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Direct Localization by Partly Calibrated Arrays: A Relaxed Maximum Likelihood Solution

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***Abstract*—We present a novel relaxed maximum likelihood solution to the problem of direct localization of multiple narrowband sources by partly calibrated arrays, i.e., arrays composed of fully calibrated subarrays yet lacking inter-array calibration. The proposed solution is based on eliminating analytically all the nuisance parameters in the problem, thus reducing the likelihood function to a maximization problem involving only the location of the sources. The performance of the solution is demonstrated via simulations.**

***Index Terms*—Partly calibrated arrays, direct localization, relaxed maximum likelihood, signal subspace.**

I. Introduction

Partly calibrated arrays are arrays composed of *fully calibrated subarrays, yet lacking inter-array calibration*. Such arrays are common in large scale systems composed of small subarrays with large inter-array distances, as is the case in multi-site surveillance systems, multi-site communication systems, and multi-site radar systems.

A powerful model for partly calibrated arrays, referred to as the Partly Calibrated Array (PCA) model, was introduced by See and Gershman [1] in the problem of direction finding. This model can cope with a variety of uncertainties in the direction finding problem, including unknown subarrays displacements and unknown phase offsets between the subarrays. It can be regarded as a generalization of a more limited model, introduced by Pesavento et al. [2], addressing partly calibrated arrays composed of identically oriented subarrays with unknown subarray displacements. Apart from introducing the PCA model, [1] presented a MUSIC-like technique for estimating the direction-of-arrival of multiple narrowband sources and the Cramer-Rao bound (CRB) for this problem. This work was followed by Lie et al. [3] and Mavrychev et al. [4] who introduced MVDR-like techniques. Liao and Chan [5] exploited the special structure of the uniform linear array to reduce the computational complexity. A sparse recovery approach for direction finding in partly calibrated arrays composed of subarrays with unknown displacements was introduced by Steffens and Pesavento [6].

Independently of this work on direction finding, Weiss [7] and Weiss and Amar [8] introduced the PCA model in the direct localization problem, to cope with the unknown propagation to the subarrays. Direct localization, advocated first in [10]-[11] and further developed in [12]-[21], is a localization scheme in which the location is estimated directly from the data in one-step, as opposed to the more conventional two-step scheme, where the directions-of-arrival to the subarrays are estimated in the first step and then, in the second step, the location is estimated using triangulation. Direct localization provides not only higher accuracy at low signal-to-noise and low signal-to-interference ratios, but not less importantly, reduced ambiguity. This is because the data association step, needed in the two-step procedure and prone to ambiguity errors, is eliminated. Apart from introducing the PCA model, [7] introduced the maximum likelihood solution for a single narrowband source, while [8]-[9] extended this approach to wideband sources, introduced MUSIC-like solution for sources with unknown waveforms and a maximum likelihood solution for sources with known waveforms, as well as the CRB for these problems.

In this paper we present a novel and computationally efficient relaxed maximum likelihood solution to the problem of *direct localization* by partly calibrated arrays. Note that since direction finding can be considered as a special case of direct localization, corresponding to the case that the sources are in the far-field of the array, our solution apples to both problems.

The rest of the paper is organized as follows. The problem formulation is presented in section II. Section III presents the "relaxed" maximum likelihood solution. Simulation results are discussed in section IV, and the conclusions are provided in section V.

II. Problem Formulation

Consider an array composed of L fully calibrated subarrays, each composed of M_l antennas with arbitrary locations and arbitrary directional characteristics. Let $M = \sum_{l=1}^{L} M_l$ denote the total number of antennas. Assume that Q sources, located at locations $\{\mathbf{p}_q\}_{q=1}^{Q}$, with $\mathbf{p}_q \in R^{D\times 1}$, $D = 1, 2, 3$, and emitting signals $\{s_q(t)\}_{q=1}^{Q}$, are impinging on the array.

To capture both the direct localization and the direction finding problems, we allow the dimension D to be a parameter. If the sources are in the far-field of the array then either $D = 1$, if both the sources and the array are confined to a plane, or $D = 2$, if otherwise. In case the sources are in the near-field of the array, then either $D = 2$, if both the sources and the array are confined to a plane, or $D = 3$, if otherwise.

We further make the following assumptions regarding the emitted signals, the array and the noise:

A1: The number of sources Q is *known*.

A2: The emitted signals are *narrowband*, i.e, their bandwidth is much smaller than the reciprocal of the propagation time across the array, and centered around ω_c.

A3: The emitted signals are *unknown* with zero mean and uncorrelated.

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A4: The array is synchronized in time, but there is unknown phase offsets between the subarrays.

A5: The locations of the subarrays are known, but with an uncertainty of σ_a^2.

A6: The propagation model is *spherical waves* (it degenerates to plane waves if the sources are in the far-field).

A7: The steering vectors of the subarrays toward any potential location $\mathbf{p}$, given by $\{\mathbf{a}_l(\mathbf{p})\}_{l=1}^L$, are *known* and have unit norm, i.e., $\|\mathbf{a}_l(\mathbf{p})\| = 1$.

A8: The additive noises at the subarrays are independent of the signals and independent of each other, and distributed as complex Gaussian with zero mean and covariance $\sigma_n^2 \mathbf{I}_M$.

Assumptions A1-A3 and A6-A8 are conventional and do not need further justification. Assumptions A4-A5 reflect the current limitation of the Global Positioning System (GPS). A4 reflects the current accuracy of the GPS time data - typically 10 ns - which is good enough for time synchronization in the case of narrowband signals, but not good enough for phase synchronization. A5 reflects the current accuracy of the GPS location data, which is typically 5-10 meters.

Under these assumptions, the PCA model for the $M_l \times 1$ vector of the complex envelopes of the received signals at the l-th subarray is given by

$$\mathbf{x}_l(t) = \sum_{q=1}^{Q} b_{l,q} \mathbf{a}_l(\mathbf{p}_q) s_q(t - \tau_l(\mathbf{p}_q)) + \mathbf{n}_l(t), \tag{1}$$

where $b_{l,q}$ is a complex coefficient associated with the propagation of the q-th signal to the l-th subarray, $\mathbf{a}_l(\mathbf{p}_q)$ is the steering vector of the l-the subarray toward location $\mathbf{p}_q$, $\tau_l(\mathbf{p}_q)$ is the delay from $\mathbf{p}_q$ to the l-th subarray, and $\mathbf{n}_l(t)$ is the noise at the l-th subarray.

The partly calibrated nature of the array is embodied by the set of QL complex coefficient $\{b_{l,q}\}$, $q = 1, ..., Q;\ l = 1, ..., L$, assumed to be *unknown* parameters. In our problem these parameter capture the combined effect of the unknown propagation to the subarrays, the unknown subarrays displacement due to subarrays location error, and the unknown phase offset between subarrays.

The narrowband assumption A2 implies that the time delays are well approximated by phase shifts, which allow us to rewrite (1) as

$$\mathbf{x}_l(t) = \sum_{q=1}^{Q} b_{l,q} \mathbf{a}_l(\mathbf{p}_q) s_q(t) e^{-j\omega_c \tau_l(\mathbf{p}_q)} + \mathbf{n}_l(t), \tag{2}$$

where $j = \sqrt{-1}$. Assuming the array is sampled N times, we can express the signals received by the l-th subarray as

$$\mathbf{X}_l = \mathbf{A}_l(\mathbf{P}) \mathbf{B}_l \mathbf{S} + \mathbf{N}_l, \tag{3}$$

where $\mathbf{X}_l$ is the $M_l \times N$ matrix

$$\mathbf{X}_l = [\mathbf{x}_l(t_1), ..., \mathbf{x}_l(t_N)], \tag{4}$$

$\mathbf{A}_l(\mathbf{P})$ is the $M_l \times Q$ matrix of the steering vectors towards the Q locations (to simplify the notation, the explicit dependence on the locations $\mathbf{P} = \{\mathbf{p}_1, ..., \mathbf{p}_Q\}$ will be sometimes dropped)

$$\mathbf{A}_l(\mathbf{P}) = \mathbf{A}_l = [\mathbf{a}_l(\mathbf{p}_1) e^{-j\omega_c \tau_l(\mathbf{p}_1)}, ..., \mathbf{a}_l(\mathbf{p}_Q) e^{-j\omega_c \tau_l(\mathbf{p}_Q)}], \tag{5}$$

$\mathbf{B}_l$ is a $Q \times Q$ diagonal matrix

$$\mathbf{B}_l = diag(\mathbf{b}_l), \tag{6}$$

with

$$\mathbf{b}_l = [b_{l,1}, ..., b_{l,Q}]^T, \tag{7}$$

$\mathbf{S}$ is the $Q \times N$ signals matrix

$$\mathbf{S} = [\mathbf{s}(t_1), ..., \mathbf{s}(t_N)] = [\mathbf{s}_1^T, ..., \mathbf{s}_Q^T], \tag{8}$$

with

$$\mathbf{s}(t) = [s_1(t), ..., s_Q(t)]^T, \tag{9}$$

and $\mathbf{N}_l$ is the $M_l \times N$ matrix of the noise

$$\mathbf{N}_l = [\mathbf{n}_l(t_1), ..., \mathbf{n}_l(t_N)]. \tag{10}$$

To equalize the contributions of the subarrays, we normalize their power, namely set

$$\mathrm{tr}(\mathbf{X}_l \mathbf{X}_l^H) = 1, \qquad l = 1, ..., L \tag{11}$$

where $\mathrm{tr}()$ denotes the trace operator and H denotes the conjugate transpose.

We can now state the direct localization problem as follows: *Given the received data* $\{\mathbf{X}_l\}_{l=1}^L$, *estimate the Q locations* $\{\mathbf{p}_q\}_{q=1}^Q$.

III. Relaxed Maximum Likelihood Solution

Regarding the signals matrix $\mathbf{S}$ and the coefficient matrices $\{\mathbf{B}_l\}$ as unknown parameters, it follows from (3) and the Gaussian noise assumption A8 that the maximum likelihood cost function is given by

$$\hat{\mathbf{P}} = \underset{\mathbf{P}, \{\mathbf{B}_l\}, \mathbf{S}}{\operatorname{argmin}} \sum_{l=1}^{L} \| \mathbf{X}_l - \mathbf{A}_l(\mathbf{P}) \mathbf{B}_l \mathbf{S} \|_F^2. \tag{12}$$

Note that this cost function is a multidimensional nonlinear minimization with a total of $DQ + 2QL + 2QN$ real unknown parameters, corresponding to $\mathbf{P}$, $\{\mathbf{B}_l\}$, and $\mathbf{S}$, respectively. Out of this large number of unknowns, only the DQ unknowns corresponding to the locations $\mathbf{P}$ are of our interest, while the other are considered as *nuisance parameters*.

Since the exact solution of (12) yields a complicated expression which does not seem to enable the elimination of all the nuisance parameters, we next present a relaxed maximum likelihood solution which enables the desired elimination and yields a concentrated likelihood involving only the unknown locations of the sources.

Our first step is to eliminate the unknown coefficients $\{\mathbf{B}_l\}$ by expressing them in terms of the other parameters $\mathbf{P}$ and $\mathbf{S}$. To this end, note that $\mathbf{B}_l$ appears only in the l-th term in (12), implying that it can be estimated by the following minimization problem:

$$\hat{\mathbf{B}}_l = \underset{\mathbf{B}_l}{\operatorname{argmin}} \| \mathbf{X}_l - \mathbf{A}_l \mathbf{B}_l \mathbf{S} \|_F^2 \tag{13}$$

where we hold $\mathbf{A}_l$ and $\mathbf{S}$ fixed. Denoting by J_l the cost function of (13), we have

$$\begin{aligned} J_l &= \mathrm{tr}((\mathbf{X}_l - \mathbf{A}_l\mathbf{B}_l\mathbf{S})^H(\mathbf{X}_l - \mathbf{A}_l\mathbf{B}_l\mathbf{S})) \\ &= \mathrm{tr}(\mathbf{X}_l^H\mathbf{X}_l) - \mathrm{tr}(\mathbf{B}_l^H\mathbf{A}_l^H\mathbf{X}_l\mathbf{S}^H) - tr(\mathbf{A}_l\mathbf{B}_l\mathbf{S}\mathbf{X}_l^H) \\ &+ \mathrm{tr}(\mathbf{B}_l^H\mathbf{A}_l^H\mathbf{A}_l\mathbf{B}_l\mathbf{S}\mathbf{S}^H). \end{aligned} \tag{14}$$

Dropping the terms which do not contain $\mathbf{B}_l$, we can rewrite it as

$$\begin{aligned} J_l &= -\mathrm{tr}(\mathbf{A}_l^H\mathbf{X}_l\mathbf{S}^H\mathbf{B}_l^H) - \mathrm{tr}(\mathbf{A}_l\mathbf{S}\mathbf{X}_l^H\mathrm{diag}(\mathbf{b}_l)) \\ &+ \mathrm{tr}(\mathbf{S}\mathbf{S}^H\mathbf{B}_l^H\mathbf{A}_l^H\mathbf{A}_l\mathrm{diag}(\mathbf{b}_l)). \end{aligned} \tag{15}$$

Now, equating to zero the derivative with respect to $\mathbf{b}_l$, using the well known complex differentiation rules [22] and the following matrix differentiation rule [23],

$$\frac{\partial}{\partial \mathbf{b}}\mathrm{tr}(\mathbf{A}\mathrm{diag}(\mathbf{b})) = \mathrm{diag}(\mathbf{A}), \tag{16}$$

we get

$$\mathrm{diag}(\mathbf{S}\mathbf{X}_l^H\mathbf{A}_l) = \mathrm{diag}(\mathbf{S}\mathbf{S}^H\mathbf{B}_l^H\mathbf{A}_l^H\mathbf{A}_l). \tag{17}$$

To solve this equation for $\mathbf{B}_l$, we first relax the equality of the diagonals of the two marices to an equality of the whole matrices, yielding

$$\mathbf{S}\mathbf{X}_l^H\mathbf{A}_l = \mathbf{S}\mathbf{S}^H\mathbf{B}_l^H\mathbf{A}_l^H\mathbf{A}_l. \tag{18}$$

Next, we relax the constraint that $\mathbf{B}_l$ is diagonal and allow it to be an arbitrary matrix, which enables us to straightfowardly solve this equation for $\mathbf{B}_l$, yielding

$$\hat{\mathbf{B}}_l = (\mathbf{A}_l^H\mathbf{A}_l)^{-1}\mathbf{A}_l^H\mathbf{X}_l\mathbf{S}^H(\mathbf{S}\mathbf{S}^H)^{-1}. \tag{19}$$

Now, multiplying from the left and right by $\mathbf{A}_l$ and $\mathbf{S}$, respectively, we get

$$\mathbf{A}_l\hat{\mathbf{B}}_l\mathbf{S} = \mathbf{A}_l(\mathbf{A}_l^H\mathbf{A}_l)^{-1}\mathbf{A}_l^H\mathbf{X}_l\mathbf{S}^H(\mathbf{S}\mathbf{S}^H)^{-1}\mathbf{S} \tag{20}$$

or alternatively,

$$\mathbf{A}_l\hat{\mathbf{B}}_l\mathbf{S} = \mathbf{P}_{\mathbf{A_l}}\mathbf{X}_l\mathbf{P}_{\mathbf{S}^H} \tag{21}$$

where $\mathbf{P}_{\mathbf{A_l}}$ is the projection matrix on column span of of $\mathbf{A}_l$

$$\mathbf{P}_{\mathbf{A_l}} = \mathbf{A}_l(\mathbf{A}_l^H\mathbf{A}_l)^{-1}\mathbf{A}_l^H, \tag{22}$$

and $\mathbf{P}_{\mathbf{S}^H}$ is the projection matrix on the column span of $\mathbf{S}^H$

$$\mathbf{P}_{\mathbf{S}^H} = \mathbf{S}^H(\mathbf{S}\mathbf{S}^H)^{-1}\mathbf{S}. \tag{23}$$

Substituting (21) into (12), yields

$$\hat{\mathbf{P}} = \underset{\mathbf{P},\mathbf{S}}{\mathrm{argmin}} \sum_{l=1}^{L} \| \mathbf{X}_l - \mathbf{P}_{\mathbf{A}_l(\mathbf{P})}\mathbf{X}_l\mathbf{P}_{\mathbf{S}^H}\|_F^2, \tag{24}$$

which, using the properties of the trace operator and the projection matrix, with some straightforward manipulations, reduces to

$$\hat{\mathbf{P}} = \underset{\mathbf{P},\mathbf{S}}{\mathrm{argmax}}\, tr(\mathbf{P}_{\mathbf{S}^H}\sum_{l=1}^{L}\mathbf{X}_l^H\mathbf{P}_{\mathbf{A}_l(\mathbf{P})}\mathbf{X}_l). \tag{25}$$

This expression can be interpreted as a search for the locations $\mathbf{P}$ and the signal matrix $\mathbf{S}$ for which there is maximum correlation between the signal subspace defined by $\mathbf{P}_{\mathbf{S}^H}$ and the sum of projections of $\mathbf{X}_l$ on the signal subspaces defined by $\mathbf{P}_{\mathbf{A}_l(\mathbf{P})}$, $l = 1, ..., L$.

To further eliminate the unknown parameters of the matrix $\mathbf{S}$, we next evaluate (25) for the case of noncoherent signals[1], i.e. when the signal subspace defined by $\mathbf{P}_{\mathbf{S}^H}$ is Q-dimensional. in this case we can express $\mathbf{P}_{\mathbf{S}^H}$ as

$$\mathbf{P}_{\mathbf{S}^H} = \tilde{\mathbf{S}}^H\tilde{\mathbf{S}}, \tag{26}$$

where $\tilde{\mathbf{S}}$ obeys

$$\tilde{\mathbf{S}}\tilde{\mathbf{S}}^H = \mathbf{I}_Q, \tag{27}$$

with $\mathbf{I}_Q$ denoting the $Q \times Q$ identity matrix. Substituting this expression in (25), using the properties of the trace operator, we get

$$\hat{\mathbf{P}} = \underset{\mathbf{P},\tilde{\mathbf{S}};\ \tilde{\mathbf{S}}\tilde{\mathbf{S}}^H=\mathbf{I}_q}{\mathrm{argmax}}\ \mathrm{tr}(\tilde{\mathbf{S}}(\sum_{l=1}^{L}\mathbf{X}_l^H\mathbf{P}_{\mathbf{A}_l(\mathbf{P})}\mathbf{X}_l\tilde{\mathbf{S}}^H). \tag{28}$$

Maximizing this expression over $\tilde{\mathbf{S}}$, holding $\mathbf{P}$ constant, we get

$$\hat{\tilde{\mathbf{S}}}^H(\mathbf{P}) = \tilde{\mathbf{V}}_S(\mathbf{P}) = [\tilde{\mathbf{v}}_1, ..., \tilde{\mathbf{v}}_Q], \tag{29}$$

where $\tilde{\mathbf{v}}_q$ denotes the $N \times 1$ eigenvector corresponding to the q-th eigenvalue of the $N \times N$ matrix $\sum_{l=1}^{L}\mathbf{X}_l^H\mathbf{P}_{\mathbf{A}_l(\mathbf{P})}\mathbf{X}_l$. Substituting this expression for $\hat{\tilde{\mathbf{S}}}(\mathbf{P})$ back into (28), we get

$$\hat{\mathbf{P}} = \underset{\mathbf{P}}{\mathrm{argmax}} \sum_{q=1}^{Q}\lambda_q(\sum_{l=1}^{L}\mathbf{X}_l^H\mathbf{P}_{\mathbf{A}_l(\mathbf{P})}\mathbf{X}_l), \tag{30}$$

where $\lambda_q()$ denotes the q-th eigenvalue of the bracketed matrix.

For large N, computing the eigenvalues of the $N \times N$ matrix $\mathbf{X}_l^H\mathbf{P}_{\mathbf{A_l(P)}}\mathbf{X}_l$ may be prohibitive. We next show how to reduce the dimensionality of this problem.

To this end, denote by $\mathbf{P}_{\mathbf{A}(\mathbf{P})}$ the $M \times M$ block-diagonal matrix

$$\mathbf{P}_{\mathbf{A}(\mathbf{P})} = \mathrm{blkdiag}(\mathbf{P}_{\mathbf{A}_l}), \tag{31}$$

by $\mathbf{X}$ is the $M \times N$ matrix of the sampled data

$$\mathbf{X} = (\mathbf{X}_1^T, \ldots, \mathbf{X}_L^T)^T, \tag{32}$$

and by $\hat{\mathbf{R}} = \mathbf{X}\mathbf{X}^H$ the $M \times M$ sample-covariance matrix of the array

$$\hat{\mathbf{R}} = \begin{pmatrix} \mathbf{X}_1\mathbf{X}_1^H & \cdots & \mathbf{X}_1\mathbf{X}_L^H \\ \vdots & \ddots & \vdots \\ \mathbf{X}_L\mathbf{X}_!^H & \cdots & \mathbf{X}_L\mathbf{X}_L^H \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{R}}_{1,1} & \cdots & \hat{\mathbf{R}}_{1,L} \\ \vdots & \ddots & \vdots \\ \hat{\mathbf{R}}_{L,1} & \cdots & \hat{\mathbf{R}}_{L,L} \end{pmatrix}. \tag{33}$$

Now, using the properties of the projection matrix and the fact that the eigenvalues of a product of two matrices are unchanged by their permutation, we have

$$\lambda_q(\sum_{l=1}^{L}\mathbf{X}_l^H\mathbf{P}_{\mathbf{A}_l(\mathbf{P})}\mathbf{X}_l) = \lambda_q(\mathbf{P}_{\mathbf{A}(\mathbf{P})}\hat{\mathbf{R}}\mathbf{P}_{\mathbf{A}(\mathbf{P})}), \tag{34}$$

[1]The case of coherent signals is also of interest, and will be addressed in future research.

which when substituted into (30) yields

$$\hat{\mathbf{P}} = \underset{\mathbf{P}}{\operatorname{argmax}} \sum_{q=1}^{Q} \lambda_q(\mathbf{P}_{\mathbf{A}(\mathbf{P})}\hat{\mathbf{R}}\mathbf{P}_{\mathbf{A}(\mathbf{P})}). \quad (35)$$

To further simplify this expression, let $\tilde{\mathbf{A}}$ denote the block-diagonal matrix

$$\tilde{\mathbf{A}} = \text{blkdiag}(\tilde{\mathbf{A}}_l), \quad (36)$$

where $\tilde{\mathbf{A}}_l$ is given by

$$\tilde{\mathbf{A}}_l = \mathbf{A}_l(\mathbf{A}_l^H\mathbf{A}_l)^{-1/2}. \quad (37)$$

Using this notation we can rewrite $\mathbf{P}_{\mathbf{A}}$ as

$$\mathbf{P}_{\mathbf{A}} = \tilde{\mathbf{A}}\tilde{\mathbf{A}}^H = \text{blkdiag}(\tilde{\mathbf{A}}_l\tilde{\mathbf{A}}_l^H), \quad (38)$$

which implies, using the invariance of the eigenvalues of a product of matrices to their cyclic permutation, that

$$\lambda_q(\mathbf{P}_{\mathbf{A}}\hat{\mathbf{R}}\mathbf{P}_{\mathbf{A}}) = \lambda_q(\mathbf{P}_{\mathbf{A}}\mathbf{P}_{\mathbf{A}}\hat{\mathbf{R}}) = \lambda_q(\mathbf{P}_{\mathbf{A}}\hat{\mathbf{R}}) = \lambda_q(\tilde{\mathbf{A}}^H\hat{\mathbf{R}}\tilde{\mathbf{A}}). \quad (39)$$

Substituting this result into (35), we get

$$\hat{\mathbf{P}} = \underset{\mathbf{P}}{\operatorname{argmax}} \sum_{q=1}^{Q} \lambda_q(\tilde{\mathbf{A}}^H(\mathbf{P})\hat{\mathbf{R}}\tilde{\mathbf{A}}(\mathbf{P})) \quad (40)$$

Note that since the matrix $\tilde{\mathbf{A}}^H(\mathbf{P})\hat{\mathbf{R}}\tilde{\mathbf{A}}(\mathbf{P})$ is $LQ \times LQ$, and since typically $LQ \ll N$, the computational complexity of the solution (40) is significantly smaller than that of (30). To reduce the computational load of the Q-dimensional search over $\mathbf{P}$, we propose to use the alternating projections (AP) algorithm [24], which transforms a Q-dimensional search into an iterative process involving only *single source* searches. Denote the cost function by

$$g(\mathbf{P}) = \sum_{q=1}^{Q} \lambda_q(\tilde{\mathbf{A}}^H(\mathbf{P})\hat{\mathbf{R}}\tilde{\mathbf{A}}(\mathbf{P})). \quad (41)$$

In the proposed iterative search algorithm, the number of sources is increased from $q = 1$ to $q = Q$, with the q-th step involving a maximization over $\mathbf{p}_q$, with the other $q-1$ pre-computed locations held fixed:

$$\hat{\mathbf{p}}_q = \underset{\mathbf{p}_q}{\operatorname{argmax}}\, g(\mathbf{P}_q^{(0)}) \quad (42)$$

where

$$\mathbf{P}_q^{(0)} = (\hat{\mathbf{p}}_1, ..., \hat{\mathbf{p}}_{q-1}, \mathbf{p}_q). \quad (43)$$

IV. Simulation Results

The performance of the proposed solution was evaluated for the cases of two and three sources, randomly positioned in a squared area of 1 square kilometer. Four uniform linear arrays of 8 elements (each) were positioned in the four corners of the squared area. The sources were simulated as uncorrelated unit power QPSK signals, each one transmitting 1,000 i.i.d. QPSK symbols. The standard free space path loss (FSPL) model was used to compute the received power in each array position: $FSPL = (\frac{\lambda}{4\pi d})^2$, with $\lambda = 12.5cm$ (carrier frequency of 2.4GHz), and d is the source-to-array distance in meters.

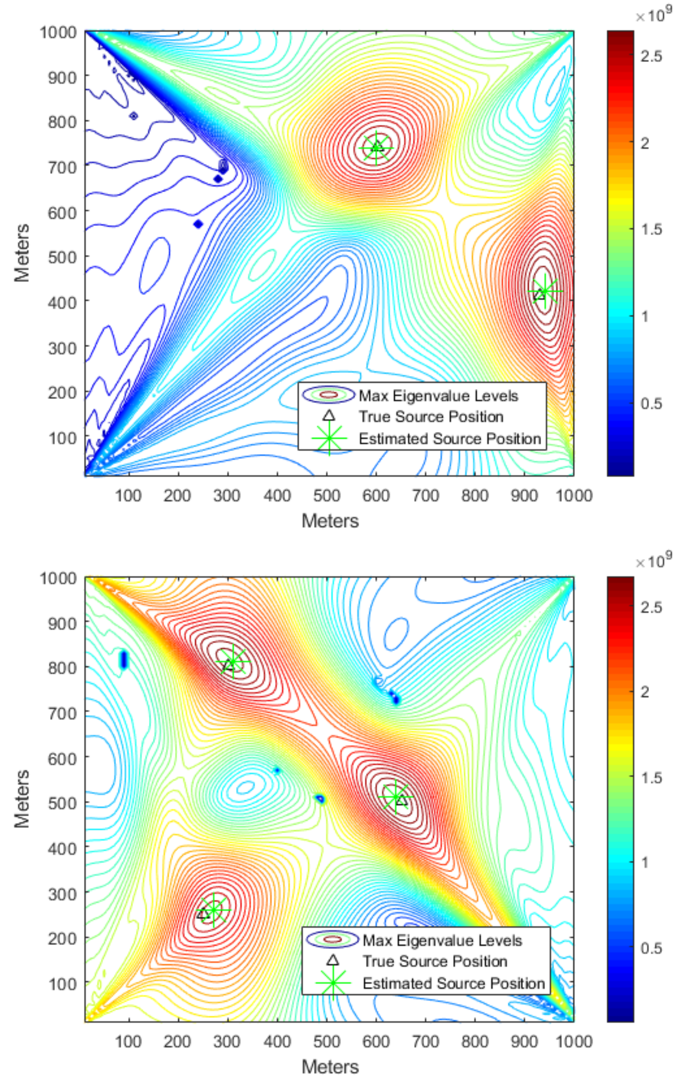

Fig. 1. Localization of 2 (top) and 3 (bottom) sources, by 4 uniform linear arrays of 8 elements, positioned in the 4 corners of the area.

In addition, an additive white Gaussian noise (AWGN) with variance 0.1 was simulated for all arrays elements. The results, presented in Fig. 1 clearly demonstrate the high accuracy of the proposed solution.

V. Conclusions

We have presented a novel relaxed maximum likelihood solution to direct localization in partly calibrated arrays which, by eliminating analytically all the nuisance parameters in the problem, reduced the solution to a maximization problem over the sources' locations. The resulting maximization problem is computationally complex for more than one source. To simplify the computation, we propose using the AP technique [24], which transforms the multidimensional maximization to iterative one-dimensional maximizations.

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