

Compressive Chirp Transform for Estimation of Chirp Parameters

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Abstract—This paper develops a new algorithm for estimating the parameters of multiple chirp signals in noise. The proposed method uses Compressive Sensing (CS) formulation of the Discrete Chirp Fourier Transform (DCFT) basis to achieve superior estimator performance. Unlike Fourier or time-frequency based approaches, DCFT incorporates the underlying chirp signal model parameters in formulating the transform [1]–[4]. In this work a CS formulation exploits the parametric DCFT basis for fast recovery to achieve highly accurate parameter estimation results in polynomial time using Orthogonal Matching Pursuit (OMP). The performance of the proposed algorithm has been compared with existing methods via simulations.

Index Terms—Chirp Parameter Estimation, Compressive Sensing, Discrete Chirp Fourier Transform (DCFT)

I. INTRODUCTION

Wide-band chirp, or Linear Frequency Modulated (LFM) signals have a broad range of applications, especially in radar [5], sonar [6], and medical imaging [7]. Parameter estimation of Chirp signals has been studied extensively in literature [6], [8]–[12]. In one of the earliest reported work, [8] used phase un-wrapping followed by linear regression to estimate the parameters of a single chirp signal. In [9] rank reduction and total least-squares fit were used for maximum likelihood estimates. [10] uses Signal Reconstructing Least-Square (SRLS) for the estimation of initial frequency and chirp rate by first unwrapping the signal phase using subsequent time samples and a modified least square to update the estimate. Time-frequency analysis have been used in [6], [12]. The work in [11] uses a de-chirp method based on the initial guess of estimated parameters, followed by a multistage joint least squares approach. [6] uses Short Time Fourier Transform (STFT) and Zoom-Fractional Fourier Transform (FRFT) in a multi-stage process.

The concept of exploiting chirp basis was introduced in [1]–[4], by developing DCFT and modified DCFT. Xia proposed DCFT [1], [13] as a tool similar to Discrete Fourier Transform (DFT) but specifically for chirp signals. They showed that if the number of observation samples (N) is a prime number, then the DCFT side-lobes magnitude is 1 and the mainlobe magnitude is \sqrt{N} [see (4)]. Later, a modified DCFT (mDCFT) was proposed in [14] to circumvent the requirement of prime number signal samples. Also, letters exchanged between the authors of [13] and [15] suggested a modification to DCFT to increase the chirp rate resolution that avoids the prime number of samples issue by increasing the sample rate of the quadratic term. Given its general applicability, in this work the

mDCFT basis space [13]–[15] will be used in a CS framework to enable chirp parameters estimation with a limited number of observation samples.

Chirp parameter estimation using CS has also appeared in the literature [16]–[20]. Applebaum et al [16] used a recursive least square of de-chirp and DFT to find minimum energy change and estimate the initial frequency and chirp rate. Guo [17] uses Gabor dictionary as sparse representation and then Hough Transform (HT) to estimate the parameters without computing the time-frequency distribution. Kang et al [18] used a transformation matrix similar to DCFT but with down chirp embedded in it. Their algorithm performed well at low SNR. Sward [19] introduced an iterative sparse reconstruction method to estimate the parameters of linear or harmonically related chirp signals. The algorithm uses re-weighted group-sparsity approach, followed by an iterative relaxation-based refining step. The work in [20] exploits sparsity in the Polynomial Fourier Transform (PFT) domain, where the discrete chirp transform is presented as a special linear case. It uses a sequential process of de-chirping of the signal and then applies CS with DFT basis and thresholding to select spectral peaks. The chirp parameter estimation and reconstruction process appears to rely on visual inspection by user to identify spectral concentration in Fourier/PFT domain plots, *i.e.*, the process is not automated.

The CS based DCFT has appeared in [2], [21]. [21] uses recursive DCFT to find the nonzero locations of chirp signal parameters followed by an updated least squares solutions. Their algorithm was applied to MRI images, which exhibited good sparsity properties using chirp bases. In [2], [4] Alkishriwo et al discuss the sparsity of a different version of modified DCFT approach named as DLCT and applied it primarily in the context of data compression of real chirp-like signals produced by bats and birds. Extension to CS was pointed out as future work. Their work did not discuss the complete CS process including the transformation matrix construction and the recovery process. Finally, in [21] the original DCFT and not MDCFT was used but no optimization was clearly discussed with regards to the range of the DCFT transformation or its resolution. Also, their work dealt with reconstruction of images that may contain multiple DCFT peaks and not formulated as typical parameter estimation problem.

II. PROBLEM FORMULATION AND BACKGROUND

A. Discrete Chirp Fourier Transform (DCFT)

A discrete linear chirp signal is defined as,

$$s(n) = ae^{j\frac{2\pi}{N}(\beta_0 n^2 + f_0 n)}, \quad n = 0, \dots, N-1 \quad (1)$$

where, β_0 and f_0 are the unknown chirp rate and start frequency, respectively, that are to be estimated.

The forward and inverse DCFT are defined as, [1], [2],

$$S(f, \beta) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} s(n) e^{-j\frac{2\pi}{N}(fn + \beta n^2)} \quad (2)$$

$$s(n) = \frac{1}{\sqrt{N}} \sum_{f=f_1}^{f_U} \sum_{\beta=\beta_1}^{\beta_V} S(f, \beta) e^{j\frac{2\pi}{N}(\beta n^2)} e^{j\frac{2\pi}{N}fn} \quad (3)$$

where, $f \in [f_1, \dots, f_U]$ and $\beta \in [\beta_1, \dots, \beta_V]$. Unlike DFT, DCFT has two variables β and f . For a single chirp, the DCFT magnitude should have peaks at the true β_0 and f_0 , but side-lobes will also be present. It has been shown [1] that when N is a prime number, the main-lobe and side-lobe values of $S(f, \beta)$ are given by,

$$|S(f, \beta)| = \begin{cases} |a|\sqrt{N} & \text{when } f = f_0 \text{ and } \beta = \beta_0 \\ |a| & \text{when } \beta \neq \beta_0 \\ 0 & \text{when } f \neq f_0 \text{ and } \beta = \beta_0 \end{cases} \quad (4)$$

Equation (4) shows that DCFT coefficients will be sparse as peaks are formed at true $S(f_0, \beta_0)$ of amplitude $|a|\sqrt{N}$ and maximum side-lobes of values $|a|$ when N is prime [1]. For non-prime N , sidelobes can be more significant.

Next, consider multiple chirp signals modeled as,

$$s_L(n) = \sum_{i=1}^{L} a_i e^{j\frac{2\pi}{N}(\beta^i n^2 + f^i n)} \quad (5)$$

where, L is the number of chirp signals and a_i is the amplitude of the i -th chirp signal. Since DCFT is linear [1], DCFT of multiple chirps is the sum of individual DCFTs,

$$S_t(f, \beta) = \sum_{i=1}^L S_i(f, \beta). \quad (6)$$

where $S_i(f, \beta)$, defined in (2), is the DCFT coefficients of i -th chirp signal. At i -th chirp signal location (f_i, β_i) , the main-lobe components of $S_j(f_j, \beta_j)$ are expressed as,

$$S_t(f_j, \beta_j) = S_j(f_j, \beta_j) + \sum_{i=1, i \neq j}^L S_i(f_j, \beta_j). \quad (7)$$

Using results from equation (4) it can be shown that,

$$|S_t(f_j, \beta_j)| \leq |a_j|\sqrt{N} + \sum_{i=1, i \neq j}^L |a_i| \quad (8)$$

The bounds of the maximum side-lobes for $f \neq f_i$ and $\beta \neq \beta_i$ can be shown to be,

$$|S_t(f, \beta)| \leq \sum_{i=1}^L |a_i|. \quad (9)$$

It can be concluded that DCFT for multi-chirp signals is less sparse, as main-lobe is affected by $\sum_{i=1, i \neq j}^L S_i(f_j, \beta_j)$ representing side-lobes due to other chirp signals at (f_j, β_j) . At the same time, side-lobes tend to be higher because it's the sum of contributions from all side-lobe components.

B. Modified Discrete Chirp Fourier Transform (MDCFT):

It is important to note that the original DCFT in (2) is not restricted to prime N but it performs optimally when N is prime [13]. Exchange of letters between the authors of [15] and [13] suggested useful modifications to (2) where it was shown that increasing the chirp rate resolution by increasing the sampling rate to n/N [15], DCFT can be generally effective for non-prime N . With this modification, the discretization of the chirp signal in equation (1) becomes,

$$s(n) = ae^{j\frac{2\pi}{N}(\frac{\beta_0}{N}n^2 + f_0 n)}. \quad (10)$$

The Modified DCFT is given by [15],

$$S(f, \beta) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} s(n) e^{-j2\pi(f(\frac{n}{N}) + \beta(\frac{n}{N})^2)} \quad (11)$$

$$\triangleq \psi^H(f, \beta)\mathbf{s}, \quad \text{where,} \quad (12)$$

$$\psi(f, \beta) \triangleq [e^{j\frac{2\pi}{N}[f(0) + \frac{\beta}{N}(0)^2]} \dots e^{j\frac{2\pi}{N}[(N-1)f + \frac{\beta}{N}(N-1)^2]}]^T \quad (13)$$

represents MDCFT basis vector at (f, β) and $\mathbf{s} \triangleq [s(0) \ s(1) \ \dots \ s(N-1)]^T$. Several other modifications of DCFT have also been presented in [13], [14]. In the rest of this paper, the general MDCFT version defined in equation (12) will be used and will be referred to interchangeably as DCFT.

For both DCFT and MDCFT, increase in side-lobe levels limits the usefulness of DCFT for multi-chirp signals, especially at low SNR [15]. In this work, this limitation will be overcome by the use of Compressive Sensing (discussed next), as CS will perform as time-variant filter to reduce the side-lobes and increase the detection resolution of the chirp rate parameters of DCFT transformation. CS can also reduce the number of measurements required for implementing DCFT transformation.

C. Compressed Sensing (CS)

CS is a signal acquisition and recovery framework that supports signal estimation from compressive measurements, i.e., fewer measurements than unknowns. The underlying assumption making CS possible is that the signal is sparse in a known basis. Performance largely depends on the signal basis structure because higher sparsity signal results in more efficient and reduced cost of measurement, collection and processing [22], [23]. The basic CS process can be expressed in matrix notation as [22], [23],

$$\mathbf{y} = \Phi \mathbf{x} = \Phi \Psi \boldsymbol{\alpha}, \quad (14)$$

where \mathbf{y} is an $(M_s \times 1)$ measurement vector, Φ is a random projection measurement matrix of size $M_s \times N$ where, $M_s \ll N$, and \mathbf{x} is an $N \times 1$ signal to be recovered. The signal \mathbf{x} is modeled as a linear combination of columns from a potentially

overcomplete dictionary (FFT, wavelet, DCFT, etc.) matrix Ψ , $\mathbf{x} = \Psi\alpha$. The coefficient vector α is assumed to be sparse (i.e., the number, K , of non-zero elements is small), such that only a few columns of Ψ contribute to \mathbf{x} . Letting $\mathbf{A} = \Phi\Psi$, and $\|\cdot\|_1$ denote the ℓ_1 norm, the convex optimization

$$\hat{\alpha} = \operatorname{argmin} \|\alpha\|_1 \quad \text{s.t.} \quad \mathbf{A}\alpha = \mathbf{y}, \quad (15)$$

may be used to recover α . When \mathbf{A} satisfies the restricted isometry property (RIP), the reconstruction is guaranteed to succeed with high probability [24]. From estimates of the sparse coefficients $\hat{\alpha}$, the signal estimate is computed $\hat{\mathbf{x}} = \Psi\hat{\alpha}$.

In the current work, we choose $\Phi = \mathbf{I}$, and let the dictionary, Ψ , be formed from the mDCFT vectors (13), as explained in detail next.

D. CS-DCFT Formulation

For CS-DCFT implementation, noisy observations are denoted by, $\mathbf{y} = [y(0) \dots y(N-1)]^T$, with $y(n) = s(n) + w(n)$ where, $w(n)$ is AWGN and $s(n)$ is a number of chirp. In this case, the desired sparsity is in the DCFT domain $\alpha \equiv \mathbf{X}(f, \beta) \in \mathcal{C}^{UV \times 1}$, which is formed by applying the transformation in (12) on noisy signal \mathbf{y} repeatedly with probing pairs of (f_i, β_j) for $i = 1, \dots, U$ and $j = 1, \dots, V$ and stacking the transformed vectors,

$$\mathbf{X}(f, \beta) = \Psi^H \mathbf{y} \quad \text{and} \quad (16)$$

$$\Psi = [\psi(f_1, \beta_1) \dots \psi(f_1, \beta_V) \mid \dots \quad (17)$$

$$\mid \psi(f_U, \beta_1) \dots \psi(f_U, \beta_V)] \in \mathcal{C}^{N \times UV} \quad (18)$$

Using these notations, the optimization in (15) has the form,

$$\hat{\mathbf{X}}(f, \beta) = \operatorname{argmin} \|\mathbf{X}(f, \beta)\|_1 \quad \text{s.t.} \quad \Phi\Psi\mathbf{X}(f, \beta) = \mathbf{y}. \quad (19)$$

In this paper, Orthogonal Matching Pursuit (OMP) [25] will be used as the convex recovery algorithm. Several other recovery algorithms can also be used, but OMP demonstrated respectable performance with lower computation time than other convex recovery algorithms. OMP can reliably recover a signal with K nonzero entries in dimension N_s given $O(k \ln N_s)$ random linear measurements of that signal [25]. This is a significant improvement over other algorithms, which require $O(k^2)$ measurements. One disadvantage of OMP is that it is less stable than traditional Basis Pursuit (BP) algorithms like ℓ_1 [22] that were also attempted with good results in our work with minor changes to accommodate complex numbers. ℓ_1 minimization is a little computation intensive compared to other techniques but it is more stable [26]. ℓ_1 minimization can recover perfectly using M_s random measurements if K satisfies $K < \frac{cM_s}{\log(N/M_s)}$, where, K can be considered as the number of sparse peaks and c is a known constant.

III. SIMULATION RESULTS

Sum of three wide-band chirp signals were simulated with the starting frequencies, $f_{01} = 6.0$, $f_{02} = 10.0$, $f_{03} = 15.0 \text{ MHz}$ and chirp rates $\beta_{01} = 4 \text{ MHz}$, $\beta_{02} =$

10 MHz , $\beta_{03} = 3 \text{ MHz}/\mu\text{sec}$, respectively. 200 samples at a sampling rate $F_s = 50 \text{ MHz}$ were generated with $\text{SNR} = -7 \text{ dB}$. The plot on the left of Figure 1 shows 3-D view of the magnitude of the original DCFT spectrum $|\mathbf{X}(f, \beta)|$ at various f and β values by processing all of the 200 samples. For this highly noisy case, it can be observed that the performance of regular DCFT was quite poor when compared with the CS-DCFT implementation proposed in this work, as shown at the right. No compression was used for these experiments.

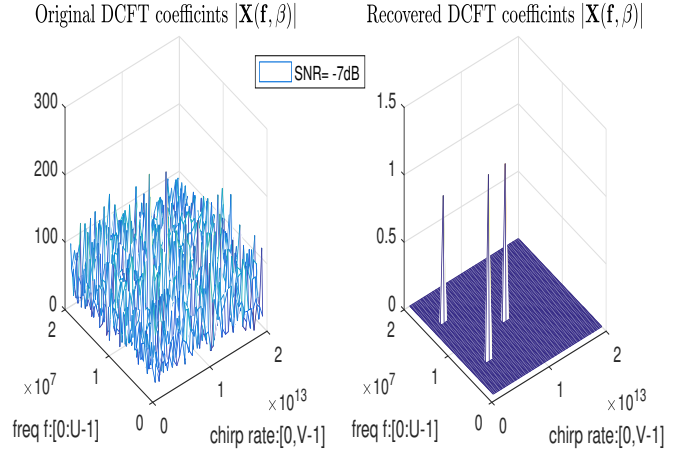


Fig. 1. Performance with $\text{SNR} = -7 \text{ dB}$. Left: Original DCFT Spectrum; Right: Recovered $\hat{\mathbf{X}}(f, \beta)$ using CS-DCFT.

Sequential Processing: The estimation process for multiple chirps as described above can be considered as *batch processing*, where all the chirp parameters are estimated in one step. However, if fine search grid is used then the processing of 2-D search can be intensive. Instead, the estimation process can be made sequential, where coarse search can be used in the first step to determine approximate peaks and then fine search grid can be used around the strongest peak for lower computational expense. The estimated chirp signal is subtracted from the original signal, and the process can be repeated until no further significant peaks are found. This approach is similar to the Cyclic Algorithm proposed for frequency estimation [27], except in the present case using CS, there is no need for repeat pass as the first pass using CS itself produces excellent estimates.

Analysis of Compression Ratio with increasing Number of Chirps: As in case of regular DCFT [see equation (9)] increasing the number of chirp signals intensifies the side-lobe affecting the performance of the CS algorithm if high compression ratio is used. Figure 2 shows RMSEs from recovery of one, three and four chirp signals. Therefore, multiple chirp estimation requires more measurements than single chirp to produce similar RMSE, which is reasonable as stated before because signal sparsity is reduced from $N - 1$ to $N - L$, where L is the number of chirp signals.

Performance Comparison with an existing CS-based Method: Next, we compare the CS-DCFT method against the Applebaum algorithm [12] for chirp parameter estimation. A

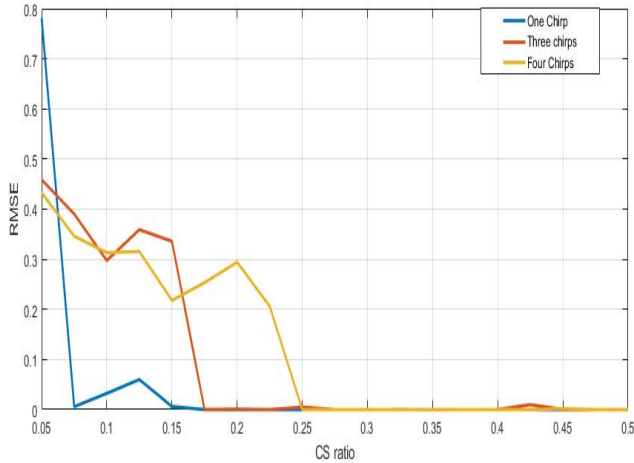


Fig. 2. RMSE with different CS ratio for a single and multiple chirp signal.

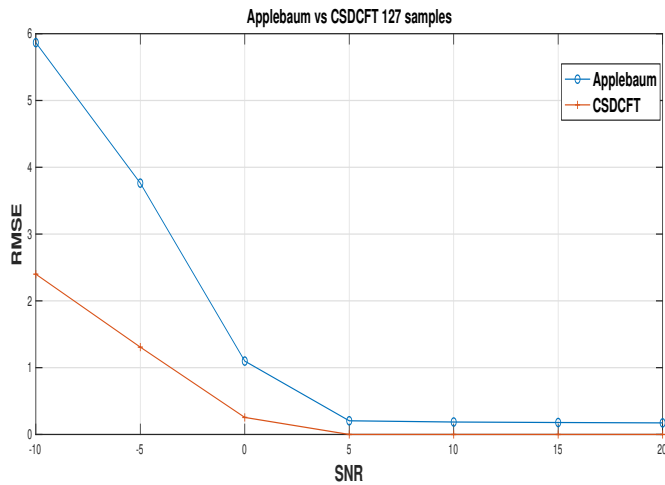


Fig. 3. CS-DCFT parameter estimation compared to the Applebaum algorithm.

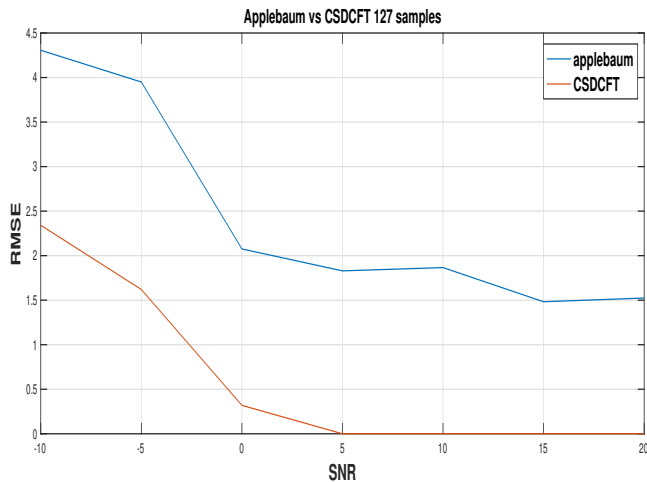


Fig. 4. CS-DCFT parameter estimation compared to the Applebaum algorithm when two signals out of three share the same chirp rate.

signal with three chirp sources was generated with $N = 127$, and the RMSE vs. SNR is shown in Figure 3. Each point in the figure is the result of averaging 20 Monte Carlo runs for the given SNR. The CS-DCFT method outperforms the Applebaum algorithm overall at all SNRs considered, however the performance difference is greatest for low SNR. The Applebaum algorithm is also limited in that the signal length N must be prime and performance degrades when two or more signals have the same chirp rates, as demonstrated in Figure 4.

IV. CONCLUDING REMARKS

This work introduces parameter estimation of multiple wideband chirp signals by incorporating a modified DCFT basis in Compressive Sensing formulation (CS-DCFT). The chirp basis requires two-dimensional search over chirp rate and chirp frequency. Simulation results show superior performance as compared to original DCFT and an existing CS-based method.

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