Coded Aperture Design for Super-Resolution Phase Retrieval

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***Abstract*—Phase retrieval is an inverse problem which consists on estimating a complex signal from intensity-only measurements. Recent works have studied the problem of retrieving the phase of a high-resolution image from low-resolution phaseless measurements, under a setup that records coded diffraction patterns. However, the attainable resolution of the image depends on the sensor characteristics, whose cost increases in proportion to the resolution. Also, this methodology lacks theoretical analysis. Hence, this work derives a super-resolution model from low-resolution coded phaseless measurements, that in contrast with prior contributions, the attainable resolution of the image directly depends on the resolution of the coded aperture. For this model we establish that an image can be recovered (up to a global unimodular constant) with high probability. Also, the theoretical result states that the image reconstruction quality directly depends on the design of the coded aperture. Therefore, a strategy that designs the spatial distribution of the coded aperture is developed. Simulation results show that reconstruction quality using designed coded aperture is higher than the non-designed ensembles.**

I. Introduction

PHASE retrieval (PR) is a common problem in diffractive optical imaging (DOI) [1], where intensity-only measurements are sensed. Recently, a coded diffraction patterns (CDP) approach, which modifies the traditional DOI system introducing an optical element called coded aperture has been proposed [2], [3]. This element modulates the object to then record the intensity of its coded diffraction patterns at the sensor, as illustrated in Fig. 1, [3], [4]. In fact, if the spatial configuration of the coded aperture is changed, this acquisition scheme allows multiple projections of the same scene.

One of the main limitations in a DOI typical setup is the spatial resolution, which is limited by the optics and the sensor resolution [5], [6]. In particular, the super-resolution phase retrieval problem, which consists of estimating a high-resolution image from low-resolution phaseless measurements, has been previously studied in [7], under the setup illustrated in Fig. 1. Specifically, this work developed a computational super-resolution phase retrieval algorithm that estimates a high resolution image from noisy phaseless measurements using a forward-backward method. Additionally, this method has a filtering step, based on the block-matching 3D filtering (BM3D) [8], to reduce the effect of the noise. However, the attainable resolution of the image in this approach still depends on the sensor characteristics, whose cost increases in proportion to the resolution. Further, this method does not provide image recovery guarantees.

This paper derives a super-resolution model from low-resolution coded phaseless measurements, that in contrast with previous works, the attainable resolution of the image directly depends on the spatial resolution of the coded aperture. Additionally, we establish that any image can be recovered (up to a global unimodular constant) with high probability. This theoretical result shows that image reconstruction quality directly depends on the spatial structure of the coded aperture through the projections. Therefore, a strategy to design the coded aperture based on temporal and spatial correlation is developed in this paper. Specifically, the proposed design criteria generate uniform spatially distributed coded apertures that equality sense each pixel of the image along the projections. Simulation results suggest that the proposed coded aperture designed overcomes random distributions. Specifically, the designed coded apertures attain a gain of up to 5dB of PSNR compared with random coded apertures.

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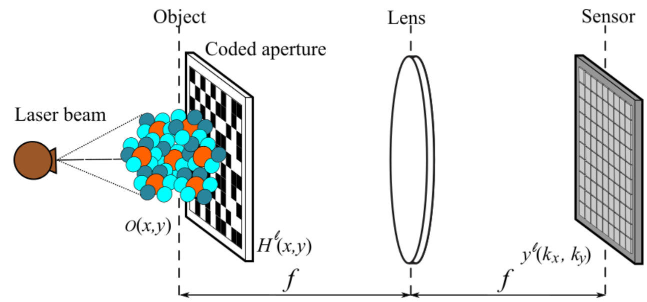

Fig. 1. Optical system to obtain coded diffraction patterns.

II. Super-resolution phase retrieval problem

The phase retrieval problem from coded diffraction patterns is formulated as the finding of a complex image $o(x,y)$ from phaseless measurements of the form

$$y^\ell(k_x,k_y) \propto |\mathcal{F}\{H^\ell(x,y)o(x,y)\}|^2, \tag{1}$$

where $\ell = 1,\cdots,L$ indexes the projections, $H^\ell(x,y)$ models the ℓ-th configuration of the coded aperture, with (x,y), (k_x,k_y) as the spatial and frequency coordinates, respectively. In a discrete form, the pixels of the sensor and coded aperture are assumed to be squared. Specifically, let Δ_s and $\Delta_{\tilde{o}}$ be the pixel sizes of the sensor and the modulated object $\tilde{o}^\ell(x,y) = H^\ell(x,y)o(x,y)$ at the ℓ-th projection, respectively. Given the relationship presented in [9], [10] to avoid signal overlapping at a distance z, the sampling period of the object must satisfy the following equality

$$\Delta_{\tilde{o}} = \frac{\lambda z}{M\Delta_s}, \tag{2}$$

where $M \times M$ are the number of pixels of the sensor. Under this setup, the discrete version of (1) can be expressed as

$$y^\ell_{s,r} = \left|\sum_{k,l}^{M} \tilde{o}_{k,l} e^{2j\pi\left(\frac{ks}{M}+\frac{lr}{M}\right)}\right|^2, \tag{3}$$

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where $\tilde{o}_{k,l}$ is the discrete form of $\tilde{o}(x,y)$ and $y^{\ell}_{s,r}$ are the discrete measurements. Further, considering Δ_h as the pixel size of the coded aperture, its transmittance function can be expressed as

$$H^{\ell}(x,y) = \sum_{k',l'} H^{\ell}_{k',l'} \text{rect}\left(\frac{x}{\Delta_h} - k', \frac{y}{\Delta_h} - l'\right), \quad (4)$$

where $H^{\ell}_{k',l'}$ represents the value at the pixel k', l' for the ℓ-th projection. Usually, the pixel sizes of the coded aperture Δ_h and the sensor Δ_s are equal. For this reason if $\Delta_{\tilde{o}} > \Delta_h$ a high-resolution object can be obtained from low-resolution measurements [11]. Specifically, it is convenient to assume that $\Delta_{\tilde{o}} = r\Delta_h$ where $r \geq 1$ is an integer up-sampling factor, then the super-resolution phase retrieval problem can be modeled as

$$y^{\ell}_{s,r} = \left| \sum_{k,l}^{M} \left(\sum_{s=kr}^{(k+1)r} \sum_{r=lr}^{(l+1)r} H^{\ell}_{s,r} o_{s,r} \right) e^{2j\pi\left(\frac{ks}{M} + \frac{lr}{M}\right)} \right|^2, \quad (5)$$

which can be seen as a discrete version of (1). Thus, expressing (5) in matrix form it can be obtained that

$$\mathbf{y}_{\ell} = |\mathbf{F}\mathbf{D}\mathbf{H}_{\ell}\mathbf{x}|^2, \ell = 1, \cdots, L, \quad (6)$$

where $\mathbf{y}_{\ell} \in \mathbb{C}^n$ is the vectorization of the low-resolution observed measurements at the ℓ-th projection, $\mathbf{F} \in \mathbb{C}^{m \times m}$ is the discrete Fourier transform matrix, $\mathbf{H}_{\ell} \in \mathbb{C}^{n \times n}$ is a diagonal matrix whose entries are $H^{\ell}_{s,r}$ and $\mathbf{x} = [o_{1,1}, \cdots, o_{M,M}]^T$ represent the target image, and $\mathbf{D} \in \mathbb{R}^{m \times n}$ represents a down-sampling matrix defined as

$$(\mathbf{D})_{i,k} = \begin{cases} \frac{1}{r_m^2}, & \text{if} \quad i = \lfloor \frac{k(\text{mod} N)}{r_m} \rfloor + 1 \text{ and} \\ & \quad k \leq N r_m + i r_m \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

Now, if $\mathbf{g} = [\tilde{\mathbf{y}}_1, \cdots, \tilde{\mathbf{y}}_L]$ is defined as the global measurement vector, we have

$$\mathbf{g} = |\mathbf{A}\mathbf{x}|^2, \quad (8)$$

where the matrix $\mathbf{A}$ is the vertical concatenation of the matrices $\tilde{\mathbf{F}}\mathbf{D}\mathbf{H}_{\ell}$ for $\ell = 1, \cdots L$ given by

$$\mathbf{A} = \left[(\tilde{\mathbf{F}}\mathbf{D}\mathbf{H}_1)^H, \cdots, (\tilde{\mathbf{F}}\mathbf{D}\mathbf{H}_L)^H \right]^H. \quad (9)$$

Considering the formulation in (8), the next section provides theoretical guarantees to recover a high-resolution image from low-resolution coded diffraction patterns.

III. Recovery Theoretical Guarantees

Taking the vector form of the super-resolution model in (8) into account, each measurement $\mathbf{g}_i$ can be expressed as

$$\mathbf{g}_i = |\mathbf{a}_i^H \mathbf{x}|^2 = \mathbf{a}_i^H \mathbf{x}\mathbf{x}^H \mathbf{a}_i, \quad (10)$$

where $\mathbf{a}_i$ is the i-th row of the matrix $\mathbf{A}$. Let $\mathcal{H} : \mathcal{S}^{n \times n} \to \mathbb{R}^{mL}$ be a linear mapping, where $\mathcal{S}^{n \times n}$ is the space of self-adjoint matrices, defined as

$$\mathcal{H}(\mathbf{W}) = [\mathbf{a}_1^H \mathbf{W} \mathbf{a}_1, \cdots, \mathbf{a}_{mL}^H \mathbf{W} \mathbf{a}_{mL}]^T. \quad (11)$$

From (8) and (11) it can be observed that $\mathbf{g} = \mathcal{H}(\mathbf{x}\mathbf{x}^H)$. Therefore, in order to guarantee that the signal $\mathbf{x}$ can be recovered from the measurements in (10), the linear operator $\mathcal{H}(\cdot)$ must be injective. With this formulation, Theorem III.1 shows that any signal $\mathbf{x}$ can be reconstructed if the set of coded apertures is properly designed. In order to prove Theorem III.1, we remark that this work assumes that the entries of $\mathbf{H}_{\ell}$ are independent and identically distributed ($i.i.d$) copies of a random variable d which satisfy the following definition.

Definition III.1. (*Admissible Random Variable*). A discrete random variable obeying $|d| \leq 1$, is said to be *admissible*.

Theorem III.1. Fix any $\delta \in (0, 1)$ and the set of coded apertures $\{\mathbf{H}_{\ell} : \ell = 1, \cdots, L\}$ with $i.i.d$ entries of an admissible random variable d. If for some constant $c > 0$ matrix $\mathbf{P} = \sum_{\ell=1}^{L} \mathbf{H}_{\ell}^H \mathbf{D}^H \mathbf{D} \mathbf{H}_{\ell}$ satisfies

$$\|\mathbf{P} - c\mathbf{I}\|_{\infty}^2 \leq \delta, \quad (12)$$

where $L \geq c_0 n$ for some sufficiently large constant $c_0 > 0$, with $\mathbf{I}$ as the identity matrix, then we have that

$$\mathcal{P}\left(\frac{1}{cmL}\|\mathbf{A}\|_{\infty}^2 \leq 1 + \delta\right) \leq 1 - ne^{-c_1 mL\epsilon^2}. \quad (13)$$

for some constant $c_1 > 0$. Also, with the same probability

$$(1 - \delta)\|\mathbf{W}\|_1 \leq \frac{1}{cmL}\|\mathcal{H}(\mathbf{W})\|_1 \leq (1 + \delta)\|\mathbf{W}\|_1, \quad (14)$$

for all positive semidefinite matrices $\mathbf{W}$.

Proof: See Appendix A. ■

Notice that Theorem III.1 essentially proves that the high-resolution image $\mathbf{x}$ can be recovered from low-resolution coded diffraction measurements if (12) is satisfied, which directly depends on the spatial distribution of the coded aperture and the super-resolution factor. This result provides that the set of coded apertures has to be designed in order to better recover the target image $\mathbf{x}$. Considering this observation, the following section provides an strategy to design the set of coded apertures that seeks to better satisfy the condition (12).

IV. Coded Aperture Design

Notice that the theoretical condition in (12) shows that the set of coded apertures defines the concentration of measure of the largest eigenvalue of the sensing matrix $\mathbf{A}$. Therefore, the structure of matrix $\mathbf{P}$ is analyzed in order to determine a design strategy for the set of coded apertures. Specifically, given the decimation matrix $\mathbf{D}$ with a down-sampling factor r, it can be noticed that

$$\left(\mathbf{D}^H\mathbf{D}\right)_{i,k} = \begin{cases} \frac{1}{r^2}, & \text{if} \quad i = \lfloor \frac{k(\text{mod} N)}{r} \rfloor + 1 \text{ and} \\ & \quad k \leq \left(\lfloor \frac{i}{rm} \rfloor + 1\right) Nr \\ 0, & \text{otherwise} \end{cases} \quad (15)$$

Observe that (15) can be decomposed as

$$\mathbf{D}^H\mathbf{D} = \frac{1}{r_m^2}\mathbf{I} + \mathbf{R}, \quad (16)$$

where $\mathbf{R}$ contains the off-diagonal terms of $\mathbf{D}^H\mathbf{D}$. Then, taking (16) into account, $\mathbf{P}$ can be equivalently expressed as

$$\mathbf{P} = \sum_{\ell=1}^{L} \mathbf{H}_\ell^H \mathbf{D}^H \mathbf{D} \mathbf{H}_\ell = \underbrace{\frac{1}{r_m^2} \sum_{\ell=1}^{L} \mathbf{H}_\ell^H \mathbf{H}_\ell}_{\mathbf{V}_1} + \underbrace{\sum_{\ell=1}^{L} \mathbf{H}_\ell^H \mathbf{R} \mathbf{H}_\ell}_{\mathbf{V}_2}. \tag{17}$$

Given the structure of $\mathbf{P}$ in (17), it can be observed that (12) can be satisfied if $\mathbf{V}_1 = c\mathbf{I}$, and $\mathbf{V}_2 = \mathbf{0}$ for some imposed constant $c > 0$, where $\mathbf{0}$ represents the zero matrix. More precisely, considering the diagonal structure of $\mathbf{H}_\ell$, the non-zero elements of the $\mathbf{V}_1$ term in (17) only depend on the spatial distribution of the coded aperture through the projections and the non-zero elements of $\mathbf{V}_2$ depend on the super-resolution factor r as follows

$$(\mathbf{V}_2)_{i,k} = \begin{cases} \frac{1}{r^2} \sum_{\ell=1}^{L} (\mathbf{H}_\ell)^*_{i,i} (\mathbf{H}_\ell)_{k,k} \text{ if } i = \lfloor \frac{k (\text{mod} N)}{r} \rfloor + 1 \text{ and} \\ \qquad k \leq \left(\lfloor \frac{i}{rm} \rfloor + 1 \right) Nr \text{ for } i \neq k \\ 0 \qquad \text{otherwise} \end{cases}. \tag{18}$$

where w^* represent the conjugate version of w. Thus, imposing the conditions, $\mathbf{V}_1 = c\mathbf{I}$, and $\mathbf{V}_2 = \mathbf{0}$, they lead to the following design criteria:

(a) Temporal correlation: Condition $\mathbf{V}_1 = c\mathbf{I}$ for some $c > 0$ can be accomplished if each pixel of the image is modulated for all the coding elements of d, along the L-projections.

(b) Spatial separation: In practical terms one can minimize the term $\mathbf{V}_2$ building a set of coded apertures with an $r \times r$ equi-spaced distribution of the coding elements, since $\mathbf{R}$ is the matrix that contains the off-diagonal terms of $\mathbf{D}^H\mathbf{D}$ as it is illustrated in (18) .

Taking this design criteria into account, recent works have developed some strategies to design the sensing matrix [12], [13]. Specifically, [13] minimizes the upper bounds of the Gershgorin theorem of a given matrix, which in this case is $\mathbf{P}$. This process generates a spatial uniform distribution of the coding elements within the coded apertures ensuring that $\mathbf{V}_1 = c\mathbf{I}$, for some $c > 0$ and $\mathbf{V}_2 \approx \mathbf{0}$. Then, in this work we follow the optimization strategy developed in [13] to design the set of coded apertures, which can be formulated as

$$\min_{\{\mathbf{H}_\ell\}} \left\| \mathbf{1}_n^T \sum_{\ell=1}^{L} (\mathbf{H}_\ell^H \mathbf{D}^H \mathbf{D} \mathbf{H}_\ell) - (U/n)\mathbf{1}_n \right\|_2^2 + \left\| \sum_{\ell=1}^{L} (\mathbf{H}_\ell^H \mathbf{D}^H \mathbf{D} \mathbf{H}_\ell)\mathbf{1}_n - (U/n)\mathbf{1}_n \right\|_2^2, \tag{19}$$

where $\mathbf{1}_n \in \mathbb{R}^n$ denotes the vector whose entries are ones, and U is a constant. This optimization problem is solved using a greedy algorithm [13]. The first term in (19) handles the sum per column of $\mathbf{P}$ that indicates the number of times a pixel of the image is sensed. Additionally, since each row of $\mathbf{P}$ indicates the number of image pixels measured by a sensor, the design criteria in (19) provides an uniform sensing as illustrated in Fig. 2, for an admissible random variable $d = \{1, 0\}$. In addition, the proposed coded aperture design prevents the formation of clusters of size $r \times r$ of a same coding element, as is illustrated in Fig. 2.

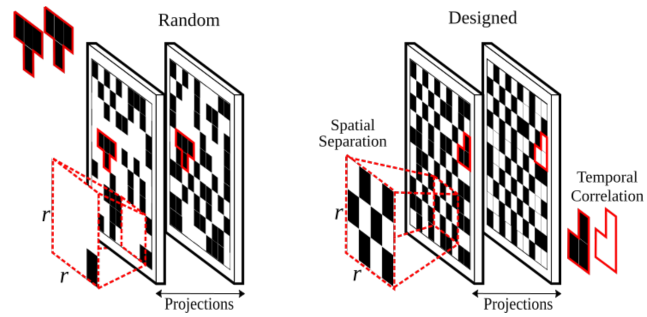

Fig. 2. Visual comparison between a designed and non-designed coded aperture for two projections with $d = \{0, 1\}$ as the random variable.

V. Phase Retrieval Reconstruction Algorithm

A high number of PR algorithms are available in the literature [14], [15], [16], [17], [18]. In particular, this work uses The Smoothing Projected Gradient Phase Retrieval Method (SPGPR)[15] since it requires less number of measurements and exhibits faster computational speed compared with competitive alternatives in the state-of-the-art[15]. This method overcomes the non-smoothness of the amplitude-based objective proposed in [14], optimizing a cost function of the form

$$\min_{\mathbf{r}, \boldsymbol{\varphi}, \mathbf{x} \in \mathbb{C}^n} f(\mathbf{x}, \mu) = \frac{1}{Lm} \sum_{i=1}^{mL} \left(\vartheta_\mu (|\mathbf{a}_i^H \mathbf{x}|) - \sqrt{\mathbf{g}_i} \right)^2, \tag{20}$$

where the function $\vartheta_\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is defined as $\vartheta_\mu(w) = \sqrt{w^2 + \mu^2}$, with $\mu \in \mathbb{R}_{++}$, smooths the term $|\mathbf{a}_i^H \mathbf{x}|$. Notice that if $\mu = 0$, then (20) reduces to the non-smooth formulation proposed in [14]. In practice, μ is a tunable parameter that decreases at each iteration.

Algorithm 1 Algorithm to estimate $\mathbf{x}$

1: **Input:** constants $\tau, \gamma, \gamma_1 \in (0, 1)$, and the number of iterations S_1.
2: Initial point $\mathbf{x}^{(0)} = \sqrt{\frac{\sum_{i=1}^{mL} \mathbf{g}_i}{mL}} \tilde{\mathbf{x}}^{(0)}$, where $\tilde{\mathbf{x}}^{(0)}$ is the leading eigenvector of $\mathbf{Y}_0 := \frac{1}{|I_0|} \sum_{i \in I_0} \frac{\mathbf{a}_i \mathbf{a}_i^H}{\|\mathbf{a}_i\|_2^2}$ given by the power iteration method.
3: **for** $s = 0 : S_1 - 1$ **do**
4: $\quad \mathbf{x}^{(s+1)} \leftarrow \mathbf{x}^{(s)} - \tau \partial f(\mathbf{x}^{(s)}, \mu^{(s)})$
5: $\quad$ **if** $\|\partial f(\mathbf{x}^{(s+1)}, \mu^{(s)})\|_2 \leq \gamma \mu^{(s)}$ **then**
6: $\quad\quad \mu^{(s+1)} = \gamma_1 \mu^{(s)}$
7: $\quad$ **else**
8: $\quad\quad \mu^{(s+1)} = \mu^{(s)}$
9: $\quad$ **end if**
10: **end for**
11: **Output:** $\mathbf{x}^{(S_1)}$

Algorithm 1 summarizes the method to solve (20). Specifically, in line 2, this method uses the orthogonal-promoting initialization proposed in [18]. This initialization consists on calculating $\mathbf{x}^{(0)}$, which is the leading eigenvector $\tilde{\mathbf{x}}^{(0)}$ of the matrix

$$\mathbf{Y}_0 := \frac{1}{|I_0|}\sum_{i\in I_0}\frac{\mathbf{a}_i\mathbf{a}_i^H}{\|\mathbf{a}_i\|_2^2},$$

scaled by the quantity $\lambda_0 := \sqrt{\frac{\sum_{i=1}^{mL}\mathbf{g}_i}{mL}}$, i.e, $\mathbf{x}^{(0)} = \lambda_0\tilde{\mathbf{x}}^{(0)}$, where the set I_0 contains the values of k associated with the $\lfloor\frac{mL}{6}\rfloor$ largest values of $|\mathbf{a}_i^H\mathbf{x}|^2/\|\mathbf{a}_i\|_2$. The notation $|I_0|$ is the cardinality of the set I_0, which is usually chosen as $\lfloor\frac{mL}{6}\rfloor$, where $\lfloor w\rfloor$ denotes the largest integer number less than w. Also, [18] guarantees that the initial guess $\mathbf{x}^{(0)}$ is close to the solution with high probability.

The outcome of the initialization step is refined using a gradient descent strategy based on the Wirtinger derivative as introduced in [16], computed in Line 4, which is defined as

$$\partial f(\mathbf{x},\mu) = \frac{1}{Lm}\sum_{i=1}^{mL}\left(\mathbf{a}_i^H\mathbf{x} - \sqrt{\mathbf{g}_i}\frac{\mathbf{a}_i^H\mathbf{x}}{\varphi_\mu(|\mathbf{a}_{u,i}^H\mathbf{x}|)}\right)\mathbf{a}_i. \quad (21)$$

Finally, from line 5 to 9, the value of μ decreases if $\|\partial f\left(\mathbf{x}^{(s+1)},\mu^{(s)}\right)\|_2 \leq \gamma\mu^{(s)}$ is satisfied.

VI. Experimental Results

This section numerically evaluates the performance of the designed coded apertures following the proposed strategy. The metric used was the recovery error computed as relative error := $\frac{\text{dist}(\mathbf{w},\mathbf{x})}{\|\mathbf{x}\|_2}$, where $\mathbf{x}$ is the underlying signal and the distance between two complex values is defined as

$$dist(\mathbf{w}_1,\mathbf{w}_2) = \underset{\theta\in[0,2\pi)}{min}\|\mathbf{w}_1e^{-j\theta} - \mathbf{w}_2\|_2. \quad (22)$$

Three different uniform admissible random variables $d_1 = \{0,1\}, d_2 = \{-1,1\}$ and $d_3 = \{-1,1,-j,j\}$ were tested. The diffraction patterns of a simulated crystal structure called Rhombic Dodecahedron are the tested images of size 256×256. All simulations were implemented in Matlab 2017a on an Intel Core i7 3.41Ghz CPU, with 32 GB RAM.

In order to verify that the proposed design reduce δ, which lead to better estimation of the true signal, the minimum constant δ in (14) are presented in Table I, varying the super-resolution factor r, when $L = 3$. Specifically, observe that the attained value of δ obtained with the designed coded apertures is smaller than random distribution of the coded aperture. These numerical tests validate that the proposed design strategy allows a better estimation of a high-resolution image from low-resolution coded diffraction measurements.

TABLE I. Value of δ using designed coded apertures for random variables d_1, d_2 and d_3, when $L = 4$ varying r

δ		d_1	d_2	d_3
$r=2$	Proposed	**0.0151**	**0.1283**	**0.0219**
	Random	0.0243	0.1304	0.0235
$r=4$	Proposed	**0.0547**	**0.2281**	**0.1157**
	Random	0.1216	0.2498	0.1215
$r=8$	Proposed	**0.3750**	**0.7105**	**0.6200**
	Random	0.6748	0.7514	0.6256

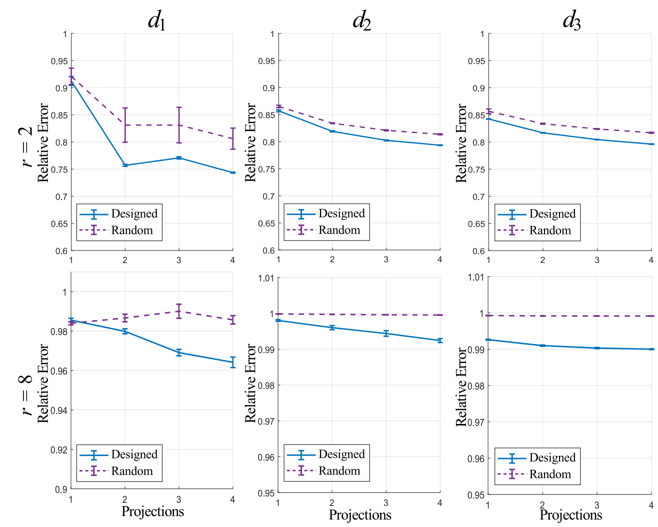

Fig. 3. Relative error of the returned initialization using designed and non-designed coded apertures when the number of projections is varied.

In addition, the relative error of the returned initialization (Line 2, Algorithm 1) for the three random variables and $r = 2$ and $r = 4$ are summarized in Fig. 3. Notice that the designed coded apertures generate a more accurate initialization of the true image compared with non-designed ensembles for any diffraction zone and for all d_1, d_2 and d_3 and the super-resolution factors. We provide these results since the initialization is required to guarantee convergence of Algorithm 1 [15].

Finally, Fig. 4 illustrates the reconstructed magnitude and phase obtained using random and designed coded apertures, for $L = 4$ and $r = 2$. Notice that the quality of the reconstruction obtained using designed coded apertures provide a more detailed image compared with random ensembles. Specifically the designed coded apertures attain a gain of up to 5dB of Peak-Signal-to-Noise-Ration (PSNR) compared with random coded apertures. These experiments provide the effectiveness of the design strategy to retrieve a high-resolution image from low-resolution coded diffraction patterns.

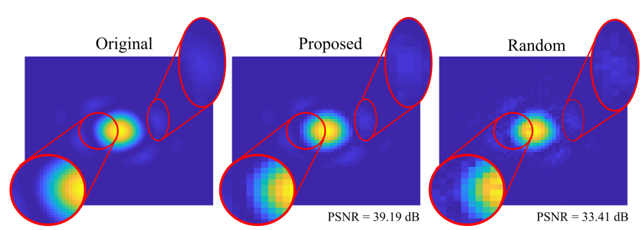

Fig. 4. Reconstructed images using random and designed coded apertures via Algorithm 1.

VII. Conclusion

This paper studied the super-resolution phase retrieval problem from coded diffraction patterns, where a mathematical model was derived. Further, theoretical recovery guarantees for a high-resolution image from low-resolution phaseless measurements were provided establishing that the set of coded apertures need to be designed through the projections in order to better reconstruct the scene image. Thus, a strategy to design the coded aperture based on temporal correlation and spatial separation was developed. Numerical experiments were conducted to evaluate the performance of the proposed method for different admissible random variables. The proposed design was compared with random distribution showing better results for any super-resolution factor. Specifically the designed coded apertures attain a gain of up to 5dB of PSNR compared with random coded apertures.

APPENDIX A
PROOF OF THEOREM I

Proof: Before proving the theorem, let us introduce some notation that will be useful for the proof. Let $\mathcal{T}_{\mathbf{x}}$ be the set of symmetric matrices of the form $\mathcal{T}_{\mathbf{x}} = \{\mathbf{W} = \mathbf{x}\mathbf{w}^H + \mathbf{w}\mathbf{x}^H\}$, which may be interpreted as the tangent space of the manifold of all rank-1 Hermitian matrices at the point $\mathbf{x}\mathbf{x}^H$.

Let $\mathbf{W} \in \mathcal{T}_{\mathbf{x}}$, with rank at most two. For a normalized eigenvector of $\mathbf{W}$, the eigenvalue decomposition can be expressed as $\mathbf{W} = \lambda_1 \mathbf{b}\mathbf{b}^H + \lambda_2 \mathbf{v}\mathbf{v}^H$, with nonnegative eigenvalues λ_1 and λ_2. Observe that from the definition of the linear map $\mathcal{H}_u(\cdot)$ in (11) we have that

$$\begin{aligned} \|\mathcal{H}_u(\mathbf{W})\|_1 &= \sum_{i=1}^{mL} \left|\lambda_1 |\mathbf{h}_{i,u}\mathbf{b}|^2 + \lambda_2 |\mathbf{h}_{i,u}\mathbf{v}|^2\right| \\ &\leq (|\lambda_1| + |\lambda_2|)\|\mathbf{H}_u\|_\infty^2 = \|\mathbf{W}\|_1 \|\mathbf{H}_u\|_\infty^2, \end{aligned} \tag{23}$$

in which the first and second inequalities are obtained using the triangular inequality, and the last claim with the fact that $\sum_j |\lambda_j| = \|\mathbf{W}\|_1$. On the other hand, notice that from the definition of $\mathbf{H}_u$, it can be obtained that

$$\begin{aligned} \|\mathbf{H}_u\|_\infty^2 &= \lambda_{max}\left(\mathbf{H}_u^H \mathbf{H}_u\right) \\ &= \left\| \sum_{\ell=1}^{L} \tilde{\mathbf{M}}_\ell^H \mathbf{D}^H \mathbf{D} \tilde{\mathbf{M}}_\ell \right\|_\infty^2 = \|\mathbf{P}\|_\infty^2, \end{aligned} \tag{24}$$

where $\lambda_{max}(\cdot)$ denotes the largest eigenvalue of a matrix. Additionally, assuming the condition in (12) holds for $L \geq c_0 n$, for some sufficiently large constant $c_0 > 0$, then from Theorem 5.44 in [19] it can be obtained that

$$\mathcal{P}\left(\frac{1}{cmL}\|\mathbf{H}_u\|_\infty^2 \leq 1 + \delta\right) \leq 1 - ne^{-c_1 mL\epsilon^2}, \tag{25}$$

for some constant $c_1 > 0$. Thus, combining (23) and (25), the right side of the inequality, in (14) is expressed as

$$\frac{1}{cmL}\|\mathcal{H}_u(\mathbf{W})\|_1 \leq (1+\delta)\|\mathbf{W}\|_1, \tag{26}$$

with probability at least $1 - ne^{-c_1 mL\epsilon^2}$.

On the other hand, since (25) holds, then from Lemma 5.36 in [19] it can be obtained that

$$\begin{aligned} \frac{1}{c}\|\mathcal{H}_u(\mathbf{W})\|_1 &= \frac{1}{c}\left(\lambda_1 \|\mathbf{H}_u\mathbf{b}\|_2^2 + \lambda_2 \|\mathbf{H}_u\mathbf{v}\|_2^2\right) \\ &\geq (1-\delta)(\lambda_1 + \lambda_2) = (1-\delta)\|\mathbf{W}\|_1, \end{aligned} \tag{27}$$

with probability at least $1 - ne^{-c_1 mL\epsilon^2}$, where the last equality comes from considering that $\mathbf{W}$ is positive semidefinite. Thus, from (27) it get

$$\frac{1}{cmL}\|\mathcal{H}_u(\mathbf{W})\|_1 \geq \frac{1}{mL}(1-\delta)\|\mathbf{W}\|_1, \tag{28}$$

Finally, combining the left side of (26) and the right side of (28), the result in (14) holds. ■

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