlling points. The maximum order of multipole and circular harmonics N were set as 4 in this simulation. The superscript $[\cdot]^{\mathrm{H}}$ is the Hermitian transpose of a matrix. λ is a regularization parameter to prevent the inverse matrix of $\mathbf{G}$ from being unstable [25].

To estimate the accuracy of the reproduced sound fields, the error distributions between the original and reproduced sound fields at position $\mathbf{r}$ were defined as

$$Err(\mathbf{r}) = 10 \log 10 \left(\frac{|S_{\mathrm{org}}(\mathbf{r}) - S_{\mathrm{syn}}(\mathbf{r})|^2}{|S_{\mathrm{org}}(\mathbf{r})|^2} \right), \tag{22}$$

where $S_{\mathrm{org}}(\mathbf{r})$ and $S_{\mathrm{syn}}(\mathbf{r})$ are the original and synthesized sound field at position $\mathbf{r}$, respectively. We used randomly generated $2N+1$ complex numbers with amplitudes of less than 1 for C_μ to compute the original sound field $S_{\mathrm{org}}(\mathbf{r})$ by (5), the synthesized sound fields by the proposed method using (10) and (14), and the corresponding sound fields by the existing method using (10) and (21). For the simulation, a multipole loudspeaker array was placed at the origin of the coordinate. The interval between adjacent monopole sources in the multipole loudspeaker array $(2d)$ was set as 0.01 m.

Figure 3 shows the results for the original sound field and the sound fields synthesized by the least-squares-based pressure-matching method and proposed method. The temporal frequency was 2 kHz. Averaged synthesis error up to 3400 Hz computed in the region with $-1.5 \le x \le 1.5$ and $1.0 \le y \le 3.0$ for the pressure-matching method and proposed method was plotted in Fig. 4.

The results indicated that at frequencies higher than 1100 Hz, the proposed method reproduced the sound field of the target directional source more accurately than the pressure-matching method did.

VII. Conclusion

We proposed a method for synthesizing the 2D exterior sound field of a directional sound source on the basis of multipole superposition. By using a Cartesian multipole loudspeaker array, the proposed method can reproduce the sound field of a directional sound source modeled by circular harmonic expansion coefficients. The proposed method analytically converts the circular harmonic expansion coefficients into weighting coefficients for multipoles. The conversion method is based on a comparison of coefficients derived from circular harmonic expansion coefficients and partial derivatives of the free field

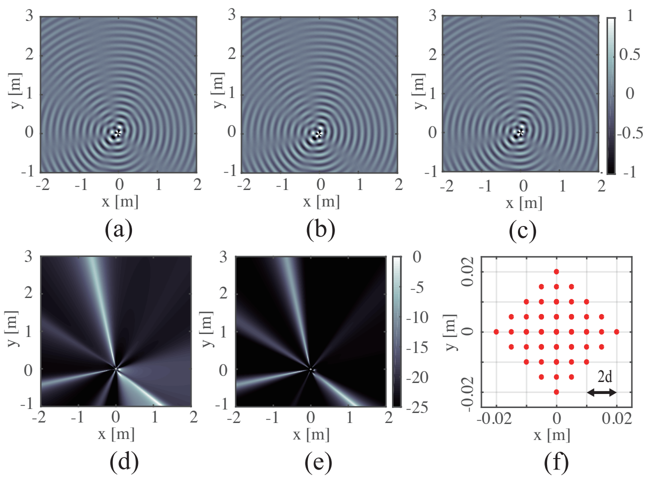

Fig. 3. Reproduced sound fields and error distributions for $N = 4$. (a) Original sound field. (b) Sound field reproduced by existing pressure-matching-based method. (c) Sound field reproduced by proposed method. (d) Error distribution between original sound field and that reproduced by existing method. (e) Error distribution between original sound field and that reproduced by proposed method. (f) Positions of loudspeakers in a multipole array.

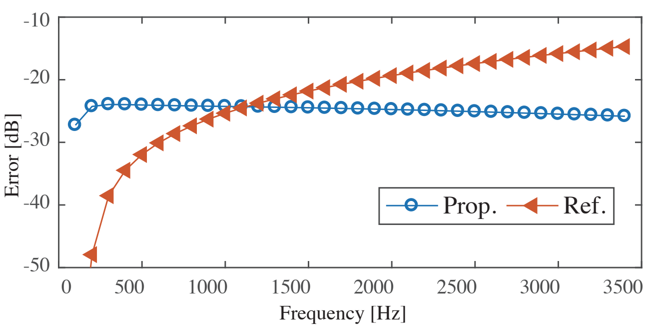

Fig. 4. Results of averaged synthesis error for $N = 4$. **Prop**: error of proposed method. **Ref**: error of existing pressure-matching-based method.

Green's function of the Helmholtz equation. Weighting coefficients obtained by the existing method [16] were weighted by the Green's function calculated along the unit circle compared with the coefficients obtained by the proposed method, thereby reproducing only the desired directivity pattern. Using numerical simulations, we showed that the sound field reproduced by the proposed method was more accurate than that reproduced by the existing pressure-matching-based method, at regions with frequencies higher than 1100 Hz.

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