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Bayesian Sequential Joint Signal Detection and Signal-to-Noise Ratio Estimation

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***Abstract*—Jointly detecting a signal in noise and, in case a signal is present, estimating the Signal-to-Noise Ratio (SNR) is investigated in a sequential setup. The sequential test is designed such that it achieves desired error probabilities and Mean-Squared Errors (MSEs), while the expected number of samples is minimized. This problem is first converted to an unconstrained problem, which is then reduced to an optimal stopping problem. The solution, which is obtained by means of dynamic programming, is characterized by a non-linear Bellman equation. A gradient ascent approach is then presented to select the cost coefficients of the Bellman equation such that the desired error probabilities and MSEs are achieved. A numerical example concludes the work.**

***Index Terms*—sequential analysis, joint detection and estimation, signal-to-noise ratio estimation, Monte Carlo, optimal stopping**

I. Introduction

Sequential analysis is a field of statistics initially introduced by Abraham Wald in the late 1940s [1]. The aim of sequential analysis is to perform statistical inference, e.g., estimation or detection, with a minimum number of samples while ensuring a certain inference quality. An overview on sequential detection methods is given in [2] and on sequential estimation methods in [3]. Sequential inference is an area of ongoing research. Especially for low power or time critical applications, sequential methods are preferable to conventional ones.

In most applications, detection and estimation are intrinsically coupled and both are of primary interest. This means that we want to decide between two or more hypotheses and, depending on the test outcome, estimate one or more parameters. This problem, referred to as joint detection and estimation, was first investigated by Middleton and Esposito [4] in the late 1960s, using a fixed number of samples. They used a combined risk function in a Bayesian framework which was then minimized to obtain the optimal detector and estimator. Fredriksen *et al.* extended that framework to multiple hypotheses [5]. More recent solutions for the problem of joint detection and estimation involve a combined Neyman-Pearson and Bayesian approach [6] or Bayesian decision-dependent costs [7]. Joint detection and estimation gained attention in a lot of applications such as speech processing [8], biomedical applications [9], communications [10] or change point detection [11].

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Combining the ideas of sequential analysis and joint detection and estimation leads to a flexible and powerful framework applicable for example to state estimation in smart grids [12]. The problem of sequential joint detection and estimation was addressed, for example, in [12], where the aim was to minimize the number of samples subject to a constraint on a weighted sum of detection and estimation errors. In [13], we investigated the problem under distributional uncertainties. Recently, we developed a framework in a Bayesian context, which minimizes the average number of samples with constraints on the detection and estimation errors [14].

The problem of estimating the Signal-to-Noise Ratio (SNR) arises in a variety of fields such as communications [15], audio processing [16] or detectors with power constraints [17]. In communications, for example, estimating the SNR is important for adaptive demodulation schemes or power control [15]. In audio processing, estimating the SNR is crucial for speech enhancement [16].

In this work, we investigate the problem of sequentially detecting a signal in Gaussian noise and, in the case of a signal being present, estimating the SNR. This was also addressed in [18], where the true SNR was treated as an unknown and deterministic parameter. The aim of [18] is to minimize the expected run-length under constraints on the error probabilities as well as a constraint on the relative Mean-Squared Error (MSE). Contrary to [18], we treat both, the variances of the signal and the noise, as random variables with known distribution. Moreover, the constraint on the estimation accuracy is imposed on the MSE instead of the relative MSE.

The paper strongly builds on our previous work [14], in which a general framework for Bayesian sequential joint detection and estimation is derived. Hence, we do not cover all aspects of the underlying theory in detail, but refer the interested reader to [14] for an in-depth discussion.

The remainder of the paper is structured as follows. First, a detailed problem formulation is given in Section II. Then, the solution is presented in Section III. A numerical example is discussed in Section IV and Section V concludes the work.

II. Problem Statement

Let $\mathbf{X}_N = (X_1, \dots, X_N)$ be a set of random variables which is observed sequentially. This sequence can be generated under two different hypotheses, either H_0 or H_1. Under the null hypothesis, the received signal $\mathbf{X}_N$ is assumed to be composed of zero-mean Gaussian noise only, whereas under the alternative, the received signal $\mathbf{X}_N$ is assumed to be the signal of interest disturbed by zero-mean Gaussian noise. Using a linear model, the two hypotheses can be written as

$$\begin{aligned} \mathrm{H}_0: \quad & \mathbf{X}_N = \mathbf{W}_N \,, \\ \mathrm{H}_1: \quad & \mathbf{X}_N = \mathbf{S}_N + \mathbf{W}_N \,, \end{aligned} \tag{1}$$

where $\mathbf{S}_N = (S_1, \ldots, S_N)$ and $\mathbf{W}_N = (W_1, \ldots, W_N)$ denote the signal of interest and the noise, respectively. Both, the noise and the signal of interest, are zero-mean Gaussian distributed and their respective variances are denoted by σ_S^2 and σ_W^2. The variances σ_S^2 and σ_W^2 are random variables with known distributions which have a disjoint support. In addition, the occurrence of each hypothesis H_i, $i = 0, 1$, is random with probability $p(\mathrm{H}_i)$. It is further assumed, that the random variables σ_S^2 and σ_W^2 as well as $\mathbf{S}_N$ and $\mathbf{W}_N$ are statistically independent. Moreover, conditioned on σ_S^2 respective σ_W^2, the sequences $\mathbf{S}_N$ and $\mathbf{W}_N$ are independent and identically distributed.

Since $\mathbf{S}_N$ and $\mathbf{W}_N$ are zero-mean Gaussian distributed, the two hypotheses can be expressed in terms of the distributions of the received signal $\mathbf{X}_N$ as

$$\begin{aligned} \mathrm{H}_0: \quad & X_n \sim \mathcal{N}(0, \sigma_W^2),\, \sigma_W^2 \sim p(\sigma_W^2)\,, \\ \mathrm{H}_1: \quad & X_n \sim \mathcal{N}(0, \sigma_S^2 + \sigma_W^2),\, \sigma_S^2 \sim p(\sigma_S^2),\, \sigma_W^2 \sim p(\sigma_W^2)\,. \end{aligned}$$

The aim is to decide between these two hypotheses, and, in case we decide in favor of H_1, to estimate the SNR $\theta = \frac{\sigma_S^2}{\sigma_W^2}$. Since optimal methods for detection and estimation do not necessarily result in an overall optimal performance [6], we have to *jointly* solve the detection and estimation problem. Moreover, we are interested to find a procedure with a minimum number of average used samples, while the error probabilities and the MSE of the SNR estimate are bounded. Therefore, a *sequential* method is needed. Hence, we end up with a *sequential joint detection and estimation* problem.

In order the solve this problem, we observe a second sequence $\tilde{\mathbf{W}}_N = (\tilde{W}_1, \ldots, \tilde{W}_N)$, which consists of noise samples only. This sequence can, for example, be gathered by a sensor which is shielded against the signal of interests but has the same environmental conditions, like background noise or the noise of the communication channel. It is further assumed that the underlying hypothesis and the random parameters σ_W^2, σ_S^2 remain constant during the observation time. The realization of the random parameter σ_W^2 is the same for the observed sequences $\mathbf{X}_N$ and $\tilde{\mathbf{W}}_N$.

Before a more technical problem formulation can be provided, we have to introduce a few important quantities. The stopping and the decision rules at time n are denoted by Ψ_n, $\delta_n \in \{0, 1\}$, respectively. The estimator of the SNR at time n is denoted by $\hat{\theta}_n$. The tuple of stopping rule, decision rule and estimator is referred to as policy in the following and denoted by $\pi = \{\Psi_n, \delta_n, \hat{\theta}_n\}_{0 \leq n \leq N}$. Furthermore, the time instant at which the test stops, the so called stopping time, is defined as

$$\tau = \min\{n \geq 1 : \Psi_n = 1\}\,.$$

As performance measures of the sequential joint detection and estimation scheme, we use the expected run-length $\mathbf{E}[\tau]$ along the error probabilities and the MSE, which are given by

$$\begin{aligned} \alpha_i &= P(\delta_\tau = 1 - i \,|\, \mathrm{H}_i)\,, i = 0, 1\,, \\ \beta &= \mathbf{E}[\mathbf{1}_{\{\delta_\tau = 1\}}(\theta - \hat{\theta}_\tau)^2 \,|\, \mathrm{H}_1]\,. \end{aligned}$$

In this work, we consider a truncated sequential scheme, i.e., we force the test to stop at latest at a fixed time N, which implies that $\Psi_N = 1$. Hence, we can formulate the task of designing the test as the following constrained optimization problem:

$$\begin{aligned} \min_\pi \; \mathbf{E}[\tau] \quad & \text{s.t. } \Psi_N = 1 \\ & P(\delta_\tau = 1 \,|\, \mathrm{H}_0) \leq \kappa_0 \\ & P(\delta_\tau = 0 \,|\, \mathrm{H}_1) \leq \kappa_1 \\ & \mathbf{E}[\mathbf{1}_{\{\delta_\tau = 1\}}(\hat{\theta} - \theta)^2 \,|\, \mathrm{H}_1] \leq \kappa_2 \end{aligned} \tag{2}$$

In (2), the constants $\kappa_0, \kappa_1 \in (0, 1)$ are upper bounds on the detection errors and $\kappa_2 \in (0, \infty)$ is the upper bound on the MSE. As long as N is large enough and $\mathbf{E}[\theta^2 \,|\, \mathrm{H}_1]$ is finite, (2) always admits a solution.

III. Solution Methodology

To solve problem (2), we proceed as in [14]: First, the constrained optimization problem is converted to an unconstrained problem. This unconstrained problem is then reduced to an optimal stopping problem, which can be solved by means of dynamic programming. By exploiting the results in [14], we arrive at a projected gradient ascent to obtain the set of optimal cost coefficients. Finally, the gradients are evaluated numerically using Monte Carlo simulations.

A. Reduction to an Optimal Stopping Problem

In order to solve (2), the problem first has to be converted to an optimal stopping problem. To keep the problem tractable, it is not solved directly on the observation space but on a space of sufficient statistics. Conditioned on the variance, both random variables $\mathbf{X}_N$ and $\tilde{\mathbf{W}}_N$ are Gaussian distributed with a continuous prior, hence, we use a sufficient statistic $\mathbf{t}_n$, such that

$$p(\sigma_S^2, \sigma_W^2 \,|\, \mathbf{x}_n, \tilde{\mathbf{w}}_n) = p(\sigma_S^2, \sigma_W^2 \,|\, \mathbf{t}_n)\,.$$

Since the two sequences, $\mathbf{X}_N$ and $\tilde{\mathbf{W}}_N$, are Gaussian distributed with zero mean,

$$\mathbf{t}_n = [t_n^x, t_n^{\tilde{w}}]^\top = \left[\frac{1}{n}\sum_{i=1}^n x_i^2, \frac{1}{n}\sum_{i=1}^n \tilde{w}_i^2\right]^\top$$

can be used as a sufficient statistic. The transition kernel and the initial statistic are denoted by $\mathbf{t}_{n+1} = \xi_{\mathbf{t}_n}(x_{n+1}, \tilde{w}_{n+1})$ and $\mathbf{t}_0 = [0, 0]^\top$, respectively. First, (2) has to be converted to an unconstrained optimization problem in which the Lagrange multipliers of the inequality constraints are denoted by C_i, $i \in \{0, 1, 2\}$. Following the ideas from [14], we have to solve the unconstrained problem first with respect to the decision rule and then with respect to the estimator. It can be shown that when using the optimal decision rule and the optimal estimator, the design problem reduces to the optimal stopping problem

$$\min_{\Psi} \sum_{n=0}^{n} \mathbf{E}[\Phi_n(n + g(\mathbf{t}_n))] \,, \tag{3}$$

where we use the short hand notation $\Phi_n = \Psi_n \prod_{i=0}^{n-1}(1-\Psi_i)$ and define the instantaneous cost as

$$g(\mathbf{t}_n) = \min\{D_{0,n}(\mathbf{t}_n), D_{1,n}(\mathbf{t}_n)\} \,.$$

This means that the instantaneous cost for stopping at time n consists of the minimum of $D_{0,n}$ and $D_{1,n}$. These are the costs for stopping at time n and deciding in favor of H_0 or H_1 and are given by

$$\begin{aligned} D_{0,n}(\mathbf{t}_n) &= C_1 p(\mathrm{H}_1 \,|\, \mathbf{t}_n) \,, \\ D_{1,n}(\mathbf{t}_n) &= C_0 p(\mathrm{H}_0 \,|\, \mathbf{t}_n) + C_2 p(\mathrm{H}_1 \,|\, \mathbf{t}_n) \, \mathbf{Var}[\theta \,|\, \mathrm{H}_1, \mathbf{t}_n] \,. \end{aligned}$$

It can be seen that the cost of deciding in favor of H_0 consists of the detection error only, whereas the cost for deciding in favor of H_1 consists of a weighted sum of a detection and estimation error. Once the test has stopped, it decides in favor of the hypothesis whose cost is lower.

B. Characterization of the Cost Function

The cost function of a truncated optimal sequential test can be described by a system of non-linear integral equations, see, e.g., [14]. We assume for now fixed and finite cost coefficients C_i, $i \in \{0, 1, 2\}$. It can then be shown that the cost function of the sequential joint detection and estimation problem is characterized by the recursion

$$\begin{aligned} \rho_n(\mathbf{t}_n) &= \min\{g(\mathbf{t}_n), d_n(\mathbf{t}_n)\} \,, \text{ for } n < N \,, \\ \rho_N(\mathbf{t}_N) &= g(\mathbf{t}_N) \,, \end{aligned} \tag{4}$$

where $d_n(\mathbf{t}_n)$ are the costs for continuing the test, i.e.,

$$d_n(\mathbf{t}_n) = 1 + \mathbf{E}[\rho_{n+1}(\mathbf{t}_{n+1}) \,|\, \mathbf{t}_n] \,.$$

The optimal sequential scheme stops as soon as the costs for stopping $g(\mathbf{t}_n)$ are lower than the costs for continuing $d_n(\mathbf{t}_n)$. The optimal policy can hence be summarized as:

$$\begin{aligned} \Psi_n^\star &= \mathbf{1}_{\{\rho_n(\mathbf{t}_n) = g_n(\mathbf{t}_n)\}} \\ \delta_n^\star &= \mathbf{1}_{\{D_{0,n}(\mathbf{t}_n) > D_{1,n}(\mathbf{t}_n)\}} \\ \hat{\theta}_n^\star &= \mathbf{E}[\theta \,|\, \mathrm{H}_1, \mathbf{t}_n] \end{aligned} \tag{5}$$

It has to be mentioned that the optimal policy in (5) depends on the cost coefficients $C_i, i \in \{0, 1, 2\}$, which are, thus, crucial for the performance of the test.

C. Optimal Cost Coefficients

To obtain the set of optimal cost coefficients, one could in principle use the linear programming approach presented in [14]. However, the linear programming approach can be computationally demanding and memory consuming. Therefore, we present an alternative, computationally more efficient approach that exploits the properties of the particular problem at hand.

Algorithm 1 Calculation of the Optimal Cost Coefficients

inputs: $\kappa_0, \kappa_1, \kappa_2, C_0^0, C_1^0, C_2^0$
initialize: Set $k \leftarrow 0$
repeat
  Set $k \leftarrow k + 1$
  Calculate the policy π^k according to (5) using C^{k-1}
  Perform Monte Carlo simulation
  Calculate ∇C according to (8)
  Set $C^k = \max\{C^{k-1} + \gamma \cdot \nabla C, 0\}$
until $\tilde{\alpha}_0^k \approx \kappa_0$ and $\tilde{\alpha}_1^k \approx \kappa_1$ and $\tilde{\beta}^k \approx \kappa_2$
return π^k

It can be shown that the cost function at time $n = 0$ and using an initial sufficient statistic $\mathbf{t}_0$ is equivalent to a weighted sum of the average run-length, the error probabilities and the estimation errors, namely

$$\rho_0(\mathbf{t}_0) = \mathbf{E}[\tau] + \sum_{i=0}^{1} C_i p(\mathrm{H}_i)\alpha_i + C_2 p(\mathrm{H}_1)\beta \,. \tag{6}$$

Due to the strong connection between the derivative of the cost function with respect to the cost coefficients and the performance measures [14, Theorem 4.2] [19, Theorem 3.2], one can obtain the optimal cost coefficients by solving

$$\max_{C \geq 0} \left\{ \rho_0(\mathbf{t}_0) - p(\mathrm{H}_0) C_0 \kappa_0 - p(\mathrm{H}_1) C_1 \kappa_1 - p(\mathrm{H}_1) C_2 \kappa_2 \right\} ,$$

subject to

$$\begin{aligned} &\rho_n(\mathbf{t}_n) = \min\{g(\mathbf{t}_n), 1 + \mathbf{E}[\rho_{n+1}(\mathbf{t}_{n+1}) \,|\, \mathbf{t}_n]\} \,, \;\; n < N \,, \\ &\rho_N(\mathbf{t}_N) = g(\mathbf{t}_N) \,. \end{aligned}$$

As mentioned before, solving this problem by, for example, linear programming, can become tedious with increasing dimensionality of $\mathbf{t}_n$ and/or maximum number of samples N. Hence, we use an approximate but more tractable formulation to obtain the optimal cost coefficients. We first replace the term $\rho_0(\mathbf{t}_0)$ in the previous equation by the weighted sum given in (6). Once a test with fixed cost coefficients C_i, $i \in \{0, 1, 2\}$, is designed, the empirical expected run-length, error probabilities and estimation errors can be obtained by Monte Carlo simulations. Replacing the expected run-length, the error probabilities and the estimation errors by their Monte Carlo estimates, which are denoted by a tilde, we end up with the final optimization problem

$$\begin{aligned} &\max_{C \geq 0} \left\{ \tilde{\mathbf{E}}[\tilde{\tau}] + \sum_{i=0}^{1} C_i p(\mathrm{H}_i)(\tilde{\alpha}_i - \kappa_i) + C_2 p(\mathrm{H}_1)(\tilde{\beta} - \kappa_2) \right\} , \\ &\text{subject to} \\ &\rho_n(\mathbf{t}_n) = \min\{g(\mathbf{t}_n), 1 + \mathbf{E}[\rho_{n+1}(\mathbf{t}_{n+1}) \,|\, \mathbf{t}_n]\} \,, \;\; n < N \,, \\ &\rho_N(\mathbf{t}_N) = g(\mathbf{t}_N) \,. \end{aligned} \tag{7}$$

In order to solve (7), we propose an iterative algorithm which is summarized in Algorithm 1. In each iteration, the cost functions in (4) are calculated and hence, a policy as stated in (5) is obtained for a given set of cost coefficients. Next, a Monte Carlo simulation using this policy is performed. To update the set of cost coefficients, the gradient of the objective function in (7) is calculated and the set of cost coefficients is shifted in the direction of the gradient. The gradient is given by [14, Theorem 4.2] [19, Theorem 3.2]

$$\nabla C = [p(\mathrm{H}_0)(\tilde{\alpha}_0 - \kappa_0), p(\mathrm{H}_1)(\tilde{\alpha}_1 - \kappa_1), p(\mathrm{H}_1)(\tilde{\beta} - \kappa_2)] \,. \tag{8}$$

Since the cost coefficients have to be non-negative, the shifted set of cost coefficients has to be projected on the set of feasible coefficients. Moreover, in Algorithm 1, the gradient is scaled by a scalar γ which is used to control the convergence speed of the algorithm.

According to [14], the cost coefficients act as a slack variable, i.e., an optimal cost coefficient may be equal to zero if the corresponding constraint is implicitly fulfilled by another detection/estimation constraint. For the case, that all cost coefficients $C_i, i \in \{0, 1, 2\}$, are non-zero, all constraints are fulfilled with equality. To simplify notation, we focus on the latter case, though the procedure can be easily extended to deal with the general problem.

IV. Numerical Results

To validate the proposed approach and to illustrate some characteristics of the optimal test, we present a numerical example. Similar to [14], we use a two-step procedure for benchmarking, more precisely, a truncated Sequential Probability Ratio Test (SPRT) followed by an Minimum Mean-Squared Error (MMSE) estimator. The SPRT updates the likelihood ratio at every time instant and compares it to two predefined thresholds to decide whether to stop or to continue sampling. The likelihood ratio at time n can be calculated as

$$\Lambda(\mathbf{t}_n) = \frac{\int\int p(\mathbf{t}_n \,|\, \sigma_S^2, \sigma_W^2, \mathrm{H}_1) p(\sigma_S^2, \sigma_W^2) \mathrm{d}\sigma_S^2 \mathrm{d}\sigma_W^2}{\int p(\mathbf{t}_n \,|\, \sigma_W^2, \mathrm{H}_0) p(\sigma_W^2) \mathrm{d}\sigma_W^2}$$

and the thresholds are chosen according to Wald [1]

$$A = \frac{1-\kappa_1}{\kappa_0} \quad \text{and} \quad B = \frac{\kappa_1}{1-\kappa_0} \,.$$

Once the SPRT has stopped, we use an MMSE estimator to estimate the SNR.

In order to illustrate the proposed approach we chose the following example

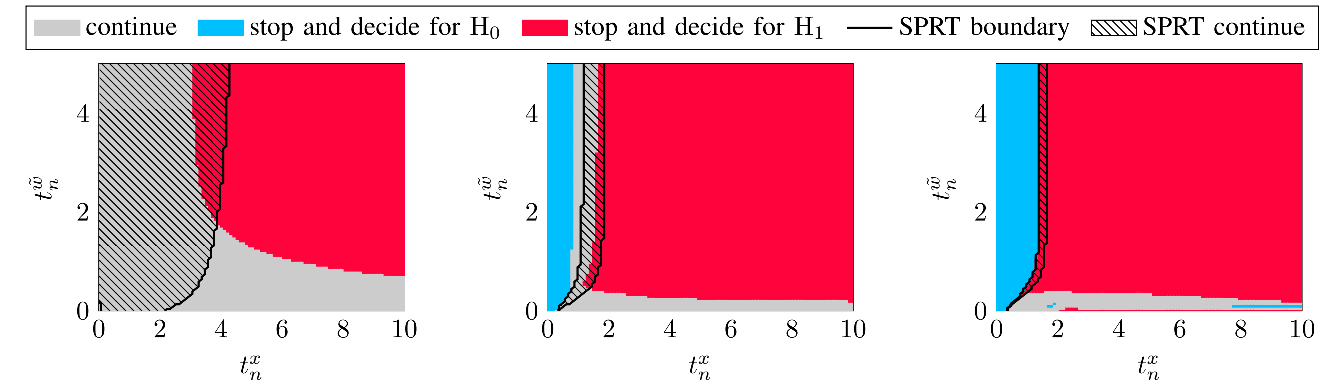
$$\begin{aligned} \mathrm{H}_0: \quad & X_n \sim \mathcal{N}(0, \sigma_W^2),\ \sigma_W^2 \sim \mathcal{U}(0.1, 1)\,, \\ \mathrm{H}_1: \quad & X_n \sim \mathcal{N}(0, \sigma_S^2 + \sigma_W^2),\ \sigma_S^2 \sim \mathcal{U}(1.2, 2)\,, \\ & \qquad\qquad\qquad\qquad \sigma_W^2 \sim \mathcal{U}(0.1, 1)\,, \end{aligned}$$

where $\mathcal{U}(a, b)$ denotes the uniform distribution on $[a, b]$. This corresponds to an SNR ranging approximately from 0.8 dB to 13 dB. For this example, we want to design a test that uses at most 80 samples, with constraints on the error probabilities of $\kappa_i = 0.05$, $i \in \{0, 1\}$, and a constraint on the MSE of $\kappa_2 = 1.25$. The prior probabilities of both hypotheses were set to $p(\mathrm{H}_0) = p(\mathrm{H}_1) = 0.5$. In order to solve the problem numerically, all quantities have to be discretized. This discretization is summarized in Table I. To obtain the posterior mean and the posterior variance, we refer to importance sampling [20], since standard numerical integration techniques suffer from large numerical inaccuracies especially for large n. For importance sampling, we use the prior $p(\sigma_S^2, \sigma_W^2)$ as proposal distribution and generate 10^5 samples of σ_S^2 and σ_W^2. The detection and estimation errors are estimated using $5 \cdot 10^5$ Monte Carlo runs in each iteration. Furthermore, we stop the iterative algorithm if the error probabilities do not differ more than 10^{-3} from the constraints, whereas the estimation errors can differ $5 \cdot 10^{-2}$ from the constraint. The initial cost coefficients were set to $C^0 = [50, 150, 500]$. The optimal cost coefficients obtained by Algorithm 1 are $C \approx [44.8,\ 572.1,\ 111.6]$.

TABLE I
Simulation Setup

quantity	domain	#grid points
σ_W^2	$[0.1, 1.1]$	101
σ_S^2	$[0.6, 2, 2]$	161
$\mathbf{t}_n$	$[0, 10] \times [0, 3]$	100×103
x_n	$[-8, 8]$	100
w_n	$[-6, 6]$	100

In Table II, the Monte Carlo results of the approximately optimal test are shown along the two-step procedure, where 10^5 Monte Carlo runs are used for evaluation. As one can learn from Table II(a), the optimal test hits the constraints exactly, except for the tolerance used during the design process. The two-step procedure, on the other hand, provides much smaller error probabilities than the constraints at the cost of exceeding the constraint on the MSE by a factor of approximately 3.5. In Table II(b) it can further be seen that the average run-length of the two-step procedure is much smaller than the one of the optimal scheme at the cost of violating the estimating constraint. Fig. 1 displays the evolution over time of both, the optimal and the SPRT policy for three distinct time instances. For the SPRT, only the corridor for continuing the test is shown, the decision region in favor of H_0 and H_1 are found left and right of this corridor, respectively. Starting with Fig. 1(a), one can see that at small time instances, the optimal policy consists of only a single region in which the test continues and a region in which the test stops and decides in favor of H_1. This is somehow counterintuitive, as one would expect the optimal test to not allow for an early decision in favor of H_1, due to the high posterior variance. Contrary to [14], where the boundary of the SPRT followed more or less the policy of the optimal scheme, but with a much broader corridor for continuing, the corridor for continuing here clearly differs between the optimal procedure and the SPRT. For $n = 25$, which is depicted in Fig. 1(b), the corridor for continuing the SPRT becomes similar to the vertical corridor of the optimal test, but is shifted. The optimal scheme also has a horizontal corridor for continuing, which is not the case for the SPRT. The policy for $n = 70$ is shown in Fig. 1(c), where one can see that the vertical corridor for continuing the optimal test, present in Fig. 1(b) is now closed. In contrast to that, the SPRT still has a small vertical corridor in which the test continues.



(a) $n = 3$ (b) $n = 25$ (c) $n = 70$

Fig. 1. Evolution of the optimal policy over time.

TABLE II
SIMULATION RESULTS

(a) Detection and estimation errors

	constraints		empirical	
	κ	tolerance	optimal	two-step
$P(\delta_\tau = 1 \,\vert\, \mathrm{H}_0)$	0.050	± 0.001	0.050	0.024
$P(\delta_\tau = 0 \,\vert\, \mathrm{H}_1)$	0.050	± 0.001	0.049	0.042
$\mathbf{E}[\mathbf{1}_{\{\delta_\tau = 0\}}(\hat{\theta}_\tau - \theta)^2 \,\vert\, \mathrm{H}_1]$	1.25	± 0.05	1.27	5.69

(b) Run-lengths

	optimal	two-step
$\mathbf{E}[\tau \,\vert\, \mathrm{H}_0]$	17.7	11.2
$\mathbf{E}[\tau \,\vert\, \mathrm{H}_1]$	29.8	8.7
$\mathbf{E}[\tau]$	23.8	9.9

Again, the horizontal corridor for continuing the test is only present for the optimal scheme and not for the SPRT. This horizontal corridor is due to the fact that the posterior variance of the SNR is very high in this region and would hence lead to large estimation errors if the test stopped.

V. CONCLUSION

Based on a linear model, we have addressed the problem of sequential joint detection and SNR estimation in a Bayesian framework. The solution is characterized by a non-linear Bellman equation. The latter characterizes the optimal policy, which is parametrized by three cost coefficients. To achieve a predefined performance in terms of error probabilities and MSE, we propose a scheme for choosing the cost coefficients. The performance of the optimal test and the gap to a suboptimal scheme, as well as the evolution of the optimal policy over time, are illustrated by a numerical example.

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