Signal Recovery from Phaseless Measurements of Spherical Harmonics Expansion

Arya Bangun, Arash Behboodi, and Rudolf Mathar
Institute for Theoretical Information Technology
RWTH Aachen University
{bangun,behboodi,mathar}@ti.rwth-aachen.de

***Abstract*—In this work, we study the problem of recovering spherical harmonics coefficients from phaseless measurements and evaluate the empirical performance of several well-known algorithms. Apart from trivial ambiguities that arise naturally from the properties of spherical harmonics, we will show that when a specific class of equiangular sampling patterns is chosen to construct the measurement matrix, another ambiguity appears as well. Nevertheless, we will numerically show that recovery can be achieved by carefully choosing the appropriate sampling patterns. Furthermore, an application of this work in phaseless spherical near-field antenna measurements will be addressed.**

***Index Terms*—Phase retrieval, spherical harmonics, spherical near-field measurements**

I. Introduction

The classical problem of recovering a signal from magnitude measurements, often called phase retrieval, is vastly studied in different fields of research, from optical-imaging, crystallography to wireless communications [1]–[3]. The original problem deals with the estimation of Fourier coefficients from phaseless Fourier transform. Various works try to tackle this problem, either by finding the condition for uniqueness or by designing an efficient algorithm to reconstruct the signal. However, it has been observed in [4]–[6] that only multidimensional Fourier transform has a unique signal recovery up to trivial ambiguities. In 1-D case, even if the trivial ambiguities, i.e., rotation, shifting, and reflected conjugate, are excluded, different signals might have the same magnitude of Fourier transform. Such ambiguities are classified as non-trivial ambiguities [7]–[9]. Existing works mainly show that uniqueness up to trivial ambiguities can be guaranteed by adding a constraint on the signal, such as minimum phase [9], masking measurements [10], [11], and sparsity [12], [13].

Different from Fourier phase retrieval, related works on phaseless measurements by random matrices or frames have also been considered. For instance, in [14] the authors show that for a signal with size N, then $m \geq 2N - 1$ and $m \geq 4N - 2$ generic measurements are needed in order to attain recovery guarantee up to global phase for any real and complex signal, respectively. Along the same line, the authors in [15] prove that $\mathcal{O}(N)$ and $\mathcal{O}(N \log N)$ subgaussian measurements are enough to have recovery guarantee in the noiseless and noisy condition for real signals. For Gaussian measurements, it has been shown in [16] that $\mathcal{O}(N \log N)$ phaseless measurements are sufficient for perfect recovery using a semidefinite program called PhaseLift. Recently, extensive works have been done for both theoretical and algorithmic perspective, as well as for different applications; see [6], [17], [18], and the references therein.

In this paper, we study phaseless measurements of signals on the 2-dimensional unit-sphere represented as spherical harmonics expansion. Spherical harmonics form an orthogonal basis for square-integrable functions on the unit sphere [19]. It is widely used in different applications, for example, spherical near-field antenna measurements [20], [21], earth magnetic fields [22], neural networks [23], 3-D model descriptors [24]–[26] and spherical microphone array [27].

Two issues arise in phase retrieval problems. First, to classify ambiguities that are incurred by phaseless measurements, and second, to design an efficient recovery algorithm up to trivial ambiguities. The authors in [28] considered ambiguities for the solution of Helmholtz equations, which includes d-dimensional spherical harmonics. They showed under which conditions the ambiguities are restricted to general phase ambiguity and complex conjugate operation. These cases include particularly real-valued spherical harmonics and 2-dimensional spherical harmonics. However, a complete characterization of ambiguities still remains open in general.

Phase retrieval algorithms for spherical harmonics are studied from an empirical perspective, mainly in antenna measurement applications. In this setup, far-field acquisition from near-field measurements is formulated as a linear inverse problem. In [29], the authors develop an iterative signal recovery algorithm from phaseless near-field measurements. Similar to some other works, in [30], [31], the techniques are mainly tailored to its application in near-field to far-field transformation. Nonetheless, they do not specify the sampling pattern used to take phaseless measurements, nor they discuss the required number of measurements. In [32], [33], the authors acquire magnitude measurements from random combinations of probe signals and use PhaseCut and Wirtinger flow algorithm to reconstruct far-field patterns. It is shown numerically that the number of required measurements is indicated to be around $5N$ for more than 90% success rate.

In this work, we consider the phase retrieval problem for band-limited square-integrable functions on 2-sphere. We consider different deterministic sampling patterns which are widely used in applications. After classifying trivial ambi-

This work has been funded by DFG project -CoSSTra-MA1184|31-1

978-9-0827-9703-9/19/$31.00 ©2019 IEEE

guities, it is shown that additional ambiguities arise if an inappropriate sampling pattern, i.e., equiangular-type sampling patterns, is chosen to construct the measurement matrix. Afterwards, we empirically show that $m \geq 2N-1$ phaseless measurements from several well-known sampling patterns are sufficient to recover the signal by using PhaseLift [16].

This paper is organized as follows. In section II, the spherical harmonics are introduced. In section III, we will discuss the ambiguity of phaseless measurements of spherical harmonics expansion. Numerical experiments in terms of phase transition diagrams and reconstruction of far-field patterns of the antenna are depicted in section IV. Finally, the conclusion and future works are discussed in section V.

A. Notation

The elevation and azimuth angles are denoted by θ and ϕ. Vectors and matrices are presented by small bold and capital bold letters, respectively. The set $\{1, ..., m\}$ is denoted by $[m]$.

II. Definitions and Backgrounds

Definition 1 (Complex spherical harmonics)**.** Spherical harmonics of degree $0 \leq l \leq \infty$ and order $-l \leq k \leq l$ are defined as follows:

$$\mathrm{Y}_l^k(\theta,\phi) = N_l^k P_l^k(\cos\theta) e^{jk\phi} \tag{1}$$

where $N_l^k = \sqrt{\frac{2l+1}{4\pi}\frac{(l-k)!}{(l+k)!}}$ is the normalization factor and $P_l^k(\cos\theta)$ is the associated Legendre polynomials of degree l and order k. For $k=0$, the associated Legendre polynomials become Legendre polynomials $P_l(\cos\theta)$.

Spherical harmonics can also be expressed in the real case as given below.

Definition 2 (Real spherical harmonics)**.** For given degree $0 \leq l \leq \infty$ and order $-l \leq k \leq l$, real spherical harmonics are given as:

$$\mathrm{Y}_l^k(\theta,\phi) = \begin{cases} (-1)^k \sqrt{2} N_l^k P_l^k(\cos\theta)\sin(|k|\phi) & \text{if } k<0 \\ N_l^0 P_l(\cos\theta) & \text{if } k=0 \\ (-1)^k \sqrt{2} N_l^k P_l^k(\cos\theta)\cos(k\phi) & \text{if } k>0 \end{cases} \tag{2}$$

Spherical harmonics are a basis for square-integrable functions over S^2. In other words, each function $f \in L^2(\mathrm{S}^2)$ can be written in terms of spherical harmonics as:

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{k=-l}^{l} \hat{f}_l^k \, \mathrm{Y}_l^k(\theta,\phi). \tag{3}$$

This is also called the S^2-Fourier expansion with Fourier coefficient $\hat{f}_l^k$ where

$$\hat{f}_l^k = \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)\,\overline{\mathrm{Y}_l^k(\theta,\phi)}\sin\theta \mathrm{d}\theta\mathrm{d}\phi. \tag{4}$$

Spherical harmonics are orthonormal with respect to the uniform measure on the sphere $\mathrm{d}\nu = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$, namely:

$$\int_0^{2\pi}\int_0^{\pi} \mathrm{Y}_l^k(\theta,\phi)\overline{\mathrm{Y}_{l'}^{k'}(\theta,\phi)}\sin\theta\mathrm{d}\theta\mathrm{d}\phi = \delta_{ll'}\delta_{kk'} \tag{5}$$

where $\delta_{ll'}$ is Kronecker delta. In this work, instead of infinite expansion, we suppose the functions are band-limited.

Definition 3 (Band-limited functions)**.** A function $f \in L^2(\mathrm{S}^2)$ is band-limited with bandwidth B if it is expressed in terms of spherical harmonics of degree less than B. The degree and order of spherical harmonics can be written as $0 \leq l \leq B-1$ and $-l \leq k \leq l$.

The sensing matrix $\mathbf{A}$ is constructed using following entries

$$\mathrm{A}_{p,q} = \mathrm{Y}_{l^{(q)}}^{k^{(q)}}(\theta_p,\phi_p), \tag{6}$$

where $p \in [m]$ is the index of a sampling point and $q \in [N]$ is an index used to provide the degree and order of each basis, $l^{(q)}, k^{(q)}$, belonging to the set $\mathcal{J}_\mathcal{S}$:

$$\mathcal{J}_\mathcal{S} = \{(l,k) \,|\, 0 \leq l \leq B-1, -l \leq k \leq l\} \tag{7}$$

where $|\mathcal{J}_\mathcal{S}| = N = B^2$. It can be seen that the set $\mathcal{J}_\mathcal{S}$ contains combinations between permissible degrees and orders, each pair corresponding to a column.

Definition 4 (Mapping and measurement)**.** The absolute and squared measurement are defined as following:

$$\mathcal{A}_1(\mathbf{x})_p := |\langle \mathbf{a}_p, \mathbf{x}\rangle| \text{ and } \mathcal{A}_2(\mathbf{x})_p := |\langle \mathbf{a}_p, \mathbf{x}\rangle|^2 \quad \text{for } p \in [m] \tag{8}$$

where the collection of measurements vector from spherical harmonics expansion are given by $\mathbf{A} = \{\mathbf{a_p}\}_{p\in[m]} \in \mathbb{C}^N$.

III. Ambiguities in phaseless measurements

A. Ambiguity in phaseless complex spherical harmonics

Identical to the Fourier case, ambiguities could also be observed in phaseless spherical harmonics measurement due to certain properties in spherical harmonics. Consider measurement matrix $\mathbf{A} = \{\mathbf{a_p}\}_{p\in[m]} \in \mathbb{C}^N$ from band-limited spherical harmonics expansion as given in (6) and its vector coefficients $\mathbf{x} \in \mathbb{C}^N$, with degree and order as in (7). Then the following ambiguities occur:

1) The rotated signal $\mathbf{y} = \mathbf{x}e^{j\alpha} \in \mathbb{C}^N$ for $\alpha \in [0, 2\pi)$ has the same intensity measurement.
2) A reflected conjugate signal $\mathbf{y} = \overline{\mathbf{x}}$ has the same intensity measurement.

Property 1 is a trivial implication of phaseless measurements. For the second part, note that the complex conjugate of spherical harmonics is given by:

$$\overline{\mathrm{Y}_l^k(\theta,\phi)} = \mathrm{Y}_l^{-k}(\theta,\phi) \tag{9}$$

Therefore a function f with coefficients $\hat{f}_l^k$ has the same phaseless measurement as the function g defined by coefficients $\hat{g}_l^k = (-1)^k f_l^{-k}$ for all elements in the set $\mathcal{J}_\mathcal{S}$. These coefficients $\hat{g}_l^k$ and $\hat{f}_l^k$ are different in general for complex signals. Conjugate symmetry in complex spherical harmonics produces an ambiguity between positive and negative order k as well as their sign. Since we are dealing with magnitude measurement, this ambiguity cannot be avoided and could indeed yield a wrong estimation.

B. Ambiguity in phaseless real spherical harmonics

The above discussion on complex spherical harmonics implies directly that if the coefficients $\hat{f}_l^k$ are real, the ambiguities are restricted to the mere phase ambiguity. Therefore, if we construct a matrix from real spherical harmonics, the trivial ambiguities consist of phase ambiguities.

An inappropriate sampling pattern, however, can aggravate the set of ambiguities significantly even in the real case. Consider, as an example, the equiangular sampling pattern, which is widely used in many applications because of its simplicity. Although the following proposition generally holds for complete measurements, the result also holds for phaseless measurements.

Proposition 5 (Ambiguity-incurring sampling patterns)**.** *Consider real spherical harmonics expansions of bandwidth B. Let the sampling points (θ_p, ϕ_p) be chosen as $(\theta_p, (B-2)\theta_p)$ for $p \in [m]$. Suppose that the coefficients of a function f is defined as follows:*

$$\hat{f}_l^k = \begin{cases} c_l & k = 0,\ l+B \text{ is an odd number} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Then there is function g with a single non-zero coefficient $\hat{g}_1^1 = d_1^1$ such that for all $p \in [m]$:

$$\sum_{l=0}^{B-1} N_l^0 P_l(\cos\theta_p) c_l = N_1^1 P_1^1(\cos\theta_p) \sin\big((B-2)\theta_p\big) d_1^1$$

In other words, the functions f and g cannot be distinguished using neither complete nor phaseless measurements.

Proof. Let assume we have an odd B, even l and $x = \cos\theta$. The Legendre polynomials can be written as the following [34] [p.24, Eq.1-62] : $P_l(x) = \sum_{r=0}^{\lfloor \frac{l}{2} \rfloor} \eta_r^l x^{l-2r}$ where $\eta_r^l = \frac{(-1)^m (2l-2r)!}{2^l r!(l-r)!(l-2r)!}$. Hence, we will obtain:

$$\sum_{\substack{l=0\\ l,even}}^{B-1} N_l^0 c_l \sum_{r=0}^{\lfloor \frac{l}{2} \rfloor} \eta_r^l x^{l-2r} = \Bigg(\sum_{\substack{l=0\\ l,even}}^{B-1} N_l^0 c_l \eta_{\frac{l}{2}}^l \Bigg) + \ldots + \Bigg(\sum_{\substack{l=B-3\\ l,even}}^{B-1} N_l^0 c_l \eta_{\frac{l}{2}-\frac{B-3}{2}}^l \Bigg) x^{B-3} + N_{B-1}^0 \eta_0^{B-1} c_{B-1} x^{B-1} \tag{11}$$

In addition, since $P_1^1(\cos\theta) = -\sin\theta$, we can write as the following

$$\begin{aligned} P_1^1(\cos\theta)\sin\big((B-2)\theta\big) &= \frac{\cos\big((B-1)\theta\big) - \cos\big((B-3)\theta\big)}{2} \\ &= \frac{1}{2}\Big(T_{B-1}(x) - T_{B-3}(x)\Big) \end{aligned} \tag{12}$$

where $T_l(x)$ is the Chebyshev polynomials of the first kind and can be represented as the following [35][Chap.5, Eq 5.34] : $T_l(x) = \sum_{q=0}^{\lfloor \frac{l}{2} \rfloor} \epsilon_q^l x^{l-2q}$ where $\epsilon_q^l = (-1)^q 2^{l-2q-1} \frac{l}{l-q} \binom{l-q}{q}$ for $l > 0$. From this expression, we can expand (12) as the following

$$N_1^1 \frac{d_1^1}{2} \Bigg(\epsilon_0^{B-1} x^{B-1} + \sum_{q=0}^{\frac{B-3}{2}} \Big(\epsilon_{q+1}^{B-1} - \epsilon_q^{B-3} \Big) x^{B-3-2q} \Bigg) \tag{13}$$

A linear system of equations can be constructed from (11) and (13). By imposing the structure of a lower triangular matrix and a forward substitution, the ratio of the coefficients can be derived.

$$\begin{bmatrix} N_{B-1}^0 \eta_0^{B-1} & 0 & \cdots & 0 \\ N_{B-1}^0 \eta_1^{B-1} & N_{B-3}^0 \eta_0^{B-3} & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ N_{B-1}^0 \eta_{\frac{B-1}{2}}^{B-1} & N_{B-3}^0 \eta_{\frac{B-3}{2}}^{B-3} & \cdots & N_0^0 \eta_0^0 \end{bmatrix} \begin{bmatrix} \frac{c_{B-1}}{d_1^1} \\ \frac{c_{B-3}}{d_1^1} \\ \vdots \\ \vdots \\ \frac{c_0}{d_1^1} \end{bmatrix} = \begin{bmatrix} \frac{N_1^1 \epsilon_0^{B-1}}{2} \\ \frac{N_1^1 (\epsilon_1^{B-1} - \epsilon_0^{B-3})}{2} \\ \vdots \\ \vdots \\ \frac{N_1^1 \left(\epsilon_{\frac{B-1}{2}}^{B-1} - \epsilon_{\frac{B-3}{2}}^{B-3} \right)}{2} \end{bmatrix}$$

Hence, we will get the ratio starting from $\frac{c_{B-1}}{d_1^1}$, $\frac{c_{B-3}}{d_1^1}$ until $\frac{c_0}{d_1^1}$.

The same argument can be used to prove for even B and odd l. $\square$

Remark 6. For $B = 4$ we have well-known equiangular sampling pattern, where $\theta_p = \frac{(p-1)\pi}{m-1}$ and $\phi_p = \frac{2(p-1)\pi}{m-1}$ for $p \in [m]$. Therefore, for arbitrary $B \geq 3$, there are coefficients c_l and d_1^1 with respect to the ratio as discussed in the previous proposition.

Proposition 5 shows that if there is a linear dependency between angles on elevation θ and azimuth ϕ, then there are linearly dependent columns from real spherical harmonics expansion. The condition in proposition 5 can be tailored to phaseless measurements. One can directly show the ratio of the coefficients, i.e., $\pm\frac{c_{B-1}}{d_1^1}, \pm\frac{c_{B-3}}{d_1^1}, \ldots, \pm\frac{c_0}{d_1^1}$, will be obtained as in the proof of proposition 5. As a result, ambiguities appear in the reconstruction of spherical harmonics coefficients. However, it is quite challenging to certify a general condition of sampling pattern that preserves uniqueness. In the following, we will observe the numerical evaluation with several well-known sampling patterns and algorithms.

IV. Numerical Example

In this section, numerical experiments of phase retrieval by using algorithms in phasepack library [17] are performed. For the semidefinite program, we use CVX platform [36]. Several well-known sampling patterns that are implemented in this paper are also discussed in [37], [38]. In addition, we will impose structured spherical harmonics coefficients as in the property 2, i.e., conjugate symmetry of coefficients $\hat{f}_l^{-k} = (-1)^k \overline{\hat{f}_l^k}$. This structure is not contrived since it appears in the spherical near-field antenna measurements [21], [39], [40] and transforms the complex spherical harmonics into real spherical harmonics. In the next section, we will numerically evaluate the recovery performance by using phase transition diagram. Furthermore, the application in the phaseless spherical near-field antenna measurements is given.

A. *Phase transition diagram*

In Figure 1, numerical experiments of phaseless measurements by using several sampling patterns and algorithms are evaluated. In this setup, we consider band-limited functions of bandwidth $B = 6$. We run 10 trials for a given signal size $N = B^2$ and classify the recovery as correct if the following holds :

$$\frac{\|\mathbf{x} - \hat{\mathbf{x}} e^{j\alpha}\|_2}{\|\mathbf{x}\|_2} \leq \varepsilon \tag{14}$$

where $\mathbf{x}, \hat{\mathbf{x}} \in \mathbb{C}^N$ are the original and its estimated signal respectively. The variable $\alpha \in [0, 2\pi)$ is a phase ambiguity and the threshold is given as $\varepsilon = 10^{-3}$. It is clear that for all sampling patterns, PhaseLift gives higher success recovery than other algorithms. Nonetheless, PhaseLamp delivers a similar performance as PhaseLift for spiral sampling. On the contrary, all algorithms fail to recover the signal for equiangular sampling, in which confirms the result in proposition 5. Figure 2 presents specific phase transition diagrams for PhaseLift. Band-limited $B = \{4, 5, \ldots, 10\}$ and $N = B^2$ with 10 trials are performed in this setting. The experiments show that if sampling patterns are properly chosen, the number of measurements $m \geq 2N - 1$ is enough to recover the coefficients using PhaseLift. (14).

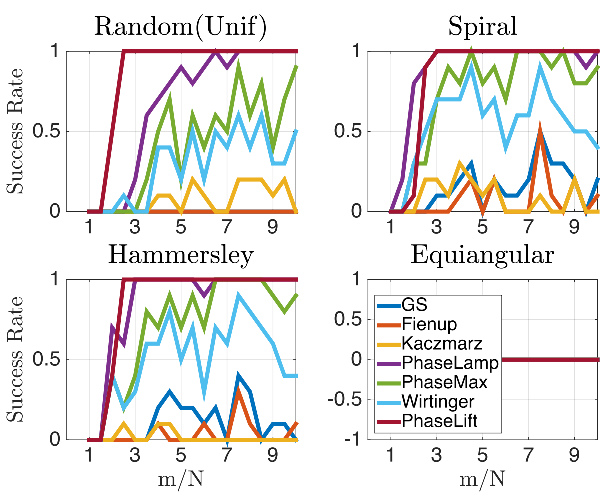

Fig. 1. Phase transition of different algorithms

B. *Phaseless spherical near-field antenna measurements*

One of the applications of phaseless measurements of spherical harmonics expansion is the spherical near-field measurements. In this paper, the array of dipole antennas with a size of coefficients $N = 96$ will be used. It can be seen from Figure 3 that $m = 2.5N$ measurements are enough to recover near-field coefficients by using PhaseLift, which is smaller than the works in [29], [32], i.e., $m > 4N$. In the same manner, equiangular fails to recover the correct coefficients as well as the reconstruction of the correct far-field pattern.

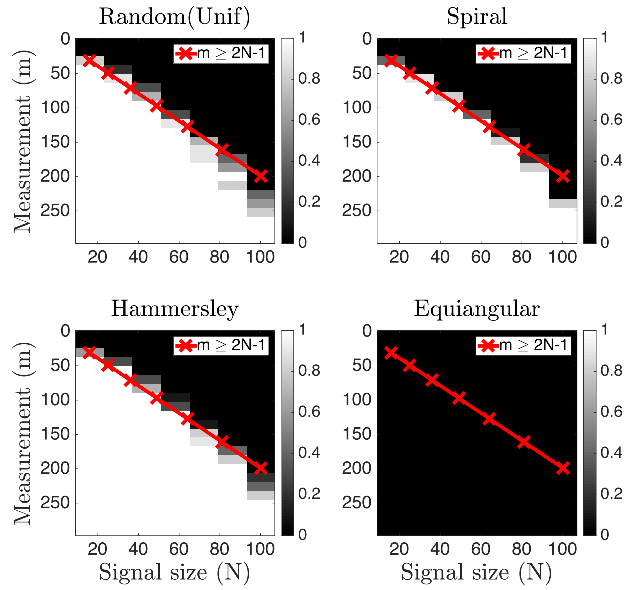

Fig. 2. Phase transition of different sampling patterns with PhaseLift

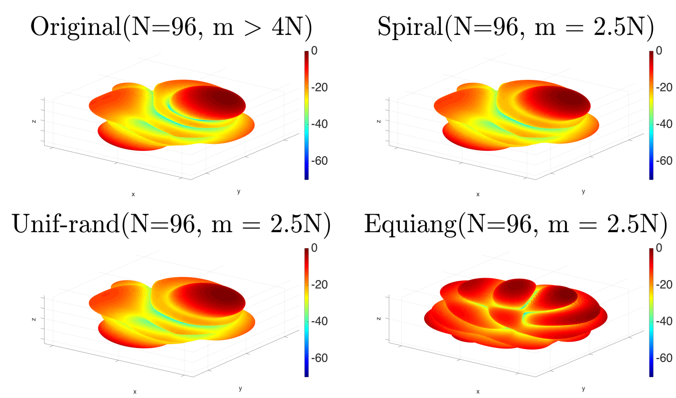

Fig. 3. Far-field reconstruction from phaseless measurement

V. Conclusion and Future Works

In this work, we discuss several ambiguities when considering phaseless measurements in spherical harmonics expansion, which includes trivial ambiguities and amiguities that arise from imporper sampling patterns. Moreover, it has been numerically shown that recovery of spherical harmonics coefficients are possible by carefully choosing appropriate sampling patterns. In future works, analyzing the number of measurements to guarantee a unique recovery would also be beneficial. In most cases, the coefficients of the antenna under test are sparse, which opens up the possibility to extend the problem to compressive phase retrieval.

References

[1] Rick P Millane, "Phase retrieval in crystallography and optics," *JOSA A*, vol. 7, no. 3, pp. 394–411, 1990.

[2] Yoav Shechtman, Yonina C Eldar, Oren Cohen, Henry Nicholas Chapman, Jianwei Miao, and Mordechai Segev, "Phase retrieval with application to optical imaging: a contemporary overview," *IEEE signal processing magazine*, vol. 32, no. 3, pp. 87–109, 2015.

[3] Philipp Walk, Henning Becker, and Peter Jung, "Ofdm channel estimation via phase retrieval," *arXiv preprint arXiv:1512.04252*, 2015.

[4] MH Hayes, "The reconstruction of a multidimensional sequence from the phase or magnitude of its fourier transform," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 30, no. 2, pp. 140–154, 1982.

[5] Dani Kogan, Yonina C Eldar, and Dan Oron, "On the 2d phase retrieval problem," *IEEE Transactions on Signal Processing*, vol. 65, no. 4, pp. 1058–1067, 2017.

[6] Yonina C Eldar, Nethaniel Hammen, and Dustin G Mixon, "Recent advances in phase retrieval [lecture notes]," *IEEE Signal Processing Magazine*, vol. 33, no. 5, pp. 158–162, 2016.

[7] Tamir Bendory, Robert Beinert, and Yonina C Eldar, "Fourier phase retrieval: Uniqueness and algorithms," in *Compressed Sensing and its Applications*, pp. 55–91. Springer, 2017.

[8] Robert Beinert and Gerlind Plonka, "Ambiguities in one-dimensional discrete phase retrieval from Fourier magnitudes," *Journal of Fourier Analysis and Applications*, vol. 21, no. 6, pp. 1169–1198, 2015.

[9] Kejun Huang, Yonina C Eldar, and Nicholas D Sidiropoulos, "Phase retrieval from 1D Fourier measurements: Convexity, uniqueness, and algorithms," *IEEE Transactions on Signal Processing*, vol. 64, no. 23, pp. 6105–6117, 2016.

[10] Emmanuel J Candes, Xiaodong Li, and Mahdi Soltanolkotabi, "Phase retrieval from coded diffraction patterns," *Applied and Computational Harmonic Analysis*, vol. 39, no. 2, pp. 277–299, 2015.

[11] Kishore Jaganathan, Yonina Eldar, and Babak Hassibi, "Phase retrieval with masks using convex optimization," in *Information Theory (ISIT), 2015 IEEE International Symposium on*. IEEE, 2015, pp. 1655–1659.

[12] Kishore Jaganathan, Samet Oymak, and Babak Hassibi, "Sparse phase retrieval: Uniqueness guarantees and recovery algorithms.," *IEEE Trans. Signal Processing*, vol. 65, no. 9, pp. 2402–2410, 2017.

[13] Robert Beinert and Gerlind Plonka, "Sparse phase retrieval of one-dimensional signals by Prony's method," *Frontiers in Applied Mathematics and Statistics*, vol. 3, pp. 5, 2017.

[14] Radu Balan, Pete Casazza, and Dan Edidin, "On signal reconstruction without phase," *Applied and Computational Harmonic Analysis*, vol. 20, no. 3, pp. 345–356, 2006.

[15] Yonina C Eldar and Shahar Mendelson, "Phase retrieval: Stability and recovery guarantees," *Applied and Computational Harmonic Analysis*, vol. 36, no. 3, pp. 473–494, 2014.

[16] Emmanuel J Candes, Thomas Strohmer, and Vladislav Voroninski, "Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming," *Communications on Pure and Applied Mathematics*, vol. 66, no. 8, pp. 1241–1274, 2013.

[17] Rohan Chandra, Ziyuan Zhong, Justin Hontz, Val McCulloch, Christoph Studer, and Tom Goldstein, "Phasepack: A phase retrieval library," *arXiv preprint*, vol. 1711, 2017.

[18] Kishore Jaganathan, Yonina C Eldar, and Babak Hassibi, "Phase retrieval: An overview of recent developments," *arXiv preprint arXiv:1510.07713*, 2015.

[19] Kendall Atkinson and Weimin Han, *Spherical harmonics and approximations on the unit sphere: an introduction*, vol. 2044, Springer Science & Business Media, 2012.

[20] Arya Bangun, Arash Behboodi, and Rudolf Mathar, "Coherence bounds for sensing matrices in spherical harmonics expansion," in *IEEE ICASSP'18*, Calgary, Canada, Apr 2018, IEEE.

[21] Cosme Culotta-Lopez, Dirk Heberling, Arya Bangun, Arash Behboodi, and Rudolf Mathar, "A compressed sampling for spherical near-field measurements," in *2018 40th (AMTA)*, Williamsburg Virginia, USA, 2018.

[22] Erwan Thébault, Christopher C Finlay, Ciarán D Beggan, Patrick Alken, Julien Aubert, Olivier Barrois, Francois Bertrand, Tatiana Bondar, Axel Boness, Laura Brocco, et al., "International geomagnetic reference field: the 12th generation," *Earth, Planets and Space*, vol. 67, no. 1, pp. 79, 2015.

[23] Taco S Cohen, Mario Geiger, Jonas Köhler, and Max Welling, "Spherical cnns," *arXiv preprint arXiv:1801.10130*, 2018.

[24] Michael Kazhdan, Thomas Funkhouser, and Szymon Rusinkiewicz, "Rotation invariant spherical harmonic representation of 3 d shape descriptors," in *Symposium on geometry processing*, 2003, vol. 6, pp. 156–164.

[25] Ramakrishna Kakarala and Dansheng Mao, "A theory of phase-sensitive rotation invariance with spherical harmonic and moment-based representations," in *2010 IEEE Computer Society Conference on Computer Vision and Pattern Recognition*. IEEE, 2010, pp. 105–112.

[26] Ramakrishna Kakarala, "The bispectrum as a source of phase-sensitive invariants for fourier descriptors: a group-theoretic approach," *Journal of Mathematical Imaging and Vision*, vol. 44, no. 3, pp. 341–353, 2012.

[27] Boaz Rafaely, "Analysis and design of spherical microphone arrays," *IEEE Transactions on speech and audio processing*, vol. 13, no. 1, pp. 135–143, 2005.

[28] Philippe Jaming and Salvador Pérez-Esteva, "The phase retrieval problem for solutions of the Helmholtz equation," *Inverse Problems*, vol. 33, no. 10, pp. 105007, 2017.

[29] Carsten H Schmidt, S Farhad Razavi, Thomas F Eibert, and Yahya Rahmat-Samii, "Phaseless spherical near-field antenna measurements for low and medium gain antennas," *Advances in Radio Science*, vol. 8, no. B. 2-1/2-2, pp. 43–48, 2010.

[30] Robert G Yaccarino and Yahya Rahmat-Samii, "Phaseless bi-polar planar near-field measurements and diagnostics of array antennas," *IEEE Transactions on Antennas and Propagation*, vol. 47, no. 3, pp. 574–583, 1999.

[31] Carsten H Schmidt and Yahya Rahmat-Samii, "Phaseless spherical near-field antenna measurements: Concept, algorithm and simulation," in *2009 IEEE Antennas and Propagation Society International Symposium*. IEEE, 2009, pp. 1–4.

[32] Alexander Paulus, Josef Knapp, and Thomas F Eibert, "Phaseless near-field far-field transformation utilizing combinations of probe signals," *IEEE Transactions on Antennas and Propagation*, vol. 65, no. 10, pp. 5492–5502, 2017.

[33] Alexander Paulus, Josef Knapp, and Thomas F Eibert, "Utilizing partial knowledge of phase differences in convex optimization for amplitude-only near-field far-field transformation," in *2017 11th European Conference on Antennas and Propagation (EUCAP)*. IEEE, 2017, pp. 3766–3770.

[34] Weikko Aleksanteri Heiskanen and Helmut Moritz, *Physical geodesy*, Institute of Physical Geodesy, Technical University, 1967.

[35] John C Mason and David C Handscomb, *Chebyshev polynomials*, Chapman and Hall/CRC, 2002.

[36] Inc. CVX Research, "CVX: Matlab software for disciplined convex programming, version 2," http://cvxr.com/cvx, Aug. 2012.

[37] Doug P Hardin, Timothy Michaels, and Edward B Saff, "A comparison of popular point configurations on $\mathbb{S}^2$," *Dolomites Research Notes on Approximation*, vol. 9, no. 1, 2016.

[38] Richard F Bass and Karlheinz Gröchenig, "Random sampling of multivariate trigonometric polynomials," *SIAM journal on mathematical analysis*, vol. 36, no. 3, pp. 773–795, 2005.

[39] Jesper E Hansen, *Spherical near-field antenna measurements*, vol. 26, IET, 1988.

[40] C Culotta-Lopez, K Wu, and D Heberling, "Radiation center estimation from near-field data using a direct and an iterative approach," in *Antenna Measurement Techniques Association Symposium (AMTA), 2017*. IEEE, 2017, pp. 1–6.