Auto-calibration of Uniform Linear Array Antennas

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***Abstract*—Calibration is instrumental to realize the full performance of a measurement system. In this contribution we consider the calibration of a uniformly linear array antenna where we assume each antenna element has an unknown complex gain. We present an algorithm which can be used to calibrate the array without full knowledge of the environment. Particularly, if the number of signal sources are known we show that we can determine the individual unknown antenna gains up to an ambiguity parametrized by a single complex scalar. If the ratio of the complex gains between two consecutive elements is also known, this ambiguity is resolved. The method is based on determining the antenna calibration parameters such that the Hankel matrix of the array snapshots has a given rank. A numerical example illustrates the performance of the method. The numerical results suggest that the method is consistent in SNR.**

***Index Terms*—Estimation, Optimization, Calibration, Linear antenna arrays, Direction-of-arrival estimation**

I. Introduction

Array antennas play an instrumental role in modern radar and communication systems. In this contribution we consider the technical problem of calibrating an antenna array to mitigate the effects of deviations in performance between different antenna elements.

The problem of estimating the unknown gains in a linear array in a blind fashion, i.e. without knowing the directions to the active signal sources is called *auto-calibration*. We can also regard this as a joint estimation of the array gain and the directions to the signal sources. This problem has been treated by many authors over the years [1]–[6] with various assumptions on the array and the target properties.

In this contribution we describe an auto-calibration method which is based on the low-rank properties of a Hankel matrix built from the snapshot data. This matrix is explored in Kung's subspace method [7] [8] for *direction of arrival (DOA)* estimation. The Hankel matrix is parametrized with the unknown calibration parameters. The calibration parameters are determined by minimizing a cost function involving the rank of the Hankel matrix. If the number of active targets are known, the unknown calibration parameters can be determined up to an inherent ambiguity involving only a complex scalar. If we assume the complex ratio between the true gain of two consecutive antenna elements to be known, both the direction of arrivals and the gain of all the antenna array elements can be recovered. The contributions of the paper are the following. We give a precise and short theoretical background of the problem based on system theory and give conditions for the possible ambiguous solutions. Compared to [1] and [6] we do not assume the sources to be uncorrelated. We do not assume any statistical properties regarding the unknown gains in contrast with the method proposed in [2].

After the problem formulation in Section II we present some system theory results in Section III. The calibration method is outlined in Section IV and numerical illustrations are given in Section V. The paper is summarized in Section VI.

A. Notation

By $(\cdot)^T$ and $(\cdot)^*$ we denote the transpose and the Hermitian transpose respectively. The Hadamard product $\odot$ is the element wise matrix product, i.e. $[\mathbf{A} \odot \mathbf{B}]_{ij} = [\mathbf{A}]_{ij}[\mathbf{B}]_{ij}$. For column vectors $\mathbf{a}$ and $\mathbf{b}$, we have $\mathbf{a} \odot \mathbf{b} = \operatorname{diag}(\mathbf{a})\mathbf{b}$, where $\operatorname{diag}(\mathbf{a})$ is a diagonal matrix with the elements in vector $\mathbf{a}$ on the diagonal.

II. Problem formulation

We consider the standard formulation by assuming P targets are emitting narrowband signals which are sensed by a uniform linear array (ULA) with M elements. After IQ demodulation the measured vector signal at snapshot n at the array can be modeled as [9]

$$\tilde{\mathbf{y}}(n) = \mathbf{y}(n) + \mathbf{v}(n) = \sum_{p=1}^{P} \mathbf{a}(\theta_p) x_p(n) + \mathbf{v}(n)$$

where θ_p is the direction of the arrival, $\mathbf{a}(\theta_p) \in \mathbb{C}^M$ is the steering vector, θ_p is the direction of arrival, $x_p(n)$ is the complex amplitude for target p, and $\mathbf{v}(n)$ is an assumed additive noise signal at snapshot n. Here $\mathbf{y}(n)$ denotes the noise free snapshot vector. In the derivation of the method below we assume noise free data and will return to the more relevant case with noise present in the numerical evaluation of the calibration method.

Due to manufacturing inaccuracies and electromagnetic effects the array steering vector $\mathbf{a}(\theta_p)$ is expressed as

$$\mathbf{a}(\theta_p) = \mathbf{g} \odot \mathbf{a}_0(\theta_p) \tag{1}$$

where $\mathbf{a}_0(\theta_p)$ is the ideal steering vector and $\mathbf{g} = [g_1, g_2, \ldots, g_M]^T \in \mathbb{C}^M$ is the static array gain and we assume $g_i \neq 0$ for all i.

The problem we address in this paper can be stated as:

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> Given N snapshots $\{\tilde{\mathbf{y}}(n)\}_{n=1}^{N}$ collected under a static scenario where P targets are active but with unknown directions of arrival and the knowledge of the ratio between the antenna gain for two consecutive antenna elements, determine the unknown antenna gain vector $\mathbf{g}$ and the direction of arrivals θ_p, $p = 1, \ldots, P$.

In a static scenario the direction of arrivals are constant during the period when all the snapshots are collected.

A. Uniform linear array

The antenna array is assumed to have a linear shape and equal spacing Δ between the antenna elements. Assume target p is located at a direction θ_p relative to the direction perpendicular to the extent of the antenna array. Define the spatial frequency $\omega_p \triangleq \frac{2\pi\Delta}{\lambda}\sin\theta_p$, where λ is the wavelength of the incoming signals. The steering vector for target p can then be described as

$$\mathbf{a}_0(\theta_p) = \begin{bmatrix} 1 & e^{j\omega_p} & e^{j2\omega_p} & \ldots & e^{j(M-1)\omega_p} \end{bmatrix}^T.$$

With the definitions

$$\mathbf{A} \triangleq \mathrm{diag}([e^{j\omega_1}, e^{j\omega_2}, \ldots, e^{j\omega_P}]), \quad \mathbf{c} \triangleq \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix} \tag{2}$$

and

$$\mathbf{x}(n) \triangleq \begin{bmatrix} x_1(n) & x_2(n) & \ldots & x_P(n) \end{bmatrix}^T \tag{3}$$

we obtain for antenna element at position m

$$y_m(n) = g_m \mathbf{c}\mathbf{A}^{m-1}\mathbf{x}(n) \tag{4}$$

By collecting the antenna responses from all the snapshots at element m we have

$$\mathbf{y}_m \triangleq \begin{bmatrix} y_m(1) & y_m(2) & \cdots & y_m(N) \end{bmatrix} = g_m \mathbf{c}\mathbf{A}^{m-1}\mathbf{X} \tag{5}$$

where

$$\mathbf{X} \triangleq \begin{bmatrix} \mathbf{x}(1) & \mathbf{x}(2) & \cdots & \mathbf{x}(N) \end{bmatrix} \tag{6}$$

We note that if $g_m = 1$ for all m then $\mathbf{y}_m$ can be seen as the Markov parameters for a linear system described by the triple $(\mathbf{A}, \mathbf{X}, \mathbf{c})$. We will rely on this fact in the derivation that follows. Finally we note that antenna gain $\mathbf{g}$ and the signal amplitude matrix $\mathbf{X}$ in the model (4) cannot be uniquely separated as an arbitrary non-zero complex scalar value can be moved between them with identical $\mathbf{y}_m$. Here we fix $g_1 = 1$ to remove this ambiguity.

III. Preliminaries

A. Some results from systems theory

Before proceeding we recall some well known results from linear systems theory [10]. We will here discuss properties of the matrix triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, where $\mathbf{A} \in \mathbb{C}^{n\times n}, \mathbf{C} \in \mathbb{C}^{p\times n}$, and $\mathbf{B} \in \mathbb{C}^{n\times m}$ defining a sequence $\mathbf{y}_i \triangleq \mathbf{C}\mathbf{A}^{i-1}\mathbf{B} \in \mathbb{C}^{p\times m}$ for all $i = 1, 2, \ldots$. The triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is called a *realization of order* n of the sequence $\{\mathbf{y}_i\}_{i=1}^{\infty}$. The *extended controllability matrix* of order s is

$$\mathcal{C}_s(\mathbf{A}, \mathbf{B}) \triangleq \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{s-1}\mathbf{B} \end{bmatrix} \tag{7}$$

and the *extended observability matrix* of order s is

$$\mathcal{O}_s(\mathbf{A}, \mathbf{C}) \triangleq \begin{bmatrix} \mathbf{C}^T & (\mathbf{C}\mathbf{A})^T & \cdots & (\mathbf{C}A^{s-1})^T \end{bmatrix}^T \tag{8}$$

A realization of the sequence $\{\mathbf{y}_i\}_{i=1}^{\infty}$ is *minimal* if there exists no other realization of the sequence that has a lower order. The following result is instrumental and can for instance be found in linear systems theory literature e.g. [10], [11].

Lemma 1 *A realization* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ *of order* n *is minimal if and only if* $\operatorname{rank}\mathcal{C}_n(\mathbf{A}, \mathbf{B}) = \operatorname{rank}\mathcal{O}_n(\mathbf{A}, \mathbf{C}) = n$.

Trivially we note that $\operatorname{rank}\mathcal{O}_s(\mathbf{A}, \mathbf{C}) = n$ for all $s > n$ if $\operatorname{rank}\mathcal{O}_n(\mathbf{A}, \mathbf{C}) = n$. Also, by the Cayley-Hamilton theorem, if $\operatorname{rank}\mathcal{O}_s(\mathbf{A}, \mathbf{C}) = n$ for some $s > n$ then $\operatorname{rank}\mathcal{O}_n(\mathbf{A}, \mathbf{C}) = n$. Dual results hold for the controllability matrix.

If we arrange the first $s + r - 1$ samples in the sequence $\mathbf{y}_i$ as a *block Hankel matrix* we obtain

$$\mathbf{Y}_{s,r} = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_r \\ \mathbf{y}_2 & \mathbf{y}_3 & \cdots & \mathbf{y}_{r+2} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{y}_s & \mathbf{y}_{s+1} & \cdots & \mathbf{y}_{s+r-1} \end{bmatrix} \in \mathbb{C}^{sp\times rm}. \tag{9}$$

The Hankel structure implies that the obtained matrix has the same block elements on all anti-diagonals. The following result ties together the minimality of a realization and the rank properties of the associated Hankel matrix. We formulate the result for the row vector case, i.e. $p = 1$ and arbitrary m. The corresponding result for the case when $p > 1$ is more involved, see e.g. [11], [12]. For the interested reader we give a compact full proof of the result.

Theorem 1 *Consider a sequence of row vectors* $\{\mathbf{y}_i\}_{i=1}^{2n}$ *and the corresponding block Hankel matrix* $\mathbf{Y}_{n+1,n+1}$*. Then* $\operatorname{rank}\mathbf{Y}_{n+1,n+1} = n$ *and* $\operatorname{rank}\mathbf{Y}_{n,n+1} = n$ *if and only if there exists a minimal realization* $(\mathbf{A}, \mathbf{B}, \mathbf{c})$ *of order* n *such that* $\mathbf{y}_i = \mathbf{c}\mathbf{A}^{i-1}\mathbf{B}$ *for* $i = 1, \ldots, 2n$.

Proof: $\Leftarrow$) Given the minimal realization $(\mathbf{A}, \mathbf{B}, \mathbf{c})$ let $\mathcal{O}_s(\mathbf{A}, \mathbf{c})$ and $\mathcal{C}_r(\mathbf{A}, \mathbf{B})$ be the extended observability and controllability matrices. It is easy to verify that $\mathcal{O}_s(\mathbf{A}, \mathbf{c})\mathcal{C}_r(\mathbf{A}, \mathbf{B}) = \mathbf{Y}_{s,r}$. Since the realization is minimal from Lemma 1 both $\mathcal{O}_s(\mathbf{A}, \mathbf{c})$ and $\mathcal{C}_r(\mathbf{A}, \mathbf{b})$ has rank n for all $s, r \geq n$ which imply that $\mathbf{Y}_{s,r}$ has rank n for all $s, r \geq n$.
$\Rightarrow$) If $\operatorname{rank}\mathbf{Y}_{n+1,n+1} = n$, the Hankel matrix $\mathbf{Y}_{n+1,n+1}$ has a one-dimensional left nullspace so there exists a non zero row vector $\mathbf{x} = \begin{bmatrix} \boldsymbol{\alpha} & \rho \end{bmatrix}$ such that $\mathbf{x}\mathbf{Y}_{n+1,n+1} = 0$ where ρ is a scalar. Since by assumption $\operatorname{rank}\mathbf{Y}_{n,n+1} = n$ we have $\boldsymbol{\alpha}\mathbf{Y}_{n,n+1} = 0$ iff $\boldsymbol{\alpha} = 0$. Hence it follows that $\rho \neq 0$ and without loss of generality we can assume $\rho = -1$. Clearly $\begin{bmatrix} \boldsymbol{\alpha} & -1 \end{bmatrix}\mathbf{Y}_{n+1,n+1} = 0$ implies the recursion

$$\mathbf{y}_{n+k} = \sum_{i=1}^{n} \alpha_i \mathbf{y}_{k+i}, \quad k = 1, \ldots, n \tag{10}$$

If we define the order n realization $(\mathbf{A}_c, \mathbf{B}_c, \mathbf{c}_c)$ with

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ \alpha_1 & \alpha_2 & \cdots & \cdots & \alpha_n \end{bmatrix}, \tag{11}$$

$\mathbf{c}_c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$, and $\mathbf{B}_c^T = \begin{bmatrix} \mathbf{y}_1^T & \mathbf{y}_2^T & \cdots & \mathbf{y}_n^T \end{bmatrix}$ we note that $\mathbf{A}_c$ is a *companion matrix*. It is easy to verify that $\mathbf{y}_i = \mathbf{c}_c \mathbf{A}_c^{i-1} \mathbf{B}_c$ for $i = 1, \ldots, 2n$ by employing the recursion (10). By construction, $\mathcal{O}_n(\mathbf{A}_c, \mathbf{c}_c) = \mathbf{I}$ which imply $\operatorname{rank} \mathcal{O}_{n+1}(\mathbf{A}_c, \mathbf{c}_c) = n$. Since by assumption $\mathbf{Y}_{n,n+1} = n$ and by construction $\mathcal{O}_n(\mathbf{A}_c, \mathbf{c}_c)\mathcal{C}_{n+1}(\mathbf{A}_c, \mathbf{B}_c) = \mathbf{Y}_{n,n+1}$ it follows that $\operatorname{rank} \mathcal{C}_{n+1}(\mathbf{A}_c, \mathbf{B}_c) = n$ and hence $\operatorname{rank} \mathcal{C}_n(\mathbf{A}_c, \mathbf{B}_c) = n$ which implies that the realization $(\mathbf{A}_c, \mathbf{B}_c, \mathbf{c}_c)$ is minimal. ■

The realization of a sequence $\mathbf{y}_i$ is not unique. Take an arbitrary non-singular $\mathbf{T} \in \mathbb{C}^{n \times n}$ then $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and $(\mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \mathbf{T}^{-1}\mathbf{B}, \mathbf{C}\mathbf{T})$ will be two realization that realize the same sequence since $\mathbf{C}\mathbf{T}(\mathbf{T}^{-1}\mathbf{A}\mathbf{T})^{i-1}\mathbf{T}^{-1}\mathbf{B} = \mathbf{C}\mathbf{A}^{i-1}\mathbf{B}$. It is well known that the set of eigenvalues for the matrix $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ and $\mathbf{A}$ is identical since the matrices are *similar* [10].

Theorem 2 *Assume the array is ideal,* $\mathbf{g}^T = \begin{bmatrix} 1, \ldots, 1 \end{bmatrix}$*, the* P *targets have distinct spatial frequencies* ω_p *and that for at least one snapshot* n' *each target* p*,* $x_p(n')$ *is non-zero. Then the realization* $(\mathbf{A}, \mathbf{X}, \mathbf{c})$ *given by* (2) *and* (6) *is minimal and* $\operatorname{rank} \mathbf{Y}_{P+1,P+1} = \operatorname{rank} \mathbf{Y}_{P,P+1} = P$.

Proof: A closer look at $\mathcal{O}_P(\mathbf{A}, \mathbf{c})$ reveal that it is a Vandermonde matrix with P distinct generators $e^{j\omega_p}$ which implies it has full rank P [13]. Consider the matrix $\mathcal{C}_P(\mathbf{A}, \mathbf{X})$ and select only the columns corresponding to the non-zero snapshot $\mathbf{x}(n')$. Collect these columns in a new matrix $\mathbf{Z}$ preserving the order. Let $\mathbf{u}$ be the element wise inverse vector to $\mathbf{x}(n')$, i.e. $\mathbf{x}(n') \odot \mathbf{u} = \begin{bmatrix} 1, 1, \ldots, 1 \end{bmatrix}^T$. Then we notice that $\operatorname{diag}(\mathbf{u})\mathbf{Z}$ is also a Vandermonde matrix with the same generators as before and hence has full rank P. Since $\operatorname{diag}(\mathbf{u})$ has full rank by construction we can conclude that $\mathbf{Z}$ has full rank P and hence also $\mathcal{C}_P(\mathbf{A}, \mathbf{X})$. The result now follows by Lemma 1 and Theorem 2. ■

The result shows that if the array is calibrated the Hankel matrix will be of rank P. This is the starting point for the calibration algorithm we describe in the next section.

IV. Auto-calibration

We will now introduce the calibration method. The calibrated array output is defined by

$$\mathbf{y}_m^c \triangleq h_m \mathbf{y}_m, \quad m = 1, \ldots, M \tag{12}$$

where $h_m \in \mathbb{C}$ compensates for the deviations from the ideal unit antenna gain. (Note that optimal calibration implies $h_m g_m = 1$ for $m = 1, \ldots, M$.) Given $\mathbf{h} = \begin{bmatrix} h_1, \ldots, h_M \end{bmatrix}$ and the snapshot data, define the calibrated Hankel matrix

$$\mathbf{Y}_{s,r}^c(\mathbf{h}) = \begin{bmatrix} \mathbf{y}_1^c & \mathbf{y}_2^c & \cdots & \mathbf{y}_r^c \\ \mathbf{y}_2^c & \mathbf{y}_3^c & \cdots & \mathbf{y}_{r+2}^c \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{y}_s^c & \mathbf{y}_{s+1}^c & \cdots & \mathbf{y}_{s+r-1}^c \end{bmatrix} \in \mathbb{C}^{s \times rN}. \tag{13}$$

By assumption $g_1 = 1$ so we fix $h_1 = 1$. We now seek values of the other elements in the vector $\mathbf{h}$. By Theorem 2 it is clear that if $h_m g_m = 1$ for $m = 1, \ldots, M$, the array is correctly calibrated, the Hankel matrices satisfy $\operatorname{rank} \mathbf{Y}_{P+1,P+1}^c = \operatorname{rank} \mathbf{Y}_{P,P+1}^c = P$ and the array response is given by a realization $(\hat{\mathbf{A}}, \hat{\mathbf{X}}, \hat{\mathbf{c}})$. Furthermore, the eigenvalues of $\hat{\mathbf{A}}$ are the same as $\mathbf{A}$ in (2) as the matrices are similar. From the eigenvalues the DOAs can be recovered.

The proposed auto-calibration method proceeds by finding a calibration vector $\mathbf{h}$ that makes the rank of the calibrated Hankel matrices $\mathbf{Y}_{P+1,P+1}^c$ and $\mathbf{Y}_{P,P+1}^c$ equal to P, the number of signal sources. A relevant question is if the rank of the two Hankel matrices can still be P when $h_m g_m \neq 1$, $m = 2, \ldots, M$. We will investigate if we can obtain solutions where $\operatorname{rank} \mathbf{Y}_{P+1,P+1}^c = \operatorname{rank} \mathbf{Y}_{P,P+1}^c = P$ although $h_m g_m \neq 1$, $m = 2, \ldots, M$. The following theorem provides two special conditions when this is the case.

Theorem 3 *Consider the scenario in Theorem 2 but with an arbitrary non-zero antenna gain for each element. If the calibration vector has either of the following two forms:*

1) $\mathbf{h}^T = \begin{bmatrix} 1 & g_2^{-1}\beta & g_3^{-1}\beta^2 & \cdots & g_M^{-1}\beta^{M-1} \end{bmatrix}$ *where* $\beta \in \mathbb{C}$ *and* $\beta \neq 0$
2) $\mathbf{h}^T = \begin{bmatrix} h_1' & h_2' & \cdots & h_P' & 0 \cdots 0 \end{bmatrix}$ *where* $h_P' \mathbf{y}_P \neq 0$ *and* $h_m' \in \mathbb{C}$ *for* $m = 1, \ldots, P-1$ *are arbitrary.*

then the calibrated Hankel matrix satisfies $\operatorname{rank} \mathbf{Y}_{P+1,P+1}^c = \operatorname{rank} \mathbf{Y}_{P,P+1}^c = P$*. The system matrix* $\hat{\mathbf{A}}$ *for the corresponding realizations will have eigenvalues:*

1) $\lambda_i = \beta e^{j\omega_i}$ *for* $i = 1, \ldots, P$
2) $\lambda_i = 0$ *for* $i = 1, \ldots, P$.

Proof: Case 1): Assume $(\mathbf{A}, \mathbf{X}, \mathbf{c})$ realize the antenna response using the correct calibration, i.e. $g_m^{-1}\mathbf{y}_m = \mathbf{c}\mathbf{A}^{m-1}\mathbf{X}$, for $m = 1, \ldots, M$. Then the calibrated response is $h_m \mathbf{y}_m = \beta^{m-1}\mathbf{c}\mathbf{A}^{m-1}\mathbf{X} = \mathbf{c}(\beta\mathbf{A})^{m-1}\mathbf{X}$ which shows that the realization $(\beta\mathbf{A}, \mathbf{X}, \mathbf{c})$ will realize the sequence $h_m \mathbf{y}_m$. Finally, the eigenvalues of $\beta\mathbf{A}$ are $\beta\lambda_i$ where λ_i are the eigenvalues of $\mathbf{A}$ and the result follows.
Case 2) The calibration vector will yield a Hankel matrix $\mathbf{Y}_{P+1,P+1}^c$ which is zero for all matrix elements where the sum of the row and column indices is larger than $P+1$. If $h_P' \mathbf{y}_P \neq 0$, trivially $\operatorname{rank} \mathbf{Y}_{P+1,P+1}^c = \operatorname{rank} \mathbf{Y}_{P,P+1}^c = P$. If a realization is constructed, as in the proof of Theorem 1, the vector $\boldsymbol{\alpha}$ at the bottom of the companion matrix (11) will be a zero vector. This implies that $\mathbf{A}_c^P = 0$, i.e. the matrix is idempotent and all eigenvalues are zero. ■

The theorem gives sufficient conditions on the calibration vector h_m such that the set of equations

$$h_m \mathbf{y}_m = h_m \mathbf{c}\mathbf{A}^{m-1}\mathbf{B} = \tilde{\mathbf{c}}\tilde{\mathbf{A}}^{m-1}\tilde{\mathbf{B}}, \quad m = 1, \ldots, M \tag{14}$$

has a solution where $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{c}})$ is a minimal realization of dimension P. If the ratio between the true gain of two consecutive antenna elements are known, the unknown scalar β in Theorem 3 Case 1 can be resolved. Without loss of generality assume the gain ratio is given by $\gamma = g_2/g_1 = g_2$ since $g_1 = 1$. Hence we obtain that $\beta = h_2\gamma$. With this information the elements of the calibration vector is modified as $\hat{h}_m = h_m\beta^{1-m}$, $m = 2, \ldots, M$, removing the previous ambiguity.

A. The calibration algorithm

The desired calibration vector $\mathbf{h}$ can be found as the solution to the following optimization problem:

$$\begin{aligned} \min_{\mathbf{h},\mathbf{L}} \; & \|\mathbf{L} - \hat{\mathbf{Y}}^c_{P+1,P+1}(\mathbf{h})\|_F^2 \\ \text{s.t.} \quad & \operatorname{rank}\mathbf{L} = P, \quad h_1 = 1 \end{aligned} \tag{15}$$

This problem is NP-hard due to the rank constraint, but could be resolved using a a relaxation method, e.g. the nuclear norm, e.g. [14], could be used. A difficulty with the nuclear norm relaxation is that an extra hyperparameter needs to be determined in order to obtain the correct rank P. Here we employ a heuristic method for solving (15) that does not involve searching over additional parameters. We suggest to iterate between solving for $\mathbf{L}$ and $\mathbf{h}$. If $\mathbf{h}$ is kept fixed the solution to the problem in (15) is given by the truncated singular value decomposition [15]. If $\mathbf{L}$ is kept fixed the problem is an ordinary least-squares problem since $\mathbf{Y}^c_{P+1,P+1}(\mathbf{h})$ is linear in $\mathbf{h}$.

Algorithm 1

1) *Initialize* $\mathbf{h} = \begin{bmatrix}1, 1, \ldots, 1\end{bmatrix}$
2) *Determine the SVD*

$$\mathbf{Y}_{P+1,P+1}(\mathbf{h}) = \begin{bmatrix}\mathbf{U}_1\mathbf{u}_2\end{bmatrix}\begin{bmatrix}\mathbf{S} & 0 & 0\\ 0 & \sigma_{P+1} & 0\end{bmatrix}\begin{bmatrix}\mathbf{V}_1^*\\ \mathbf{v}_2^*\\ \mathbf{V}_3^*\end{bmatrix} \tag{16}$$

with $\mathbf{S} = \operatorname{diag}(\sigma_1, \ldots, \sigma_P) \in \mathbb{R}^{P\times P}$ *where* $\sigma_1 \geq \ldots \geq \sigma_P \geq \sigma_{P+1}$ *are the singular values and set* $\mathbf{L} = \mathbf{U}_1\mathbf{S}\mathbf{V}_1^*$.
3) *Project* $\mathbf{L}$ *to the space of block Hankel matrices.*
4) *Solve the least-squares problem*

$$\min_{\mathbf{h}} \|\mathbf{L} - \mathbf{Y}^c_{P+1,P+1}(\mathbf{h})\|_F^2 \tag{17}$$

and scale $\mathbf{h} := \mathbf{h}/h_1$.
5) *Repeat 2-4 until the* σ_{P+1}/σ_P *is below some set threshold.*
6) *Set* $\beta = h_2\gamma$ *and adjust the calibration vector* $\hat{h}_m = h_m\beta^{1-m}$, $m = 2, \ldots, M$

Step 4 in Algorithm 1 separates into M scalar least-squares problems. If a solution to the original problem (15) has been found then $\operatorname{rank}\mathbf{Y}^c_{P+1,P+1}(\mathbf{h}) = P$ and hence $\sigma_{P+1} = 0$. This property is the basis for the stopping criteria in Step 5). Note that after Step 6) we still have $\operatorname{rank}\mathbf{Y}^c_{P+1,P+1}(\hat{\mathbf{h}}) = P$.

In all numerical examples investigated, the procedure in Steps 1-5) in Algorithm 1 has resulted in solutions where $h_m \approx g_m^{-1}\beta^{m-1}$, i.e. Case 1) in Theorem 3, often with β close to 1. The fact that the numerical experimentation has not seen

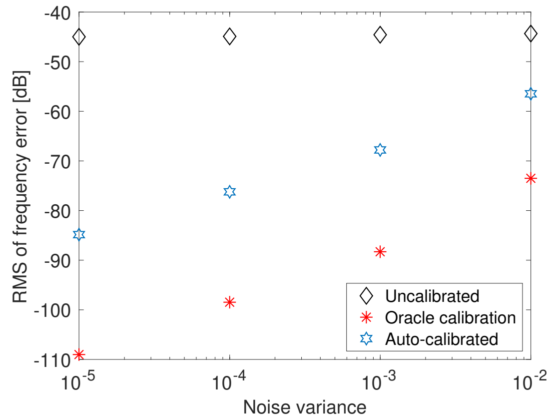

Fig. 1. RMS error for estimated spatial frequencies versus variance of the noise.

solutions according to Case 2) can perhaps be explained by the way we initialize the algorithm, i.e. $h_m = 1$ for all elements. Based on the outcome of the numerical experimentations we conjecture that if we solve the problem in (15) for noise free data we end up with a solution where

$$h_m = g_m^{-1}\beta^m \tag{18}$$

for some non-zero β, i.e. the solution to (17) identifies the unknown calibration gain up to the ambiguity β^m. With the information $\gamma = g_2/g_1$ used in Step 6) in Algorithm 1 this ambiguity is removed.

B. DOA estimation

We can use the calibrated array response derived according to Algorithm 1 to perform an estimate of the directions of arrival with an arbitrary DOA estimation algorithm, e.g. maximum-likelihood, Kung's method or ESPRIT [7], [16]–[18]. Without the information $\gamma = g_2/g_1$, the ambiguity in the calibration will lead to a perturbation in the estimated direction of arrival. If the true spatial frequencies are ω_p, $p = 1, \ldots, P$ then the eigenvalues for any $\mathbf{A}$ matrix realizing the array response for an correctly calibrated array will be $e^{j\omega_p}$, $p = 1, \ldots, P$. If the calibration algorithm (steps 1-5) yields a solution with $\beta \neq 1$ the corresponding eigenvalues of a realization will, according to Theorem 3, be $|\beta|e^{j(\omega_p + \arg\beta)}$. All spatial frequencies have thus been shifted by the same amount, given by $\arg\beta$. This implies that the difference between any pairs of estimated spatial frequencies is equal to the true difference. We can interpret this ambiguity as an uncertainty in the direction of the array relative to the directions to the targets.

V. Numerical Illustration

In this section we illustrate the performance of the outlined method. We use Monte Carlo simulations with the following setup:

- A uniform linear array with $M = 16$ elements.
- Two targets with relative spatial frequencies $\omega_1 = -2\pi \cdot 0.122$ and $\omega_2 = 2\pi \cdot 0.22$ (unknown to the algorithm).
- Array responses from $N = 100$ snapshots are generated.

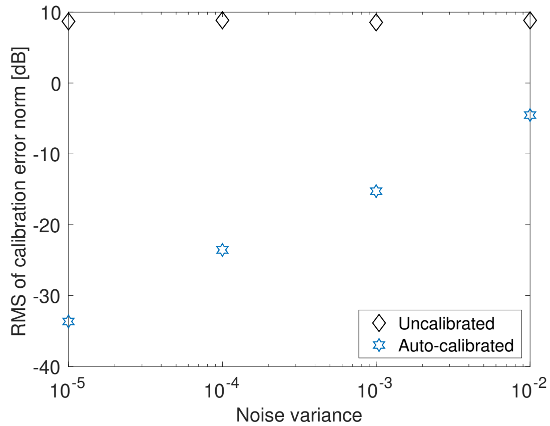

Fig. 2. RMS of calibration error norm versus variance of the noise.

- A zero mean complex circularly symmetric Gaussian noise with variance ranging from 10^{-4} to 10^{-1} is added to the noise free array response.
- For each Monte Carlo run an antenna gain vector is generated by adding a zero mean complex circularly symmetric noise with variance 0.2 and a uniform distribution to the ideal unit gain.
- The performance is evaluated by generating 100 independent realizations of the target signals, noise, and the antenna gains. The sample statistics of the performance is then evaluated.
- Algorithm 1 is used to estimate the unknown calibration vector $\hat{\mathbf{h}}$ using information $\gamma = g_2/g_1$. Kung's algorithm [7] is then used to estimate the two unknown spatial frequencies using the *auto-calibrated* data. We compare this estimate with an *oracle calibration* where we employ Kung's algorithm to array data compensated with the correct calibration vector. Finally we also compare with the estimate obtained from Kung's algorithm by directly using the *uncalibrated* raw data.

The result of the numerical evaluation is reported in figures 1–2. In Figure 1 the root mean square (RMS) errors for the spatial frequencies for the three cases are compared. We notice that the auto-calibrated case improves the performance compared to the uncalibrated case but is inferior to the oracle based estimate. This is natural since the auto-calibration algorithm needs to estimate additional $M-2$ complex values besides the 2 DOAs. In Figure 2 the RMS of the norm of the error between the optimal calibration, the auto-calibrated version, and the uncalibrated case is illustrated. In both figures we see that the error decreases with improved SNR which suggests that the method is consistent in SNR, i.e. the RMSE approaches zero as the SNR increases.

VI. Summary

In this contribution we have presented an algorithm which can be used to calibrate an ULA without full knowledge of the environment. Particularly, if the number of signal sources are known we have shown that we can determine the individual unknown antenna gains up to an ambiguity parametrized by a single complex scalar. If the ratio between the complex gains of two consecutive antenna elements is known, the ambiguity can be resolved and the calibration as well as the DOAs can be recovered. The numerical results sugest that the proposed method is consistent in SNR.

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