Path-connectedness of tensor ranks

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***Abstract*—Computations of low-rank approximations of tensors often involve path-following optimization algorithms. In such cases, a correct solution may only be found if there exists a continuous path connecting the initial point to a desired solution. We will investigate the existence of such a path in sets of low-rank tensors for various notions of ranks, including tensor rank, border rank, multilinear rank, and their counterparts for symmetric tensors.**

***Index Terms*—Tensor rank, symmetric rank, border rank, multilinear rank, symmetric multilinear rank, path-connectedness**

I. Introduction

Given a d-tensor $A \in \mathbb{F}^{n_1 \times \cdots \times n_d} := \mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_d}$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, its *tensor rank* $\operatorname{rk}(A)$ is the smallest integer r such that

$$A = \sum_{i=1}^{r} v_{1,i} \otimes \cdots \otimes v_{d,i} \tag{1}$$

for some vectors $v_{j,i} \in \mathbb{F}^{n_j}$, $j = 1, \dots, d$, $i = 1, \dots, r$. A *flattening* $\flat_i$ maps A to a matrix by 'forgetting' the tensor product structure,

$$\flat_i : \mathbb{F}^{n_1 \times \cdots \times n_d} \to \mathbb{F}^{n_i} \otimes \mathbb{F}^{n_1 \cdots n_{i-1} n_{i+1} \cdots n_d}. \tag{2}$$

In other words, $\flat_i(A)$ is the $n_i \times n_1 \cdots n_{i-1} n_{i+1} \cdots n_d$ matrix given by

$$\begin{bmatrix} A_{1\cdots 1} & \dots & A_{n_1 \cdots n_{i-1} 1 n_{i+1} \cdots n_d} \\ \vdots & \ddots & \vdots \\ A_{1 \cdots 1 n_i 1 \cdots 1} & \dots & A_{n_1 \cdots n_d} \end{bmatrix},$$

for $i = 1, \dots, d$. The *multilinear rank* $\mu\operatorname{rk}(A)$ is defined to be

$$\mu\operatorname{rk}(A) = \bigl(\operatorname{rk}(\flat_1(A)), \dots, \operatorname{rk}(\flat_d(A))\bigr).$$

Equivalently, $\mu\operatorname{rk}(A)$ is the smallest[1] tuple $(r_1, \dots, r_d)$ such that A belongs to $\mathbb{F}^{r_1 \times \cdots \times r_d}$ after a change-of-bases. When $d = 2$, the multilinear rank (r_1, r_2) of a matrix A has r_1 and r_2 given by the row and column ranks of A respectively.

For a symmetric d-tensor $A \in \mathsf{S}^d(\mathbb{F}^n)$, the *symmetric rank* $\operatorname{rk}_{\mathsf{S}}(A)$ is the smallest integer r such that

$$A = \sum_{i=1}^{r} v_i^{\otimes d} \tag{3}$$

for some vectors $v_i \in \mathbb{F}^n$, $i = 1, \dots, r$. The *symmetric multilinear rank* $\mu\operatorname{rk}_{\mathsf{S}}(A)$ is the minimum integer r such that A belongs to $\mathsf{S}^d(\mathbb{F}^r)$ after a change-of-basis. For a symmetric tensor $A \in \mathsf{S}^d(\mathbb{F}^{n+1})$, its rank $\operatorname{rk}(A)$ and symmetric rank $\operatorname{rk}_{\mathsf{S}}(A)$ may be different [18]. On the other hand, we always have $\mu\operatorname{rk}(A) = (r, \dots, r)$ where $r = \mu\operatorname{rk}_{\mathsf{S}}(A)$.

We note that symmetric d-tensors are in bijective correspondence with homogeneous degree-d polynomials; e.g., for $d = 3$, any $A = (a_{ijk}) \in \mathsf{S}^3(\mathbb{C}^n)$ corresponds to $\sum_{i,j,k=1}^{n} a_{ijk}\, x_i x_j x_k \in \mathbb{C}[x_1, \dots, x_n]$. So our results on symmetric tensors apply verbatim to homogeneous polynomials.

When $d \geq 3$ and $r > 1$, the set of tensors of rank at most r, i.e., $\{A \in \mathbb{F}^{n_1 \times \cdots \times n_d} \mid \operatorname{rk}(A) \leq r\}$, and the set of symmetric tensors of symmetric rank at most r, i.e., $\{A \in \mathsf{S}^d(\mathbb{F}^n) \mid \operatorname{rk}_{\mathsf{S}}(A) \leq r\}$, are not necessarily closed in the Euclidean topology [10], [14], which necessitates the notions of *border rank* $\operatorname{brk}(A)$: this is the smallest integer r such that A is a limit of rank-r tensors; likewise, the *symmetric border rank* $\operatorname{brk}_{\mathsf{S}}(A)$ of a symmetric tensor A is the smallest integer r such that A is a limit of symmetric rank-r tensors.

Determining best low-rank approximations of a tensor (when they exist) with respect to one of the aforementioned ranks is an important problem in applications.

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[1]Here 'smallest' is with respect to the partial order on $\mathbb{N}^d$ defined by $(t_1, \dots, t_d) \leq (s_1, \dots, s_d)$ if $t_j \leq s_j$ for all $j = 1, \dots, d$.

Riemannian manifold optimization techniques [2], [11] have often been used, particularly in the best multilinear rank approximation problem [12], [17]. In particular, the authors of [13] proposed to find a best approximation of a given tensor in the set of *fixed* multilinear rank tensors, i.e., optimize over the set

$$X_{r_1,\dots,r_d} := \{A \in \mathbb{F}^{n_1\times\cdots\times n_d} \mid \mu\mathrm{rk}(A) = (r_1,\dots,r_d)\},$$

instead of the set

$$\overline{X}_{r_1,\dots,r_d} = \{A \in \mathbb{F}^{n_1\times\cdots\times n_d} \mid \mu\mathrm{rk}(A) \le (r_1,\dots,r_d)\},$$

the reason being that $X_{r_1,\dots,r_d}$ is a smooth Riemannian manifold [20] whereas $\overline{X}_{r_1,\dots,r_d}$ has singular points. However, as these Riemannian manifold optimization techniques are path-following algorithms, we need to know if $X_{r_1,\dots,r_d}$ is path-connected or not. If not, a path-following algorithm starting from one connected component can never reach optimizers located in other components.

In this article, we summarize our path-connectedness results for the following sets:

① $\{A \in \mathbb{F}^{n_1\times\cdots\times n_d} \mid \mathrm{rk}(A) = r\}$,
② $\{A \in \mathsf{S}^d(\mathbb{F}^n) \mid \mathrm{rk}_{\mathsf{S}}(A) = r\}$,
③ $\{A \in \mathbb{F}^{n_1\times\cdots\times n_d} \mid \mathrm{brk}(A) = r\}$,
④ $\{A \in \mathsf{S}^d(\mathbb{F}^n) \mid \mathrm{brk}_{\mathsf{S}}(A) = r\}$,
⑤ $\{A \in \mathbb{F}^{n_1\times\cdots\times n_d} \mid \mu\mathrm{rk}(A) = (r_1,\dots,r_d)\}$,
⑥ $\{A \in \mathsf{S}^d(\mathbb{F}^n) \mid \mu\mathrm{rk}_{\mathsf{S}}(A) = r\}$,

for $d \ge 3$ over both $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$. These sets are path-connected over $\mathbb{C}$ if r is strictly less than the generic rank, but the situation is more subtle over $\mathbb{R}$. Roughly speaking, ①, ③, ⑤ are path-connected over $\mathbb{R}$ when r is no more than the generic rank; with the requirement that the order of the tensor d is odd, so are ②, ④, ⑥.

This article contains a digest of selected results in [9] that provide theoretical guarantees for Riemannian optimization algorithms used in low-rank tensor approximations. For complete proofs and additional topological properties (e.g., fundamental groups and higher homotopy groups), we refer readers to [9].

II. X-RANK

As we would like to study path-connectedness, a topological property, it is natural and convenient to use geometric language. In the next section, we will restate the various notions of ranks in terms of X-ranks, where X is a complex irreducible projective variety. We refer readers to [14] for more information.

A. X-rank and X-border rank

To employ the framework of classical algebraic geometry, we will work in projective spaces instead of affine spaces, where we have more powerful tools and nicer properties of varieties and morphisms between varieties. The projective space $\mathbb{CP}^n$ is the set of lines in $\mathbb{C}^{n+1}$ passing through 0. For a nonzero vector $v \in \mathbb{C}^{n+1}$, a point $[v] \in \mathbb{CP}^n$ represents the line $\{\lambda v \mid \lambda \in \mathbb{C}\}$. This definition gives rise to a quotient map

$$\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n, \quad v \mapsto [v]. \tag{4}$$

As a quotient space, $\mathbb{CP}^n$ has a natural quotient topology induced by the Euclidean topology on $\mathbb{C}^{n+1}$, and this topology on $\mathbb{CP}^n$ is also called the Euclidean topology, which also makes $\mathbb{CP}^n$ a smooth complex compact manifold. For any subset $X \subseteq \mathbb{CP}^n$, the set $\widehat{X} := \pi^{-1}(X) \cup \{0\}$ is called the *affine cone* over X.

A complex *projective variety* in $\mathbb{CP}^n$ is defined to be the zero locus of finitely many homogeneous polynomials in $n+1$ variables. Given a complex projective variety $X \subseteq \mathbb{CP}^n$, a subset $Y \subseteq X$ is called a *subvariety* of X if Y is itself a projective variety in $\mathbb{CP}^n$, i.e., the zero locus of finitely many homogeneous polynomials. As a subset of $\mathbb{CP}^n$, any projective variety $X \subseteq \mathbb{CP}^n$ inherits the Euclidean topology from $\mathbb{CP}^n$. We say X is *smooth* if it is a smooth manifold in the Euclidean topology. Our discussions up to this point also apply verbatim to $\mathbb{RP}^n$, i.e., with $\mathbb{R}$ in place of $\mathbb{C}$.

Given a projective variety X, the union of two subvarieties of X is a projective subvariety, and the intersection of a family of subvarieties of X is also a projective subvariety. This property gives a new topology on X where closed subsets of X are exactly projective subvarieties of X. This new topology is called the *Zariski topology* on X.

A projective variety X is called *nondegenerate* if X is not contained in a hyperplane, and X is called *irreducible* if it is not a union of two nonempty proper projective subvarieties. It is known that any projective variety can be decomposed as a union of finitely many irreducible projective subvarieties.

To connect projective varieties with tensors, we look at some examples that we will use in the next sections. The *Segre map* is

$$\begin{aligned}\mathrm{Seg}\colon \mathbb{CP}^{n_1-1}\times\cdots\times\mathbb{CP}^{n_d-1} &\to \mathbb{CP}^{n_1\cdots n_d-1},\\ ([v_1],\dots,[v_d]) &\mapsto [v_1\otimes\cdots\otimes v_d].\end{aligned}$$

Its image $\mathrm{Seg}(\mathbb{CP}^{n_1-1}\times\cdots\times\mathbb{CP}^{n_d-1})$ is an irreducible smooth projective variety and is called the *Segre variety*. If we let $X = \mathrm{Seg}(\mathbb{CP}^{n_1-1}\times\cdots\times\mathbb{CP}^{n_d-1})$, then $\widehat{X}\setminus\{0\}$ is the set of rank-one d-tensors. Likewise, the *Veronese map* is

$$\nu_d\colon \mathbb{CP}^{n-1} \to \mathbb{P}\mathsf{S}^d(\mathbb{C}^n), \quad [v] \mapsto [v^{\otimes d}].$$

Its image $\nu_d(\mathbb{CP}^{n-1})$ is an irreducible smooth projective variety and is called the *Veronese variety*. If we let $X = \nu_d(\mathbb{CP}^{n-1})$, then $\widehat{X} \setminus \{0\}$ is the set of symmetric rank-one d-tensors.

The *ideal* of a projective variety $X \subseteq \mathbb{CP}^n$ is the set of those homogeneous polynomials in $\mathbb{C}[x_1, \dots, x_{n+1}]$ that vanish on X. If the ideal of X is generated by homogeneous polynomials with real coefficients, then the set of *real points* $X(\mathbb{R})$ of X is defined to be the zero locus of these polynomials in $\mathbb{RP}^n$. Although X and $X(\mathbb{R})$ are defined by the same polynomials, their properties can be vastly different. For example, let X be the smooth conic curve in $\mathbb{CP}^2$ defined by $x^2 + y^2 + z^2 = 0$, which is isomorphic to $\mathbb{CP}^1$. On the other hand, it is evident that $X(\mathbb{R})$, i.e., the zero locus of $x^2 + y^2 + z^2 = 0$ in $\mathbb{RP}^2$, is empty. To avoid such phenomena, in this article we will require that $X(\mathbb{R})$ be Zariski dense in X, which is equivalent to requiring that X has a smooth real point [5].

If X is nondegenerate in $\mathbb{CP}^n$, then any vector $v \in \mathbb{C}^{n+1}$ can be written as a finite sum of points in $\widehat{X}$. For instance, as $X = \operatorname{Seg}(\mathbb{CP}^{n_1-1} \times \cdots \times \mathbb{CP}^{n_d-1})$ is nondegenerate, any tensor in $\mathbb{C}^{n_1 \times \cdots \times n_d}$ is a finite sum of rank-one tensors, i.e., points in $\widehat{X}$ (see Section II-B). More generally, define the *sum-of-r-terms* map s_r by

$$s_r\colon \widehat{X}^r \to \mathbb{C}^{n+1}, \quad (x_1, \dots, x_r) \mapsto x_1 + \cdots + x_r, \tag{5}$$

and let $s_r(X)$ be its image. The Euclidean closure of $s_r(X)$ is in fact Zariski closed, which means that the Euclidean closure of $s_r(X)$ can be realized as the affine cone of a certain projective variety in $\mathbb{CP}^n$, i.e., the zero locus of finitely many homogeneous polynomials in $n+1$ variables. This variety is called the rth *secant variety* of X and denoted by $\sigma_r(X)$. If $\overline{s}_r(X)$ denotes the Euclidean closure of $s_r(X)$ and $\widehat{\sigma}_r(X)$ the affine cone of $\sigma_r(X)$, then

$$\overline{s}_r(X) = \widehat{\sigma}_r(X).$$

The image of $\widehat{X}(\mathbb{R})^r$ under s_r, denoted by $s_r(X(\mathbb{R}))$, is semialgebraic. However its Euclidean closure in $\mathbb{R}^{n+1}$ is not necessarily Zariski closed. The Zariski closure of $s_r(\widehat{X}(\mathbb{R}))$ over $\mathbb{R}$ is the affine cone of some real projective variety in $\mathbb{RP}^n$. We will denote this real projective subvariety by $\sigma_r(X(\mathbb{R}))$ and call it the rth *secant variety* of $X(\mathbb{R})$. If $X(\mathbb{R})$ is Zariski dense in X, then by [7], [16], the rth secant variety of $X(\mathbb{R})$ is the set of real points of the rth secant variety of X, i.e.,

$$\sigma_r(X(\mathbb{R})) = \big(\sigma_r(X)\big)(\mathbb{R}),$$

which is Zariski dense in $\sigma_r(X)$.

In this article, we deal only with irreducible nondegenerate complex projective varieties $X \subseteq \mathbb{CP}^n$ defined by real homogeneous polynomials and whose real points $X(\mathbb{R})$ are Zariski dense. Under this assumption, for any $p \in \mathbb{C}^{n+1}$, the *X-rank* of p, denoted $\operatorname{rk}_X(p)$, is the minimum integer r such that $p \in s_r(X) \setminus s_{r-1}(X)$. The *$X$-border rank* of p, denoted $\operatorname{brk}_X(p)$, is the minimum r such that $p \in \overline{s}_r(X) \setminus \overline{s}_{r-1}(X)$. Similarly, we define the *$X(\mathbb{R})$-rank* of $p \in \mathbb{R}^{n+1}$ to be the minimum r such that $p \in s_r(X(\mathbb{R})) \setminus s_{r-1}(X(\mathbb{R}))$, and the $X(\mathbb{R})$-border rank of p to be the minimum r such that $p \in \overline{s}_r(X(\mathbb{R})) \setminus \overline{s}_{r-1}(X(\mathbb{R}))$. In general, it may happen that

$$\operatorname{rk}_X(p) \neq \operatorname{rk}_{X(\mathbb{R})}(p) \quad \text{or} \quad \operatorname{brk}_X(p) \neq \operatorname{brk}_{X(\mathbb{R})}(p),$$

which requires us to study the real and complex cases separately.

For a fixed $X \subseteq \mathbb{CP}^n$, there is a unique integer $r_g(X)$, the *generic X-rank*, such that $s_{r_g(X)}(X) \setminus s_{r_g(X)-1}(X)$ contains a nonempty open subset of $\mathbb{C}^{n+1}$ in the Euclidean topology. We say that r is a *typical $X(\mathbb{R})$-rank* if $s_r(X(\mathbb{R})) \setminus s_{r-1}(X(\mathbb{R}))$ contains a nonempty Euclidean open subset of $\mathbb{R}^{n+1}$. It turns out that $r_g(X)$ always equals the smallest typical $X(\mathbb{R})$-rank [5].

B. Tensor rank and symmetric rank

We now view tensor rank and symmetric rank in Section I under the light of X-ranks introduced in Section II-A. Let $X = \operatorname{Seg}(\mathbb{CP}^{n_1-1} \times \cdots \times \mathbb{CP}^{n_d-1})$. Then $X(\mathbb{R}) = \operatorname{Seg}(\mathbb{RP}^{n_1-1} \times \cdots \times \mathbb{RP}^{n_d-1})$. For any tensor $A \in \mathbb{C}^{n_1 \times \cdots \times n_d}$, we have

$$\operatorname{rk}(A) = \operatorname{rk}_X(A) \quad \text{and} \quad \operatorname{brk}(A) = \operatorname{brk}_X(A).$$

If $A \in \mathbb{R}^{n_1 \times \cdots \times n_d}$, then

$$\operatorname{rk}_{\mathbb{R}}(A) = \operatorname{rk}_{X(\mathbb{R})}(A) \quad \text{and} \quad \operatorname{brk}_{\mathbb{R}}(A) = \operatorname{brk}_{X(\mathbb{R})}(A).$$

Now let $X = \nu_d(\mathbb{CP}^n)$. Then $X(\mathbb{R}) = \nu_d(\mathbb{RP}^n)$. For any symmetric tensor $A \in \mathsf{S}^d(\mathbb{C}^{n+1})$, we have

$$\operatorname{rk}_{\mathsf{S}}(A) = \operatorname{rk}_X(A) \quad \text{and} \quad \operatorname{brk}_{\mathsf{S}}(A) = \operatorname{brk}_X(A).$$

If $A \in \mathsf{S}^d(\mathbb{R}^{n+1})$, then

$$\operatorname{rk}_{\mathsf{S}}(A) = \operatorname{rk}_{X(\mathbb{R})}(A) \quad \text{and} \quad \operatorname{brk}_{\mathsf{S}}(A) = \operatorname{brk}_{X(\mathbb{R})}(A).$$

The use of X-rank will permit us to address the cases ①–④ in an unified manner. As a side remark, ⑤ and ⑥ are smooth manifolds, but ①–④ are not necessarily manifolds.

III. Path-connectedness of complex ranks

Now that we have formulated ranks and border ranks geometrically, the path-connectedness of border X-rank over $\mathbb{C}$ becomes clear by the following facts:

(a) $\sigma_{r-1}(X) \subsetneq \sigma_r(X)$ whenever $r \leq r_g(X)$ [3];
(b) $\sigma_r(X) \setminus \sigma_{r-1}(X)$ is path-connected if $\sigma_{r-1}(X) \neq \sigma_r(X)$ [15, Corollary 4.16].

We may then deduce a path-connectedness result for X-border rank-r points.

Theorem 1: If $r \leq r_g(X)$, then the sets $\{p \in \mathbb{C}^{n+1} \mid \operatorname{brk}_X(p) = r\}$ and $\{p \in \mathbb{C}^{n+1} \mid \operatorname{rk}_X(p) = r\}$ are path-connected.

The path-connectedness of rank, border rank, symmetric rank, and symmetric border rank over $\mathbb{C}$ are consequences of Theorem 1.

Corollary 1: If r is not more than the generic tensor rank, then

$$\{A \in \mathbb{C}^{n_1\times\cdots\times n_d} \mid \operatorname{brk}(A) = r\},$$
$$\{A \in \mathbb{C}^{n_1\times\cdots\times n_d} \mid \operatorname{rk}(A) = r\}$$

are both path-connected.

Corollary 2: If r is not more than the generic symmetric rank, then

$$\{A \in \mathsf{S}^d(\mathbb{C}^{n+1}) \mid \operatorname{brk}_{\mathsf{S}}(A) = r\},$$
$$\{A \in \mathsf{S}^d(\mathbb{C}^{n+1}) \mid \operatorname{rk}_{\mathsf{S}}(A) = r\}$$

are both path-connected.

IV. Path-connectedness of real ranks

The case of real tensors is more subtle and difficult than complex tensors. We recall from Section II-B that if $X = \nu_d(\mathbb{CP}^n)$, then we have $X(\mathbb{R}) = \nu_d(\mathbb{RP}^n)$. By Terracini Lemma [19] and Alexander–Hirschowitz Theorem [4], together with the topological fact that removing a semialgebraic subset of codimension at least two from a manifold does not change its path-connectedness, we obtain our path-connectedness result for symmetric tensor rank and border rank over $\mathbb{R}$.

Theorem 2: Let $n > 1$ and $r < \binom{n+d}{d}/(n+1)$.

(i) If d is odd, then

$$\{A \in \mathsf{S}^d(\mathbb{R}^{n+1}) \mid \operatorname{rk}_{\mathsf{S}}(A) = r\},$$
$$\{A \in \mathsf{S}^d(\mathbb{R}^{n+1}) \mid \operatorname{brk}_{\mathsf{S}}(A) = r\}$$

are both path-connected.

(ii) If d is even, then

$$\{A \in \mathsf{S}^d(\mathbb{R}^{n+1}) \mid \operatorname{rk}_{\mathsf{S}}(A) = r\},$$
$$\{A \in \mathsf{S}^d(\mathbb{R}^{n+1}) \mid \operatorname{brk}_{\mathsf{S}}(A) = r\}$$

both have $r + 1$ path-connected components.

To state our path-connectedness results for tensor rank and border rank over $\mathbb{R}$, we will need to bring in the notion of *defectivity*: We say X is not r-defective if

$$\dim\big(\sigma_r(X)\big) = \min\{r \dim X - 1, n\}$$

and r-defective otherwise.

By an argument similar to the one that led us to Theorem 2, and an additional assumption to guarantee nondefectivity of $\operatorname{Seg}(\mathbb{CP}^{n_1-1} \times \cdots \times \mathbb{CP}^{n_d-1})$, we obtain the following:

Theorem 3: Suppose $2 \le n_1 \le \cdots \le n_d$. Let $X = \operatorname{Seg}(\mathbb{CP}^{n_1-1} \times \cdots \times \mathbb{CP}^{n_d-1})$ and $r < r_g(X)$. If

$$\operatorname{codim}_{\mathbb{C}}\big(\sigma_{r-1}(X), \sigma_r(X)\big) > n_1 + \cdots + n_{d-1} - d + 2, \quad (6)$$

then

$$\{A \in \mathbb{R}^{n_1\times\cdots\times n_d} \mid \operatorname{rk}(A) = r\},$$
$$\{A \in \mathbb{R}^{n_1\times\cdots\times n_d} \mid \operatorname{brk}(A) = r\}$$

are both path-connected.

We state an alternative version of Theorem 3 that requires an explicit assumption on defectivity.

Theorem 4: If $\operatorname{Seg}(\mathbb{CP}^{n_1-1} \times \cdots \times \mathbb{CP}^{n_d-1})$ is not r-defective, then the sets

$$\{A \in \mathbb{R}^{n_1\times\cdots\times n_d} \mid \operatorname{rk}(A) = r\},$$
$$\{A \in \mathbb{R}^{n_1\times\cdots\times n_d} \mid \operatorname{brk}(A) = r\}$$

are both path-connected.

Unlike the case of symmetric tensors [4], there are still cases where the dimension of $\sigma_r(\operatorname{Seg}(\mathbb{CP}^{n_1-1} \times \cdots \times \mathbb{CP}^{n_d-1}))$, and thus its defectivity, remains unknown. On the other hand, there has been recent remarkable progress [1], [6], [8] that guarantees that when $n_d > 3$, all known cases satisfy the condition (6) in Theorem 3.

V. Path-connectedness of multilinear rank

If $(r_1, \ldots, r_d)$ is the multilinear rank of some tensor $A \in \mathbb{F}^{n_1\times\cdots\times n_d}$, then by (2) we have

$$r_i = \operatorname{rk}(\flat_i(A)) \le \min\Big\{n_i, \prod\nolimits_{j\neq i} r_j\Big\}, \quad i = 1, \ldots, d.$$

Intuitively, the connectedness of the set of tensors of a fixed multilinear rank, by virtue of its definition in terms of matrix rank, ought to be essentially the same as the connectedness of the set of matrices of a fixed rank. This intuition can be made mathematically rigorous by the so-called Kempf–Weyman desingularization [21], which leads to the following results.

Theorem 5:

(i) The set of multilinear rank-$(r_1, \ldots, r_d)$ real tensors

$$\{A \in \mathbb{R}^{n_1\times\cdots\times n_d} \mid \mu\mathrm{rk}(A) = (r_1, \ldots, r_d)\}$$

is path-connected if

$$r_i < \prod_{j\neq i} r_j \quad \text{for all } i = 1, \ldots, d,$$

or if

$$r_i = \prod_{j\neq i} r_j < n_i \quad \text{for some } i = 1, \ldots, d.$$

(ii) The set of multilinear rank-$(r_1, \ldots, r_d)$ real tensors

$$\{A \in \mathbb{R}^{n_1\times\cdots\times n_d} \mid \mu\mathrm{rk}(A) = (r_1, \ldots, r_d)\}$$

has two connected components if

$$r_i = \prod_{j\neq i} r_j = n_i \quad \text{for some } i = 1, \ldots, d.$$

Theorem 6: The set of multilinear rank-$(r_1, \ldots, r_d)$ complex tensors

$$\{A \in \mathbb{C}^{n_1\times\cdots\times n_d} \mid \mu\mathrm{rk}(A) = (r_1, \ldots, r_d)\}$$

is always path-connected.

For symmetric tensors, we may similarly deduce analogous results in terms of symmetric multilinear rank. There are four separate cases to consider over $\mathbb{R}$ but just one over $\mathbb{C}$.

Theorem 7:

(i) When $r = 1$ and d is odd, the set of symmetric multilinear rank-one real tensors

$$\{A \in \mathsf{S}^d(\mathbb{R}^n) \mid \mu\mathrm{rk}_{\mathsf{S}}(A) = 1\}$$

is a path-connected set.

(ii) When $r = 1$ and d is even, the set of symmetric multilinear rank-one real tensors

$$\{A \in \mathsf{S}^d(\mathbb{R}^n) \mid \mu\mathrm{rk}_{\mathsf{S}}(A) = 1\}$$

has two connected components.

(iii) When $d = 2$, the set of symmetric multilinear rank-r real tensors

$$\{A \in \mathsf{S}^d(\mathbb{R}^n) \mid \mu\mathrm{rk}_{\mathsf{S}}(A) = r\}$$

has $r + 1$ connected components.

(iv) When $r \geq 2$ and $d \geq 3$, the set of symmetric multilinear rank-r real tensors

$$\{A \in \mathsf{S}^d(\mathbb{R}^n) \mid \mu\mathrm{rk}_{\mathsf{S}}(A) = r\}$$

is a path-connected set.

Theorem 8: The set of symmetric multilinear rank-r complex tensors

$$\{A \in \mathsf{S}^d(\mathbb{C}^n) \mid \mu\mathrm{rk}_{\mathsf{S}}(A) = r\}$$

is always path-connected.

VI. Beyond generic rank

The case when rank exceeds generic rank is a notable omission from our list of results in this article. In general, when r is strictly greater than the generic rank, the set of (border) rank-r tensors is not path-connected. However, it is usually difficult to determine the number of connected components, and we are unaware of any technique that applies generally towards this end. We end this article with a special case, studied extensively in [10], where we are able to determine the exact number of connected components.

Proposition 1: The set

$$\{A \in \mathbb{R}^{2\times 2\times 2} \mid \mathrm{brk}(A) = 3\}$$

has four path-connected components.

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