# Detection, Enumeration and Localization of Underwater Acoustic Sources

P. V. Nagesha, G. V. Anand, N. Kalyanasundaram and Sanjeev Gurugopinath
Department of Electronics and Communication Engineering, PES University, Bengaluru 560085, India
Emails: {nagesh.uwa, gargeshwari.anand}@gmail.com, n.kalyanasundaram@yahoo.com, sanju83@gmail.com

***Abstract*—The problem of passive detection, enumeration and localization of underwater objects is of great interest in several applications of sonar. Source localization requires prior knowledge of the number of sources. Therefore, detection is followed by estimation of the number of sources, and finally localization. Normally, each of the aforementioned objectives is treated as a different problem, and separate processing techniques are employed to solve them. In this paper, we propose two techniques, viz. the embedded subspace detector (ESD) and the Bartlett processor detector (BPD), for joint detection and enumeration of underwater acoustic sources using a vertical linear array of sensors. The BPD is also capable of simultaneously estimating the range and depth of each source. Simulation results indicate that the proposed detectors can achieve good detection and enumeration at low signal-to-noise ratio, and that their enumeration performance compares very favorably with those of established source enumeration techniques such as Akaike information criterion and minimum description length.**

## I. Introduction

Passive detection, enumeration and localization of underwater acoustic sources is a problem of abiding interest. Normally, each of the afore-mentioned objectives is posed as a different problem, and separate processing techniques are employed to solve them. Narrowband detection of a source by an array of sensors may be treated as a problem of detection of a vector signal in noise. The classical detection techniques of energy detector (ED) and replica correlator (RC) [1] are not suitable for marine application. The ED is very inefficient in the low signal-to-noise ratio (SNR) environment of the ocean. The RC requires knowledge of the signal vector at the array, which is not available due to the unknown locations of the sources. In the subspace detection methods proposed recently by Hari et al. [2], the signal vectors are estimated by exploiting the fact that they belong to a lower dimensional modal subspace, and a generalized likelihood ratio test (GLRT) is employed to achieve detection. But these methods are not capable of estimating the number of sources. Several source enumeration techniques are available in the literature, including the widely used information-theoretic techniques Akaike information criterion (AIC) and minimum description length (MDL) [3]. But these methods are based on the assumption that one or more signals is present, the problem of signal detection is not addressed by them.

This work was supported through Project No. NRB-395/SSB/17-18 under the Grants-in-Aid Scheme of the Naval Research Board, Ministry of Defence, Government of India.

In this paper, we propose two methods, viz. the embedded subspace detector (ESD) and the Bartlett processor detector (BPD), for detecting one or more underwater acoustic signals and estimating the number of sources using a vertical linear array of sensors. If the number of sources is $J$, the array signal vector $\mathbf{s}$ can be decomposed as $\mathbf{s} = \sum_{j=1}^{J} \mathbf{s}_j$. We propose an alternative decomposition $\mathbf{s} = \sum_{j=1}^{J} \overline{\mathbf{s}}_j$ based on the fact that $\mathbf{s}$ belongs to the modal subspace $\mathbb{M}$ [2], and also to the $J$-dimensional signal subspace $\mathbb{S}_J$ obtained by eigendecomposition of the data correlation matrix [4]. In both methods, the objective of joint detection and enumeration is achieved by posing and solving a sequence of binary hypothesis testing problems. The $k^{\text{th}}$ test in the sequence is $\mathcal{H}_0^k : J = k-1$ versus $\mathcal{H}_1^k : J \geq k$, for $k = 1, 2, \ldots$. The GLRT for the $k^{\text{th}}$ test is formulated by finding an unbiased estimate of $\overline{\mathbf{s}}_k$ in the case of ESD, and an unbiased estimate of $\mathbf{s}_k$ in the case of BPD. The $(k+1)^{\text{th}}$ test is performed only if the decision in the $k^{\text{th}}$ test is $J \geq k$. If the decision is $J = k_0$ in the $(k_0+1)^{\text{th}}$ test, the sequence of tests is terminated and the estimate of $J$ is declared as $\widehat{J} = k_0$. The BPD provides the additional benefit of source localization because estimation of $\mathbf{s}_k$ involves a prior range-depth estimation of the $k^{\text{th}}$ source.

The outline of the paper is as follows. The problem formulation is presented in Section II. Section III contains a review the modal subspace and signal subspace. Section IV introduces the definition of embedded signal subspace, and this is followed by a description of the ESD. The BPD is described in Section V. Simulation results are presented in Section VI, and concluding remarks are presented in Section VII.

## II. Formulation of the Problem

We consider a uniform vertical linear array (VLA) of $N$ sensors in a range-independent ocean, receiving mutually uncorrelated narrowband signals of center frequency $f$ from an unknown number of sources, $J$, located at unknown ranges and depths $\{(r_j, z_j), j = 1, \ldots, J\}$. It is desired to detect the signals and estimate the number of sources using $L$ snapshots of the array output vector or data vector $\{\mathbf{x}(l), l = 1, \ldots, L\}$, which can be expressed as

$$\mathbf{x}(l) = \begin{cases} \mathbf{w}(l), & \text{if } J = 0, \\ \mathbf{s}(l) + \mathbf{w}(l), & \text{if } J > 0, \end{cases} \quad l = 1, \ldots, L. \tag{1}$$

$$\mathbf{s}(l) = \sum_{j=1}^{J} \mathbf{s}_j(l), \quad l = 1, \ldots, L. \tag{2}$$

In (1), $\mathbf{s}_j(l) \in \mathcal{C}^N$ is the signal vector received by the array from the $j^{\text{th}}$ source, and $\mathbf{w}(l) \in \mathcal{C}^N$ is the noise array vector, in the $l^{\text{th}}$ snapshot. It is assumed that the noise is uncorrelated with the signals and that $\{\mathbf{w}(l), l = 1, \ldots, L\}$ are independent and identically distributed (i.i.d.) complex circular Gaussian $\mathcal{CN}(\mathbf{0}, \mathbf{R}_w)$ vectors. The noise covariance matrix $\mathbf{R}_w$ can be constructed using the ambient noise model proposed by Buckingham [5], and the problem of signal detection in colored Gaussian noise can be converted to one of detection in white Gaussian noise by premultiplying the data vector by the whitening matrix $\mathbf{R}_w^{-1/2}$. In this paper, we shall assume that $\mathbf{R}_w = \sigma^2 \mathbf{I}_N$ for the sake of simplicity. It is known that ambient noise in the ocean is non-Gaussian under certain conditions. But the problem of joint detection and enumeration in non-Gaussian noise is beyond the scope of this paper.

Using the normal mode theory of sound propagation in a horizontally stratified channel, the signal vectors are [6]

$$\mathbf{s}_j(l) = \alpha_j(l)\mathbf{p}(r_j, z_j), \tag{3}$$

$$\mathbf{p}(r, z) = \sum_{m=1}^{M} \beta_m(r, z)\mathbf{a}_m = \mathbf{A}\boldsymbol{\beta}(r, z), \tag{4}$$

$$\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_M], \quad m=1, \ldots, M,$$
$$\mathbf{a}_m = [\psi(z), \psi(z_0+d), \ldots, \psi(z_0+(N-1)d)]^T, \tag{5}$$
$$\boldsymbol{\beta}(r, z) = [\beta_1(r, z), \ldots, \beta_M(r, z)]^T,$$

$$\beta_m(r, z) = \sqrt{2\pi} e^{\left(-i\frac{\pi}{4}\right)} \psi_m(z) \frac{\exp(i\xi_m r - \zeta_m r)}{\sqrt{\xi_m r}}. \tag{6}$$

In (3)-(6), $\alpha_j(l)$ is the unknown complex amplitude of the $j^{\text{th}}$ source signal in the $l^{\text{th}}$ snapshot, $\mathbf{p}(r, z)$ is the signal vector at the VLA if a single source of unit amplitude is present at range-depth $(r, z)$, $z_0$ is the depth of the topmost sensor in the array, $d$ is the inter-sensor spacing, $\mathbf{a}_m$ is the mode shape vector of the $m^{\text{th}}$ mode, $\boldsymbol{\beta}(r, z)$ is the vector of mode amplitudes for the signal received from a source at $(r, z)$, and $i = \sqrt{-1}$. The mode functions $\{\psi_1(\cdot), \ldots, \psi_M(\cdot)\}$, modal wavenumbers $\{\xi_1, \ldots, \xi_M\}$, and mode attenuation coefficients $\{\zeta_1, \ldots, \zeta_M\}$ can be readily computed [6] if the ocean acoustic parameters are known. We assume that $\mathcal{E}\left[||\mathbf{s}_j(l)||^2\right] \geq \mathcal{E}\left[||\mathbf{s}_k(l)||^2\right]$, for $j < k$, where $\mathcal{E}(\cdot)$ is the expectation operation. It is also assumed that $J \leq M \leq N$, i.e. the number of sources $J$ does not exceed the number of modes $M$, which in turn does not exceed the number of sensors $N$.

## III. Modal Subspace and Signal Subspace

Some basic concepts and properties of the data vectors and signal vectors are reviewed in this section. The modal subspace $\mathbb{M}$ defined as [7]

$$\mathbb{M} = \text{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_M\}. \tag{7}$$

The condition $M \leq N$ ensures that the mode shape vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_M\}$ are linearly independent and $\mathbb{M}$ is an $M$-dimensional subspace. It follows from (3) and (7) that

$$\mathbf{s}_j(l) \in \mathbb{M}, \; j = 1, \ldots, J. \tag{8}$$

The condition $J \leq M$ implies that $\{\mathbf{s}_1(l), \ldots, \mathbf{s}_J(l)\}$ are linearly independent under the reasonable assumption that $\{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_J\}$ are linearly independent for every set of $J$ distinct locations $\{(r_1, z_1), \ldots, (r_J, z_J)\}$. The $J$-dimensional signal subspace $\mathbb{S}_J$ is defined as

$$\mathbb{S}_J = \text{span}\{\mathbf{s}_1(l), \ldots, \mathbf{s}_J(l)\}. \tag{9}$$

Consider the data covariance matrix defined as

$$\mathbf{R} = \mathcal{E}\left(\mathbf{x}(l)\mathbf{x}^H(l)\right). \tag{10}$$

Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of $\mathbf{R}$, and let $\lambda_1 \geq \ldots \geq \lambda_N$. Let $[\mathbf{v}_1, \ldots, \mathbf{v}_N]$ be the unit-norm eigenvectors of $\mathbf{R}$. The signal subspace can also be defined as [4]

$$\mathbb{S}_J = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_J\} = \text{span}\{\mathbf{s}_1(l), \ldots, \mathbf{s}_J(l)\}. \tag{11}$$

Obviously, $\{\mathbf{v}_1, \ldots, \mathbf{v}_J\}$ is an orthonormal basis of $\mathbb{S}_J$, while $\{\mathbf{s}_1(l), \ldots, \mathbf{s}_J(l)\}$ is a non-orthogonal basis. It follows from (11) that the total received signal $\mathbf{s}(l)$ can be expressed either as the sum of $J$ linearly independent components $\{\mathbf{s}_1(l), \ldots, \mathbf{s}_J(l)\}$ as shown in (2), or as the sum of $J$ orthogonal components $\{\bar{\mathbf{s}}_1(l), \ldots, \bar{\mathbf{s}}_J(l)\}$:

$$\mathbf{s}(l) = \sum_{j=1}^{J} \bar{\mathbf{s}}_j(l), \text{where } \bar{\mathbf{s}}_j(l) = b_j(l)\mathbf{v}_j, \tag{12}$$

$$b_j(l) = \sum_{p=1}^{L} \mathbf{v}_j^H \mathbf{s}_p(l), j = 1, \ldots, J. \tag{13}$$

It can be readily shown that $\mathcal{E}|b_j(l)|^2 = \lambda_j$, and therefore,

$$\mathcal{E}\left[||\bar{\mathbf{s}}_j(l)||^2\right] \geq \mathcal{E}\left[||\bar{\mathbf{s}}_k(l)||^2\right], \quad \text{for } j < k, \tag{14}$$

It also follows from (8) and (11) that $\mathbb{S}_J \subset \mathbb{M}$, if $J < M$, $\mathbb{S}_J = \mathbb{M}$, if $J = M$. In practice, the covariance matrix $\mathbf{R}$ and the eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_N\}$ are not known. These are estimated from $L$ snapshots of the data vector $\{\mathbf{x}(l), l = 1, \ldots, L\}$:

$$\widehat{\mathbf{R}}_L = \frac{1}{L} \sum_{l=1}^{L} \mathbf{x}(l)\mathbf{x}^H(l). \tag{15}$$

The eigenvalues and eigenvectors of $\widehat{\mathbf{R}}_L$ are denoted by $\{\widehat{\lambda}_{1,L}, \ldots, \widehat{\lambda}_{N,L}\}$ and $\{\widehat{\mathbf{v}}_{1,L}, \ldots, \widehat{\mathbf{v}}_{N,L}\}$. Likewise, the estimate of the $J$-dimensional signal subspace is denoted by

$$\widehat{\mathbb{S}}_{J,L} = \text{span}\{\widehat{\mathbf{v}}_{1,L}, \ldots, \widehat{\mathbf{v}}_{J,L}\}. \tag{16}$$

We note that

$$\{\widehat{\mathbf{v}}_{1,L}, \ldots, \widehat{\mathbf{v}}_{N,L}\} \stackrel{\text{a.s.}}{\to} \{\mathbf{v}_1, \ldots, \mathbf{v}_N\}, \text{ as } L \to \infty. \tag{17}$$

## IV. Embedded Subspace Detector

### A. Embedded Signal Subspace

Consider the projection of the eigenvectors $\{\widehat{\mathbf{v}}_{1L}, \ldots, \widehat{\mathbf{v}}_{NL}\}$ onto the $M$-dimensional modal subspace $\mathbb{M}$:

$$\mathbf{q}_{k,L} = \mathbf{P}_{\mathbb{M}} \widehat{\mathbf{v}}_{k,L}, \quad k = 1, \ldots, N, \tag{18}$$

where $\mathbf{P}_{\mathbb{M}} = \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$ is the projection matrix. The set of vectors $\{\widehat{\mathbf{v}}_{k,L}, k = 1, \ldots, M\}$ is linearly independent. Let $\{\mathbf{h}_{k,L}, k = 1, \ldots, M\}$ be the orthonormal set of vectors generated from $\{\mathbf{q}_{k,L}, k = 1, \ldots, M\}$ by the Gram-Schmidt procedure. We define the embedded signal subspace $\mathbb{E}$ as

$$\mathbb{E} = \text{span}\{\mathbf{q}_{1,L}, \ldots, \mathbf{q}_{M,L}\} = \text{span}\{\mathbf{h}_{1,L}, \ldots, \mathbf{h}_{M,L}\}. \tag{19}$$

Using (17), it can be easily verified that

$$\{\mathbf{h}_{1,L}, \ldots, \mathbf{h}_{J,L}\} \overset{\text{a.s.}}{\to} \{\mathbf{v}_1, \ldots, \mathbf{v}_J\}, \text{ as } L \to \infty. \tag{20}$$

In the sequel, we shall denote the basis vectors of $\mathbb{E}$ by $\{\mathbf{h}_1, \ldots, \mathbf{h}_J\}$, for the sake of brevity.

### B. Embedded Subspace Detector (ESD)

It is proposed to conduct a sequence of binary hypothesis tests to detect the presence of one or more signals and to estimate the number of sources. The $k^{\text{th}}$ hypothesis test is

$$\begin{aligned} &\mathcal{H}_0^k : J = k-1, \\ &\mathcal{H}_1^k : J \geq k, \quad k = 1, \ldots, M. \end{aligned} \tag{21}$$

For $k > 1$, the $k^{\text{th}}$ test is conducted only if the decision in the $(k-1)^{\text{th}}$ test is in favour of $\mathcal{H}_1^{k-1}$. The sequence is terminated after $k_0$ tests if the decision in the $k_0^{\text{th}}$ test is in favour of $\mathcal{H}_0^{k_0}$ or if $k_0 = M$.

The first test in the sequence (21) is equivalent to the following:

$$\begin{aligned} &\mathcal{H}_0^1 : \mathbf{x}(l) = \mathbf{w}(l), \\ &\mathcal{H}_1^1 : \mathbf{x}(l) = \mathbf{s}(l) + \mathbf{w}(l), \quad l = 1, \ldots, L, \end{aligned} \tag{22}$$

where $\mathbf{s}(l) = \sum_{j=1}^J \mathbf{s}_j(l) = \sum_{j=1}^J \overline{\mathbf{s}}_j(l)$ is the sum of signals received from an unknown number of sources $J$. Since neither $\mathbf{s}(l)$ nor $J$ is known, we propose a generalized likelihood ratio test (GLRT) [1] based on an unbiased estimate of $\overline{\mathbf{s}}_1(l)$, which is the strongest component of $\mathbf{s}(l)$ in the orthogonal decomposition (13). The estimate is given by

$$\widehat{\overline{\mathbf{s}}}_1(l) = \mathbf{h}_1\mathbf{h}_1^H\mathbf{x}(l). \tag{23}$$

Since $\mathbf{h}_1 \overset{\text{a.s.}}{\to} \mathbf{v}_1$ as $L \to \infty$, and making use of (13), we get

$$\mathcal{E}\widehat{\overline{\mathbf{s}}}_1(l) = \mathbf{h}_1\mathbf{h}_1^H \sum_{j=1}^J b_j(l)\mathbf{v}_j \overset{\text{a.s.}}{\to} \overline{\mathbf{s}}_1(l), \text{ as } L \to \infty \text{ under } \mathcal{H}_1^1. \tag{24}$$

Hence, $\widehat{\overline{\mathbf{s}}}_1(l)$ is an asymptotically unbiased estimate of $\overline{\mathbf{s}}_1(l)$. The GLRT has the form

$$\overline{T}_1 \underset{\mathcal{H}_0^1}{\overset{\mathcal{H}_1^1}{\gtrless}} t_1. \tag{25}$$

The test statistic $\overline{T}_1$ is given by [2]

$$\overline{T}_1 = \begin{cases} \frac{1}{L}\sum_{l=1}^L \left[\mathbf{x}^H(l)\mathbf{x}(l) - \left(\mathbf{x}(l) - \widehat{\overline{\mathbf{s}}}_1(l)\right)^H \left(\mathbf{x}(l) - \widehat{\overline{\mathbf{s}}}_1(l)\right)\right], \\ \qquad\qquad\qquad\qquad \sigma^2 \text{ known} \\ \dfrac{\sum_{l=1}^L \mathbf{x}^H(l)\mathbf{x}(l)}{\sum_{l=1}^L \left(\mathbf{x}(l) - \widehat{\overline{\mathbf{s}}}_1(l)\right)^H \left(\mathbf{x}(l) - \widehat{\overline{\mathbf{s}}}_1(l)\right)}, \sigma^2 \text{ unknown} \end{cases} \tag{26}$$

If the decision of the GLRT (25) is in favour of $\mathcal{H}_1^1$, the second hypothesis test in the sequence (21) is performed. For $k > 1$, (21) is equivalent to the following:

$$\begin{aligned} &\mathcal{H}_0^{(k)} : \overline{\mathbf{x}}^{(k)}(l) = \overline{\mathbf{w}}^{(k)}(l), \\ &\mathcal{H}_1^{(k)} : \overline{\mathbf{x}}^{(k)}(l) = \overline{\mathbf{s}}^{(k)}(l) + \overline{\mathbf{w}}^{(k)}(l), \end{aligned} \tag{27}$$

$l = 1, \ldots, L;\ k = 1, \ldots, M$. In (27), $\overline{\mathbf{x}}^{(k)}(l)$ is the residual data vector obtained by subtracting the estimated contribution of the signals $\sum_{j=1}^{k-1} \overline{\mathbf{s}}_j(l)$ detected in the previous $k-1$ tests from the original data vector $\mathbf{x}(l)$, i.e.

$$\overline{\mathbf{x}}^{(k)}(l) = \mathbf{x}(l) - \sum_{j=1}^{k-1} \widehat{\overline{\mathbf{s}}}_j(l), \quad k = 2, \ldots, M, \tag{28}$$

where $\widehat{\overline{\mathbf{s}}}_j(l)$ is an asymptotically unbiased estimate of $\overline{\mathbf{s}}_j(l)$, analogous to that in (23):

$$\widehat{\overline{\mathbf{s}}}_j(l) = \mathbf{h}_j\mathbf{h}_j^H\mathbf{x}(l), \quad j = 1, \ldots, J, \tag{29}$$

and $\overline{\mathbf{s}}^{(k)}(l)$ and $\overline{\mathbf{w}}^{(k)}(l)$ are respectively the residual signal vector and residual noise vector:

$$\overline{\mathbf{s}}^{(k)}(l) = \mathbf{s}(l) - \sum_{j=1}^{k-1} \mathbf{h}_j\mathbf{h}_j^H\mathbf{s}(l), \quad k = 2, \ldots, M \tag{30}$$

$$\overline{\mathbf{w}}^{(k)}(l) = \mathbf{w}(l) - \sum_{j=1}^{k-1} \mathbf{h}_j\mathbf{h}_j^H\mathbf{w}(l), \ k = 2, \ldots, M, \tag{31}$$

We note that $\overline{\mathbf{s}}^{(k)}(l) \overset{\text{a.s.}}{\to} \sum_{j=k}^J \overline{\mathbf{s}}_j(l)$ as $L \to \infty$. On defining $\overline{\mathbf{x}}^{(1)}(l) = \mathbf{x}(l)$, and replacing $\mathbf{x}(l)$ and $\widehat{\overline{\mathbf{s}}}_1(l)$ in (26) by $\overline{\mathbf{x}}^{(k)}(l)$ and $\widehat{\overline{\mathbf{s}}}_k(l)$ respectively, we get

$$\overline{T}_k = \begin{cases} \frac{1}{L}\sum_{l=1}^L \overline{\mathbf{x}}^{(k)^H}(l)\mathbf{h}_k\mathbf{h}_k^H\overline{\mathbf{x}}^{(k)}(l), \qquad \sigma^2 \text{ known} \\ 1 + \dfrac{\sum_{l=1}^L \overline{\mathbf{x}}^{(k)^H}(l)\mathbf{h}_k\mathbf{h}_k^H\overline{\mathbf{x}}^{(k)}(l)}{\sum_{l=1}^L \overline{\mathbf{x}}^{(k)^H}(l)(\mathbf{I}_N - \mathbf{h}_k\mathbf{h}_k^H)\overline{\mathbf{x}}^{(k)}(l)}, \sigma^2 \text{ unknown} \end{cases} \tag{32}$$

for $k = 1, \ldots, M$. For a given threshold $t_k$, the probability of false alarm $P_{F,k}$ and probability of detection $P_{D,k}^J$ in the $k^{\text{th}}$ test are given by

$$\begin{aligned} &P_{F,k} = P(\overline{T}_k > t_k; \mathcal{H}_0^k) \\ &P_{D,k}^J = P(\overline{T}_k > t_k; \mathcal{H}_1^k), \quad k = 1, \ldots, M. \end{aligned} \tag{33}$$

In (33), $P_{D,k}^J$ denotes the probability of detecting the $k^{\text{th}}$ source, given that $J$ sources are present. For achieving a constant false alarm rate (CFAR) $\alpha$, the thresholds $t_k = t_k(\alpha)$ are chosen such that $P_{F,k} = \alpha$.

As stated earlier, the sequence of tests is terminated after $k_0$ tests if the decision in the $k_0^{\text{th}}$ test is in favour of $\mathcal{H}_0^{k_0}$ or if $k_0 = M$. The estimate of $J$ is given by

$$\widehat{J} = k_0. \tag{34}$$

## V. Bartlett Processor Detector

In the case of BPD, the $k^{\text{th}}$ hypothesis test is preceded by estimation of range-depth $(r_k, z_k)$ of the source corresponding to the $k^{\text{th}}$ strongest received signal $\mathbf{s}_k(l)$. The estimated range-depth $(\widehat{r}_k, \widehat{z}_k)$ is used to estimate $\mathbf{s}_k(l)$. The range-depth estimation is based on the Bartlett processor. Output of the Bartlett processor is given by

$$B_L(r,z) = \frac{1}{L}\sum_{l=1}^{L} \mathbf{x}^H(l)\mathbf{u}(r,z)\mathbf{u}^H(r,z)\mathbf{x}(l), \tag{35}$$

where $\mathbf{u}(r,z)$ is the replica vector corresponding to a hypothetical source position $(r,z)$:

$$\mathbf{u}(r,z) = \frac{\mathbf{p}(r,z)}{||\mathbf{p}(r,z)||}, \tag{36}$$

and $\mathbf{p}(r,z)$ is defined in (4). The location of the highest peak of $B_L(r,z)$ is chosen as the range-depth estimate $(\widehat{r}_1, \widehat{z}_1)$ of the strongest signal $\mathbf{s}_1(l)$:

$$(\widehat{r}_1, \widehat{z}_1) = \arg\max B_L(r,z) \tag{37}$$

Let the replica vector for a source at $(\widehat{r}_1, \widehat{z}_1)$ be denoted by

$$\widehat{\mathbf{u}}_1 = \mathbf{u}(\widehat{r}_1, \widehat{z}_1). \tag{38}$$

The estimate of $\mathbf{s}_1(l)$ is given by

$$\widehat{\mathbf{s}}_1(l) = \widehat{\mathbf{u}}_1\widehat{\mathbf{u}}_1^H\mathbf{x}(l). \tag{39}$$

The test statistic $T_1$ for the first hypothesis test of BPD is obtained on replacing $\widehat{\widehat{\mathbf{s}}}(l)$ by $\widehat{\mathbf{s}}(l)$ in the right-hand side of (26). If the decision is in favour of $\mathcal{H}_1^1$, we proceed to perform the second test, and so on, as explained in Section IV-B.

The $k^{\text{th}}$ hypothesis test $(k > 1)$ for the BPD is given by

$$\begin{aligned} &\mathcal{H}_0^{(k)} : \mathbf{w}^{(k)}(l), \\ &\mathcal{H}_1^{(k)} : \mathbf{s}^{(k)}(l) + \mathbf{w}^{(k)}(l), \end{aligned} \tag{40}$$

$l = 1, \ldots, L;\ k = 2, \ldots, M$, where

$$\mathbf{x}^{(k)}(l) = \mathbf{x}(l) - \sum_{j=1}^{k-1}\widehat{\mathbf{s}}_j(l),\ k = 2, \ldots, M, \tag{41}$$

and $\widehat{\mathbf{s}}_j(l)$ is the estimate of the $j^{\text{th}}$ strongest signal $\mathbf{s}_j(l)$. In the $k^{\text{th}}$ iteration, the signal vector estimates for the hypothetical source locations $(\tilde{r}_1, \tilde{z}_1), \ldots, (\tilde{r}_Q, \tilde{z}_Q)$ are given by

$$\widehat{\mathbf{s}}^{(k)}(\tilde{r}_q, \tilde{z}_q; l) = \mathbf{u}(\tilde{r}_q, \tilde{z}_q)\mathbf{u}^H(\tilde{r}_q, \tilde{z}_q)\mathbf{x}^{(k)}(l), q = 1, \ldots, Q. \tag{42}$$

The range-depth estimate of the strongest residual source is

$$(\widehat{r}_k, \widehat{z}_k) = \underset{q\in\{1,\ldots,Q\}}{\arg\max} \sum_{l=1}^{L} \widehat{\mathbf{s}}^{(k)^H}(\tilde{r}_q, \tilde{z}_q; l)\widehat{\mathbf{s}}^{(k)}(\tilde{r}_q, \tilde{z}_q; l). \tag{43}$$

The estimate of the strongest residual signal $\mathbf{s}_k(l)$ is given by

$$\widehat{\mathbf{s}}_k(l) = \widehat{\mathbf{u}}_k\widehat{\mathbf{u}}_k^H\mathbf{x}^{(k)}(l),\ \text{where}\ \widehat{\mathbf{u}}_k = \mathbf{u}(\widehat{r}_k, \widehat{z}_k). \tag{44}$$

The residual signal and noise vectors in (40) are

$$\mathbf{s}^{(k)}(l) = \mathbf{s}(l) - \sum_{j=1}^{k-1}\widehat{\mathbf{u}}_j\widehat{\mathbf{u}}_j^H\mathbf{s}(l),\ k = 2, \ldots, M \tag{45}$$

$$\mathbf{w}^{(k)}(l) = \mathbf{w}(l) - \sum_{j=1}^{k-1}\widehat{\mathbf{u}}_j\widehat{\mathbf{u}}_j^H\mathbf{w}(l),\ k = 2, \ldots, M \tag{46}$$

We note that $\mathbf{s}^{(k)}(l) \approx \sum_{j=k}^{J}\mathbf{s}_j(l)$ is a good approximation if the range-depth estimation errors are small and $\widehat{\mathbf{u}}_j^H\widehat{\mathbf{u}}_p \ll 1$, for $j \neq p$. On defining $\mathbf{x}^{(1)}(l) = \mathbf{x}(l)$, and replacing $\mathbf{x}(l)$ and $\widehat{\widehat{\mathbf{s}}}_1(l)$ by $\mathbf{x}^{(k)}(l)$ and $\widehat{\mathbf{s}}_k(l)$, respectively, we get the test statistic

$$T_k = \begin{cases} \frac{1}{L}\sum_{l=1}^{L}\mathbf{x}^{(k)^H}(l)\widehat{\mathbf{u}}_k\widehat{\mathbf{u}}_k^H\mathbf{x}^{(k)}(l), & \sigma^2 \text{ known} \\ 1+\dfrac{\sum_{l=1}^{L}\mathbf{x}^{(k)^H}(l)\widehat{\mathbf{u}}_k\widehat{\mathbf{u}}_k^H\mathbf{x}^{(k)}(l)}{\sum_{l=1}^{L}\mathbf{x}^{(k)H}(l)(\mathbf{I}_N-\widehat{\mathbf{u}}_k\widehat{\mathbf{u}}_k^H)\mathbf{x}^{(k)}(l)}, & \sigma^2 \text{ unknown} \end{cases} \tag{47}$$

Quantities $P_{F,k}$ and $P_{D,k}^J$ are calculated as shown in Sec. IV-B.

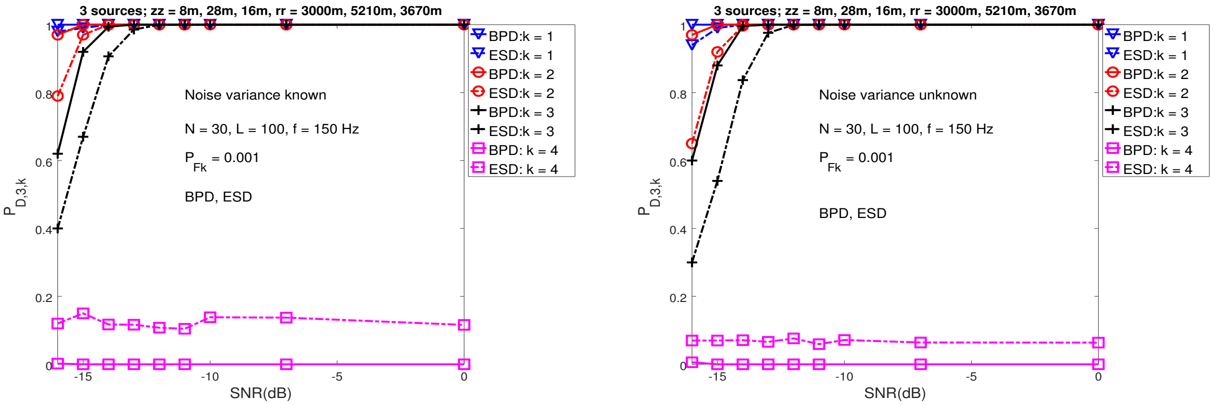
## VI. Simulation Results

We considered an ocean modelled as a Pekeris waveguide with the following parameters: ocean depth = 150 m, sound speed in water = 1500 m/s, sound speed in ocean bottom = 1700 m/s, ratio of densities in ocean bottom and water = 1.5, and attenuation in ocean bottom = 0.2 dB/wavelength. Values of the array parameters are: $N = 30$, $z_0 = 5$ m, $d = 5$ m. Three narrowband sources of centre frequency $f = 150$ Hz were assumed to be located at $(r_1, z_1) = (3000\text{ m}, 8\text{ m})$, $(r_2, z_2) = (5210\text{ m}, 28\text{ m})$, and $(r_3, z_3) = (3670\text{ m}, 16\text{ m})$. The mode functions $\psi_m(z)$, wavenumbers $\xi_m$, and attenuation coefficients $\zeta_m$ were computed using the Kraken normal mode program [8]. For the chosen channel parameters and signal frequency, the waveguide supports $M = 14$ normal modes. The amplitudes $\{\alpha_j(l); j = 1,2,3; l = 1, \ldots, L\}$ of signal vectors $\mathbf{s}_j(l)$ were modelled as complex quantities with deterministic magnitudes independent of $l$, and random phases. The signal-to-noise (SNR) for $j^{\text{th}}$ signal at the array is

$$\text{SNR}_j = 10\log_{10}\left(\frac{||\mathbf{s}_j(l)||^2}{N\sigma^2}\right), \tag{48}$$

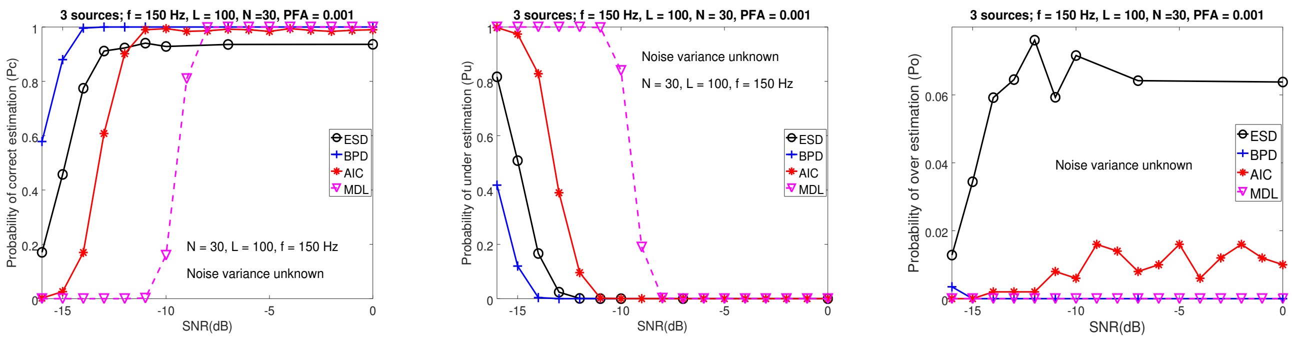
where $||\cdot||$ denotes the Euclidean norm of a vector. In all simulations, $\text{SNR}_j$ is chosen to be equal for all signals at the array. The number of snapshots is $L = 100$. In all figures, the probability of false alarm $P_{F,k} = 10^{-3}$ for all $k$.

Figures 1(a) and 1(b) show plots of $P_{D,k}^3$ vs. SNR for the cases of known noise variance and unknown noise variance respectively. Recall that $P_{D,k}^J$ denotes the probability of detecting the $k^{\text{th}}$ source when $J$ sources are present. Hence it is desirable to achieve a high value of $P_{D,k}^J$ for $k \leq J$ and a low value of $P_{D,k}^J$ for $k > J$. It is seen that both ESD and BPD perform very well if SNR exceeds $-12$ dB and that the performance of BPD is consistently better than that of ESD.




(a) Noise variance known. (b) Noise variance unknown.

Fig. 1: Plots of $P^3_{D,k}$ vs. SNR for $P_{F,k} = 10^{-3}$. Solid lines – BPD. Broken lines – ESD.




(a) Probability of correct estimation. (b) Probability of under-estimation. (c) Probability of over-estimation.

Fig. 2: Estimation of number of sources by different methods when $J = 3$ and $P_{F,k} = 10^{-3}$. Noise variance is unknown.

In Figs. 2(a)-2(c), the source enumeration performance of ESD and BPD is illustrated and also compared with the performance of the AIC and MDL methods [3]. Noise variance is assumed to be unknown. These figures show plots of probability of correct source number estimation $P_c = P(\widehat{J} = J)$, probability of under-estimation $P_u = P(\widehat{J} < J)$, and probability of overestimation $P_o = P(\widehat{J} > J)$. Ideally, we should achieve $P_c = 1$ and $P_u = P_o = 0$. Figures 2(a) – 2(c) show that BPD performs better than ESD. BPD performs significantly better than both AIC and MDL. ESD also performs better than AIC and MDL at low SNR. The excellent performance of BPD is due to the fact that the root mean square errors (RMSE) of range and depth estimation are very low. Our simulations show that, for SNR of $-13$ dB, the RMSE averaged over all sources is less than 10 m for range estimation and about 1 m for depth estimation.

## VII. Conclusion

Two methods of joint detection and enumeration of acoustic sources in a range-independent ocean, viz. ESD and BPD, were presented in this paper. The BPD is also capable of simultaneously estimating the range and depth of each source. Simulation results were presented to show that the proposed detectors can achieve good detection and enumeration at low signal-to-noise ratio, and that their enumeration performance compares very favourably with those of two widely used source enumeration techniques, viz. AIC and MDL [3]. Work on the theoretical performance analysis of the proposed methods is in progress, and the results will be reported in due course.

## References

[1] S. M. Kay, *Fundamentals of Statistical Signal Processing, Vol. II: Detection Theory*. Prentice-Hall, Upper Saddle River, New Jersey, 1998.

[2] V. N. Hari, G. V. Anand, and A. B. Premkumar, “Narrowband signal detection techniques in shallow ocean by acoustic vector sensor array,” *Digital Signal Processing*, vol. 23, pp. 1645–1661, Sep. 2013.

[3] M. Wax and T. Kailath, “Detection of signals by information theoretic criteria,” *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-33, pp. 387–392, Apr. 1985.

[4] R. O. Schmidt, “Multiple emitter location and signal parameter estimation,” *IEEE Trans. Antennas Propag.*, vol. AP-34, pp. 276–280, Mar. 1986.

[5] M. J. Buckingham, “A theoretical model of ambient noise in a low-loss, shallow water channel,” *J. Acoust. Soc. Am.*, vol. 67, pp. 1186–1192, Apr. 1980.

[6] F. B. Jensen, W. A. Kuperman, M. B. Porter and H. Schmidt, *Computational Ocean Acoustics, 2nd Ed.*, Springer, New York, 2011.

[7] S. Lakshmipathi and G. V. Anand, “Subspace intersection method of high-resolution bearing estimation in shallow ocean,” *Signal Processing*, vol. 84, pp. 1367–1384, Aug. 2004.

[8] M. B. Porter, “The Kraken normal mode program,” *SACLANTCEN Memorandum, U. S. Dept. of Commerce, National Tech. Inf. Service, Tech. Rep.*, 1991.