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One-bit compressed sensing with partial Gaussian circulant matrices

Sjoerd Dirksen, Hans Christian Jung, Holger Rauhut
RWTH Aachen University
Lehrstuhl C für Mathematik (Analysis)
Pontdriesch 10, 52062 Aachen, Germany
lastname@mathc.rwth-aachen.de

***Abstract*—We consider recent developments for memoryless one-bit compressed with structured random matrices with randomly subsampled Gaussian circulant matrices. We discuss that in a small sparsity regime and for small enough accuracy δ, $m \sim \delta^{-8} s \log(N/s\delta)$ measurements suffice to reconstruct the direction and energy of any s-sparse signal up to accuracy δ via an efficient program.**

I. Introduction

In the traditional compressed sensing literature, it is typically assumed that one can reconstruct a signal based on its analog linear measurements. In a real world sensing scenario, measurements need to be quantized to a finite number of bits before they can be transmitted, stored, and processed. Taking this assumption seriosly, leads one to consider reconstruction problems for a sparse signals x based on *non-linear* measurements, which take the form $y = Q(Ax)$, where $Q : \mathbb{R}^m \to \mathcal{A}^m$ is a quantizer and $\mathcal{A}$ denotes a finite quantization alphabet. In the following we examine the measurement model

$$y = \operatorname{sign}(Ax + \tau) \ , \tag{1}$$

where $A \in \mathbb{R}^{m\times N}$, $m \ll N$, sign is the signum function applied element-wise and $\tau \in \mathbb{R}^m$ is a random vector consisting of thresholds. The majority of the known work on signal reconstruction results for the model in (1) are restricted to standard Gaussian measurement matrices. This model is concerned with recovering sparse signals x from measurements as specified in (1), when $A \in \mathbb{R}^{m\times N}$ is a matrix with standard Gaussian entries, i.e. for all pairs i, j with $i \in [m]$ and $j \in [N]$ the entires of the matrix satisfy $a_{ij} \sim \mathcal{N}(0, 1)$.

For the case $\tau = 0$ the work [4, Theorem 2] has shown that if A is $m\times N$ Gaussian and $m \geq C\varepsilon^{-1} s \log(N/s\varepsilon)$ then, with high probability, any s-sparse x, x' with $\|x\|_2 = \|x'\|_2 = 1$ and $\operatorname{sign}(Ax) = \operatorname{sign}(Ax')$ satisfy $\|x-x'\|_2 \leq \varepsilon$. In particular, this shows that one can approximate x up to error ε by the solution of the non-convex program

$$\min \|z\|_0 \qquad \text{s.t.} \qquad \operatorname{sign}(Ax) = \operatorname{sign}(Az), \ \|z\|_2 = 1.$$

While the result [4, Theorem 1] shows that this result is near optimal in the sense that the dependency on m and ε cannot be improved in general, solving the proposed program is a NP-hard problem. The work [3] showed that by modifying the measurement matrix A the near optimal error dependence can be obtained by a polynomial time algorithm.

To recover efficiently from Gaussian one-bit measurements, Plan and Vershynin [8] proposed the linear reconstruction program

$$\min_{z\in\mathbb{R}^n} \|z\|_1 \quad \text{s.t.} \quad \operatorname{sign}(Az) = \operatorname{sign}(Ax) \quad \text{and} \quad \|Az\|_1 = 1. \tag{LP}$$

They showed that using $m \gtrsim \varepsilon^{-1} s \log^2(N/s)$ Gaussian measurements one can recover every x with $\|x\|_1 \leq \sqrt{s}$ and $\|x\|_2 = 1$ by solving (LP) with reconstruction error $\varepsilon^{1/5}$. In [9] the authors introduced a different convex program and showed that if $m \gtrsim \varepsilon^{-1} s \log(N/s)$, then one can achieve a reconstruction error $\varepsilon^{1/6}$ even if there is quantization noise present.

A central problem in the regime $\tau = 0$ is that all information on the energy of the signal x is lost in the quantization step. It was recently shown that one can recover full signals by incorporating appropriate thresholds. In [6] it was shown that by taking Gaussian thresholds τ_i one can recover energy information by slightly modifying the linear program (LP). A similar observation was made in [2] for the following second order cone program

$$\min_{z\in\mathbb{R}^N} \|z\|_1 \quad \text{s.t.} \quad \operatorname{sign}(Az+\tau) = \operatorname{sign}(Ax+\tau), \quad \text{and} \quad \|z\|_2 \leq R \ . \tag{CP}$$

The paper [6] also proposed a method to estimate $\|x\|_2$ using a single deterministic threshold $\tau_i = \tau$ that works well if one has some prior knowledge of the energy range of the signal.

In this note we want to investigate the applicability of the program (CP) for recovering s-sparse signals from structured measurements. In particular, we study the recovery problem in (1), when A is a suitable normalized, randomly subsampled Gaussian circulant matrix. The result we want to discuss shows that for an efficiently sparse signal $x \in \mathbb{R}^N$ the solution of the second-order cone program (CP) satisfies $\|x - x^{\#}\|_2 \leq R\varepsilon^{1/8}$ with high probability assuming that the sparsity s of x satisfies $s \leq c\varepsilon\sqrt{N/\log(N)}$ and $\|x\|_2 \leq R$. This result can be deduced by studying a ℓ^1/ℓ^2-restricted isometry property (RIP) for the matrix A. The Gaussian circulant measurement model is important for several real-world applications, including SAR radar imaging, Fourier optical imaging and channel estimation (see e.g. [10]).

II. Recovery garuantees for Gaussian circulant matrices

We let $\Sigma_{s,N}$ denote the set of all s-sparse vectors with unit norm. We say that $x \in \mathbb{R}^N$ is *s-effectively sparse* if $\|x\|_1 \leq$

$\sqrt{s}\|x\|_2$. We let $\Sigma_{s,N}^{\text{eff}}$ denote the set of all s-effectively sparse vectors. Clearly, if x is s-sparse, then it is s-effectively sparse. For any $x \in \mathbb{R}^N$ we let $D_x = \text{diag}(x) \in \mathbb{R}^{N\times N}$ be the diagonal matrix generated by x and let $\Gamma_x \in \mathbb{R}^{N\times N}$ denote the circulant matrix generated by x, i.e.

$$\Gamma_x = \begin{bmatrix} x_N & x_1 & x_2 & \cdots & x_{N-2} & x_{N-1} \\ x_{N-1} & x_N & x_1 & \cdots & x_{N-3} & x_{N-2} \\ x_{N-2} & x_{N-1} & x_N & \cdots & x_{N-4} & x_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_{N-1} & x_N \end{bmatrix}.$$

For a set $I \subseteq [N]$ we let $R_I : \mathbb{R}^N \to \mathbb{R}^{|I|}$ denote the coordinate projection onto the coordinates index by I.

We study the model in (1) with the following measurement matrix: We consider a vector θ of i.i.d. random selectors with mean m/N and let $I = \{i \in [N] : \theta_i = 1\}$. Let $g \sim \mathcal{N}(0, \text{id})$ be an N-dimensional standard Gaussian vector that is independent of θ. We define the *randomly subsampled Gaussian circulant matrix* by $A = R_I\Gamma_g$. Note that $\mathbb{E}|I| = m$, so m corresponds to the expected number of measurements in this model.

Theorem II.1. *Let $A = R_I\Gamma_g$ and let $\tau_1, \ldots, \tau_m$ be independent $\mathcal{N}(0, R^2)$-distributed random variables. Further, let $\eta \in [0, 1]$ and $0 < \delta \leq (\log^2(s)\log(N))^{-1/4}$. Assume that*

$$s \lesssim \min\{\sqrt{\delta^4 N/\log(N)}, \delta^2 N/\log(1/\eta)\},$$
$$m \gtrsim \delta^{-4} s\log(eN/s),$$

Then the following holds with probability exceeding $1 - \eta$: for any $x \in \mathbb{R}^N$ with $\|x\|_1 \leq \sqrt{s}\|x\|_2$ and $\|x\|_2 \leq R$, any solution $x_{CP}^{\#}$ to the second-order cone program (CP) *satisfies $\|x - x_{CP}^{\#}\|_2 \leq R\sqrt{\delta}$.*

The conclusions of Theorem II.1 are not limited to the regime $\delta \leq (\log^2(s)\log(N))^{-1/4}$ and can be extended to $\delta \in (0, 1)$. In this regime of δ the dependencies for m and s on N, η, δ are more involved. In this note we will limit our attention to the result stated above.

To deduce the result, we will use the following abstract version of the program (CP) for a matrix $C \in \mathbb{R}^{m\times(N+1)}$:

$$\min_{z\in\mathbb{R}^N} \|z\|_1 \quad \text{s.t.} \quad \text{sign}(C[z, R]) = \text{sign}(C[x, R]), \quad \text{and} \quad \|z\|_2 \leq R. \tag{2}$$

Here $[z, R] \in \mathbb{R}^{N+1}$ denotes the vector obtained by appending $R \in \mathbb{R}$ to the vector $z \in \mathbb{R}^N$. It is straight forward to verify that the program in (2) is obtained by taking $C = \frac{1}{m}\sqrt{\frac{\pi}{2}}B$ with $B = D_\theta[\Gamma_g, h]$, where $[\Gamma_g, h] \in \mathbb{R}^{m\times(N+1)}$ denotes the matrix obtained by appending the column h consisting of a standard gaussian vector $h \sim \mathcal{N}(0, \text{id})$ to Γ_g.

To deduce the result, we will first highlight the relation between one-bit compressed sensing and the ℓ^1/ℓ^2-restricted isometry property (RIP) and then have a glimpse at how to prove that the matrix $\frac{1}{m}\sqrt{\pi/2}R_I[\Gamma_g, h]$ satisfies an ℓ^1/ℓ^2-RIP estimate.

A. ℓ^1/ℓ^2-RIP and one-bit compressed sensing

The notion of ℓ^1/ℓ^2-RIP was introduced by Foucart [7]. Initially, this notion should serve as a short-cut to the theory of one-bit compressed sensing for unstructured Gaussian matrices.

Definition II.2. A matrix $A \in \mathbb{R}^{m\times N}$ *satisfies* $\text{RIP}_{1,2}(s, \delta)$ if

$$(1 - \delta)\|x\|_2 \leq \|Ax\|_1 \leq (1 + \delta)\|x\|_2, \qquad \text{for all } x \in \Sigma_{s,N}$$

and A *satisfies* $\text{RIP}_{1,2}^{\textit{eff}}(s, \delta)$ if

$$(1 - \delta)\|x\|_2 \leq \|Ax\|_1 \leq (1 + \delta)\|x\|_2, \qquad \text{for all } x \in \Sigma_{s,N}^{\text{eff}}.$$

Foucart observed that, if $A \in \mathbb{R}^{m\times N}$ satisfies $\text{RIP}_{1,2}^{\text{eff}}(9s, \delta)$ for $\delta \leq 1/5$, then, for every $x \in \Sigma_{N,s}^{\textit{eff}}$, any solution $x_{\text{LP}}^{\#}$ to (LP) satisfies the error bound $\|x - x_{\text{LP}}^{\#}\|_2 \leq 2\sqrt{5\delta}$ (cf. [7, Theorem 8]). This analysis can be extended to an analysis of (CP) by using arguments from [7, Section 8.4] and [2, Corollary 9].

Lemma II.3. *Let $\delta < 1/5$. Suppose that $C \in \mathbb{R}^{m\times(N+1)}$ satisfies $RIP_{1,2}^{\textit{eff}}(36(\sqrt{s} + 1)^2, \delta)$. Then, for any $x \in \mathbb{R}^N$ satisfying $\|x\|_1 \leq \sqrt{s}\|x\|_2$ and $\|x\|_2 \leq R$, any solution $x^{\#}$ to* (2) *satisfies*

$$\|x - x^{\#}\|_2 \leq 2R\sqrt{\delta}.$$

Proof: In the proof of [2, Corollary 9] it shown that

$$\|u - v\|_2 \leq 2\left\|\frac{[u, 1]}{\|[u, 1]\|_2} - \frac{[v, 1]}{\|[v, 1]\|_2}\right\|_2 \tag{3}$$

for any two vectors $u, v \in B_{\ell_2^N}$. Let $x \in \Sigma_{s,N}^{\text{eff}} \cap RB_{\ell_2^N}$, $x^{\#}$ be any solution to (2) and write

$$\bar{x} = [x, R]/\|[x, R]\|_2, \qquad \bar{x}^{\#} = [x^{\#}, R]/\|[x^{\#}, R]\|_2.$$

Since $x/R, x^{\#}/R \in B_{\ell_2^N}$, (3) implies that

$$\|x - x^{\#}\|_2 \leq 2R\|\bar{x} - \bar{x}^{\#}\|_2.$$

By the parallelogram identity,

$$\left\|\frac{\bar{x} - \bar{x}^{\#}}{2}\right\|_2^2 = \frac{\|\bar{x}\|_2^2 + \|\bar{x}^{\#}\|_2^2}{2} - \left\|\frac{\bar{x} + \bar{x}^{\#}}{2}\right\|_2^2. \tag{4}$$

Let us observe that $[x, R]$ and $[x^{\#}, R]$ are $(\sqrt{s}+1)^2$-effectively sparse. Indeed, by optimality of $x^{\#}$ for (2) and s-effective sparsity of x,

$$\|[x^{\#}, R]\|_1 \leq \|[x, R]\|_1 \leq \sqrt{s}\|x\|_2 + R \leq R(\sqrt{s} + 1)$$

and $\|[x, R]\|_2, \|[x^{\#}, R]\|_2 \geq R$. We claim that

$$z := \frac{\bar{x} + \bar{x}^{\#}}{2} \in \Sigma_{36(\sqrt{s}+1)^2, N+1}^{\text{eff}}. \tag{5}$$

Once this is shown, we can use $\text{sign}(C\bar{x}) = \text{sign}(C\bar{x}^{\#})$ and the $\text{RIP}_{1,2}^{\text{eff}}(36(\sqrt{s} + 1)^2, \delta)$-property of C to find

$$\left\|\frac{\bar{x} + \bar{x}^{\#}}{2}\right\|_2 \geq \frac{1}{1 + \delta}\left\|C\Big(\frac{\bar{x} + \bar{x}^{\#}}{2}\Big)\right\|_1 = \frac{\|C\bar{x}\|_1 + \|C\bar{x}^{\#}\|_1}{2(1 + \delta)} \geq \frac{(1 - \delta)}{(1 + \delta)}. \tag{6}$$

Hence, (4) implies

$$\left\|\frac{\bar{x}-\bar{x}^{\#}}{2}\right\|_2^2 \leq 1-\frac{(1-\delta)^2}{(1+\delta)^2}=\frac{4\delta}{(1+\delta)^2}.$$

Let us now prove (5). Since $[x,R]$ and $[x^{\#},R]$ are $(\sqrt{s}+1)^2$-effectively sparse,

$$\|z\|_1 \leq \frac{1}{2}\left\|\frac{[x,R]}{\|[x,R]\|_2}\right\|_1+\frac{1}{2}\left\|\frac{[x^{\#},R]}{\|[x^{\#},R]\|_2}\right\|_1 \leq \sqrt{s}+1.$$

It remains to bound $\|z\|_2$ from below. In (6) we already observed that

$$\|Cz\|_1=\frac{1}{2}\left\|\frac{C[x,R]}{\|[x,R]\|_2}\right\|_1+\frac{1}{2}\left\|\frac{C[x^{\#},R]}{\|[x^{\#},R]\|_2}\right\|_1 \geq (1-\delta). \quad (7)$$

Set $t=8s+8 \geq 4(\sqrt{s}+1)^2$. Let T_0 be the index set corresponding to the t largest entries of z, T_1 be the set corresponding to the next t largest entries of z, and so on. Then, for all $k \geq 1$,

$$\|z_{T_k}\|_2 \leq \sqrt{t}\|z_{T_k}\|_\infty \leq \|z_{T_{k-1}}\|_1/\sqrt{t}.$$

Since C satisfies $\text{RIP}^{\text{eff}}_{1,2}(36(\sqrt{s}+1)^2,\delta)$, it satisfies $\text{RIP}_{1,2}(t,\delta)$ and hence

$$\begin{aligned}\|Cz\|_1 &\leq \sum_{k\geq 0}\|Cz_{T_k}\|_1 \leq (1+\delta)\Big(\|z_{T_0}\|_2+\sum_{k\geq 1}\|z_{T_k}\|_2\Big)\\ &\leq (1+\delta)\|z\|_2+\frac{(1+\delta)}{\sqrt{t}}\|z\|_1\\ &\leq (1+\delta)\|z\|_2+\frac{(1+\delta)}{\sqrt{t}}(\sqrt{s}+1).\\ &\leq (1+\delta)\|z\|_2+\frac{1}{2}(1+\delta). \quad (8)\end{aligned}$$

Since $\delta \leq 1/5$, (7) and (8) together yield

$$\|z\|_2 \geq \frac{(1-\delta)-\frac{1}{2}(1+\delta)}{(1+\delta)}=\frac{\frac{1}{2}-\frac{3}{2}\delta}{1+\delta} \geq \frac{1}{6}.$$

■

This lemma reduces the analysis of recovery garuantees for $x \in \Sigma^{\mathit{eff}}_{N,s}$ to an analysis of the corresponding ℓ^1/ℓ^2-property of the matrix $\frac{1}{m}\sqrt{\frac{\pi}{2}}D_\theta[\Gamma_g,h]$.

B. The ℓ^1/ℓ^2-RIP for subsampled Gaussian circulant matrices

In this section we will complete the proof of Theorem II.1 by discussing that under the conditions of Theorem II.1 the matrix $\frac{1}{m}\sqrt{\frac{\pi}{2}}D_\theta[\Gamma_g,h]$ satisfies $\text{RIP}^{\mathit{eff}}_{1,2}(s,\delta)$.

Theorem II.4. *Fix $\delta>0$. Let $B=R_I[\Gamma_g\ h]$ be a randomly subsampled Gaussian circulant matrix with thresholds. Under the assumptions on s,m,N,δ,η of Theorem II.1 the matrix $\frac{1}{m}\sqrt{\frac{\pi}{2}}B$ satisfy $\text{RIP}^{\mathit{eff}}_{1,2}(s,\delta)$ with probability at least $1-\eta$.*

We will only sketch the proof of this result.

Sketch of Proof: Let $\mathcal{N}_\delta \subset \Sigma_{s,N+1}$ be a minimal δ-net for $\Sigma_{s,N+1}$ with respect to the Euclidean norm. Fix $x \in \Sigma_{s,N+1}$ and let $y \in \mathcal{N}_\delta$ be such that $\|x-y\|_2 \leq \delta$. We consider the following events:

$$E_{\text{RIP},B}=\Big\{\forall\ z \in \Sigma_{2s,N+1}\ :\ \frac{1}{\sqrt{m}}\|Bz\|_2 \leq 2+C\Big\}$$

$$E_{\Gamma,h,\ell_1}=\Big\{\forall y \in N_\delta\ :\ \Big|\frac{1}{N}\sqrt{\frac{\pi}{2}}\|[\Gamma_g\ h]y\|_1-1\Big| \leq \delta\Big\}$$

$$E_I=\Big\{\frac{m}{2} \leq |I| \leq \frac{3m}{2}\Big\}$$

$$E=\Big\{\forall y \in \mathcal{N}_\delta\ :\ \Big|\frac{1}{m}\sqrt{\frac{\pi}{2}}\|By\|_1-\frac{1}{N}\sqrt{\frac{\pi}{2}}\|[\Gamma_g,h]y\|_1\Big| \leq 2\delta\Big\}$$

We note that under the events E_I and $E_{\text{RIP},B}$ there is an absolute constant $C_0>0$, such that

$$\begin{aligned}&\Big|\frac{1}{m}\sqrt{\frac{\pi}{2}}\|Bx\|_1-\frac{1}{m}\sqrt{\frac{\pi}{2}}\|By\|_1\Big|\\ &\quad\leq \frac{\delta}{1+\kappa}\frac{1}{m}\sqrt{\frac{\pi}{2}}\Big\|B\Big(\frac{x-y}{\|x-y\|_2}\Big)\Big\|_1\\ &\quad\leq \frac{\delta}{1+\kappa}\frac{|I|}{m}\sup_{z\in\Sigma_{2s,N+1}}\frac{1}{\sqrt{|I|}}\sqrt{\frac{\pi}{2}}\|Bz\|_2 \leq C_0\delta.\end{aligned}$$

Hence, if all the events hold simultaneously, then the triangle inequality implies that there is an absolute constant $C_1>0$, such that

$$\begin{aligned}&\Big|\frac{1}{m}\sqrt{\frac{\pi}{2}}\|Bx\|_1-1\Big|\\ &\quad\leq \Big|\frac{1}{m}\sqrt{\frac{\pi}{2}}\|Bx\|_1-\frac{1}{m}\sqrt{\frac{\pi}{2}}\|By\|_1\Big|\\ &\quad+\Big|\frac{1}{m}\sqrt{\frac{\pi}{2}}\|By\|_1-\frac{1}{N}\sqrt{\frac{\pi}{2}}\|[\Gamma_g,h]y\|_1\Big|\\ &\quad+\Big|\frac{1}{N}\sqrt{\frac{\pi}{2}}\|[\Gamma_g,h]y\|_1-1\Big| \leq C_1\delta.\end{aligned}$$

With these observations the proof boilds down to show that the events $E_{\text{RIP},B}, E_{\Gamma,h,\ell_2}, E_{\Gamma,h,\ell_1}, E_I$ hold with probability at least $1-c\eta$ with respect to the net $\mathcal{N}_\delta$. The probability of the events E_I, E_{Γ,h,ℓ^2} can be estimated by standard arguments for Bernoulli random variables and Gaussian process, respectively. A sufficent estimate for the probability of the event $E_{\text{RIP},B}$ can be deduced form earlier results in [5]. Finally, the probability of the event E can be studied by using symmetrization methods. ■

Combining Lemma II.3 and Theorem II.4 the results we claimed in Theorem II.1 follows.

III. Limitations of the ℓ^1/ℓ^2-RIP

As we have seen in the preciding sections the $\text{RIP}_{1,2}(s,\delta)$ can be used to deduce recovery guarantees for randomly subsampled circulant matrices, which are generated by a Gaussian random vector. However the use of the ℓ^1/ℓ^2-RIP has certain limitations. In particular the analysis cannot be extend to the subgaussian regime. This can be seen by the following example of a matrix Γ_ε generated by a Rademacher vector ε. Suppose that the measurement matrix is a (rescaled, subsampled) Bernoulli circulant matrix, the threshold vector τ

in (1) is zero and consider, for $0 < \lambda < 1$, the normalized 2-sparse vectors

$$\begin{aligned} x_{+\lambda} &= (1+\lambda^2)^{-1/2}(1, \lambda, 0, \dots, 0), \\ x_{-\lambda} &= (1+\lambda^2)^{-1/2}(1, -\lambda, 0, \dots, 0). \end{aligned} \tag{9}$$

Then,

$$\begin{aligned} \operatorname{sign}(\langle (\Gamma_\varepsilon)_i, x_{+\lambda}\rangle) &= \operatorname{sign}(\varepsilon_{N+1-i} + \lambda \varepsilon_{N+2-i}) \\ &= \operatorname{sign}(\varepsilon_{N+1-i}) \\ &= \operatorname{sign}(\varepsilon_{N+1-i} - \lambda \varepsilon_{N+2-i}) \\ &= \operatorname{sign}(\langle (\Gamma_\varepsilon)_i, x_{-\lambda}\rangle) \ . \end{aligned}$$

This shows that $x_{+\lambda}$ and $x_{-\lambda}$ produce identical one-bit measurements. Suppose that $A = \alpha R_I \Gamma_\varepsilon$ satisfies this property for a suitable $I \subset [N]$ and scaling factor α. Since $\operatorname{sign}(Ax_{+\lambda}) = \operatorname{sign}(Ax_{-\lambda})$, we find using [7, Theorem 9] and the triangle inequality, that every solution $x^\#$ of the program (LP) has to satisfy

$$\begin{aligned} \frac{2\lambda}{(1+\lambda^2)^{1/2}} &= \|x_{+\lambda} - x_{-\lambda}\|_2 \\ &\le \|x_{+\lambda} - x^\#\|_2 \\ &+ \|x^\# - x_{-\lambda}\|_2 \\ &\le C\sqrt{\delta}. \end{aligned}$$

for some absolute constant $C > 0$. By taking $\lambda \to 1$ we find $\delta \ge \sqrt{2}/C$.

One might circumvent this counter example by excluding very sparse vectors. In fact, in the case of unstructured subgaussian random matrices, positive recovery results for sparse vectors with an additional ℓ_∞-norm constraint were shown in [1].

The ℓ^1/ℓ^2-RIP reflects this behavior by the fact that the expectations for 1-sparse vectors and s-sparse vectors (for $s > 1$) are incongruent. Let us examine this fact for the following general model: Let $A \in \mathbb{R}^{m\times N}$ denote a matrix consiting of m isotropic, independent, identically distributed subgaussian rows $a_i \in \mathbb{R}^N$. We will show that in this setting it, is impossible for A to satisfy $\text{RIP}^{\mathit{eff}}_{1,2}(s, \delta)$ for arbitrary $\delta > 0$ and large enough $s \ll N$.

For a 1-sparse vector $x = e_j \in \Sigma_{N,1}$ we have $m^{-1}\mathbb{E}\|Ax\|_1 = m^{-1}\sum_{i=1}^m \mathbb{E}|a_{ij}| = E_A$. This implies that in order to have the following inequality for all $\delta > 0$

$$(1-\delta)\|x\|_2 \le \alpha\|Ax\|_1 \le (1+\delta)\|x\|_2 \tag{10}$$

we have to demand $\alpha = E_A^{-1}$.

We claim that if $E_A \ne \sqrt{2/\pi}$, then $\text{RIP}_{1,2}(s, \delta)$ cannot hold for $\delta \to 0$.

To see this, let $\rho(A) = |\sqrt{2/\pi} - E_A|$ and for a set $J \subset [N]$ with $|J| = s$ we introduce the s-sparse vector

$$x = \frac{1}{\sqrt{s}}\mathbf{1}_J \ . \tag{11}$$

Let us observe the following fact concerning the expectation of $|\langle a_i, x\rangle|$ for all $i \in [m]$, when x is of the form given in (11):

$$\left|\mathbb{E}|\langle a_i, x\rangle| - \sqrt{\frac{2}{\pi}}\right| \le \frac{3^{\frac{3}{2}}}{\sqrt{s}} \ . \tag{12}$$

To show this, recall the following version of the Berry-Esseen inequality (see e.g. [11, Theorem 2.1.30]): Let $(X_j)_{j\in[s]}$ denote a sequence of independent random variables with mean 0, such that for all $j \in [s]$ we have $\sigma = (\sum_{j\in[s]} \|X_j\|^2_{L^2})^{1/2} < \infty$ and $\tau_i = \|X_j\|_{L^3} < \infty$, then for a standard gaussian random varible $g \sim \mathcal{N}(0, 1)$ there exists an absolute constant $C > 0$ such that

$$\int_{\mathbb{R}} \left|\mathbb{P}\left(\sigma^{-1}\sum_{j\in[s]} X_j > u\right) - \mathbb{P}\left(g > u\right)\right| du < C\frac{1}{\sigma^3}\sum_{j\in[s]}\tau_i^3 \ .$$

We observe that by the triangle inequality and the symmertry of the random varibles $\langle a_i, x\rangle$ and g the following estimate holds

$$\begin{aligned} \left|\mathbb{E}|\langle a_i, x\rangle| - \sqrt{\frac{2}{\pi}}\right| &= |\mathbb{E}|\langle a_i, x\rangle| - \mathbb{E}|g|| \\ &\le \left|\int_0^\infty \mathbb{P}(|\langle a_i, x\rangle| > u)du - \int_0^\infty \mathbb{P}(|g| \ge u)du\right| \\ &\le 2\int_0^\infty \big|\mathbb{P}(\langle a_i, x\rangle > u) - \mathbb{P}(g \ge u)\big|du \ . \end{aligned} \tag{13}$$

Further, note that $\mathbb{E}|\langle y, a_i\rangle|^2 = \|y\|_2^2$ for every $y \in \mathbb{R}^N$, implies that for all $i \in [m]$ we have $\mathbb{E}|a_{ij}|^2 = 1$ and $\|a_{ij}\|_{L^3} \le \sqrt{3}$. Since $\langle a_i, x\rangle = \frac{1}{\sqrt{s}}\sum_{j\in J} a_{ij}$ we can apply this result with $X_j = a_{ij}$. Thus, combining (13) with the Berry-Esseen inequality yields the estimate (12).

Since the $\langle a_i, x\rangle$ are subgaussian there is a constant $L > 0$, such that $\max_{i\in[m]}\|\langle a_i, x\rangle\|_{\psi_2} \le L$. Therefore, the (genearlized) Hoeffding's inequality yields,

$$\mathbb{P}\left(\Big|\frac{1}{m}\sum_{i=1}^m |\langle a_i, x\rangle - \mathbb{E}\frac{1}{m}\sum_{i=1}^m |\langle a_i, x\rangle\Big| > t\right) \le e^{-t^2 m/L} \ .$$

Thus, if $t \le 2^{-4}s^{-1/2}$, then with probability exceeding $1 - c_L e^{-m/s}$ we have

$$\Big|\frac{1}{m}\sum_{i=1}^m |\langle a_i, x\rangle| - \mathbb{E}\frac{1}{m}\sum_{i=1}^m |\langle a_i, x\rangle\Big| \le \frac{1}{2^4\sqrt{s}} \ .$$

Combining this with the prior estimate (12) shows that with owerwhelming probability

$$\Big|\frac{1}{m}\sum_{i=1}^m |\langle a_i, x\rangle| - \sqrt{\frac{2}{\pi}}\|x\|_2\Big| \le \frac{1 + 2^4 3^{\frac{3}{2}}}{2^4\sqrt{s}} \le \frac{2\cdot 3^{\frac{3}{2}}}{\sqrt{s}} \ . \tag{14}$$

In particular, on this event, we have

$$\begin{aligned} \delta_* &= \Big|E_A - \sqrt{\frac{2}{\pi}}\Big| \\ &\le \Big|\frac{1}{m}\sum_{i=1}^m |\langle a_i, x\rangle| - \sqrt{\frac{2}{\pi}}\|x\|_2\Big| \\ &\quad + \Big|\frac{1}{m}\sum_{i=1}^m |\langle a_i, x\rangle| - E_A\|x\|_2\Big| \\ &\le \frac{C}{\sqrt{s}} + \Big|\frac{1}{m}\sum_{i=1}^m |\langle a_i, x\rangle| - E_A\|x\|_2\Big| \ . \end{aligned}$$

Assuming that $s \ge 4C^2\delta_*^{-2}$, we find that $|\frac{1}{m}\sum_{i=1}^m |\langle a_i, x\rangle| - E_A\|x\|_2| \ge \delta_*/2$. This shows that the estimate in (10) cannot holds simultaneously for 1- and s-sparse vectors.

IV. CONCLUSION

We have seen that for partial Gaussian circulant matrices the program (CP) can be used to efficiently recover the direction and the energy of s-sparse signal form their one-bit measurements with probability at least $1-\eta$, provided that the sparsity does not exceed $s \lesssim \sqrt{\delta^4 N/\log(N)}$ for a polynomial scaling in N, say $N^{-\alpha}$ for $\alpha > 0$, of the probability η. Clearly the assumption $\sqrt{\delta^4 N/\log(N)}$ is a severe restriction. Note that for $\delta \le cN^{-1/4}$ this restriction excludes all reasonable scales for the sparsity s. It seem to be an interesting and necessary challange to extend the recovery result beyond the small sparsity regime.

The aspect of extending the theory beyond the theory of a Gaussian generator, i.e. the circulant matrix Γ_g is generated by a standard Gaussian $g \sim \mathcal{N}(0, \mathrm{id})$, was addressed in the last section. We have seen that in general the approach of passing through the ℓ^1/ℓ^2-RIP in order to study recovery garuantees for matrices consisting of subgaussian, isotropic rows might cause serious problems due to the fact that the expectaions for 1- and s-sparse vectors are incongruent. It might nevertheless be interesting to find scenarios in which recovery garuantees for subgaussian distributions hold.

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