Flatness and collocation based methodology for satellite trajectory planning

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Abstract— In the aim of providing Low Earth Orbit (LEO) satellite with a highly reliable path planner, we develop in this paper a methodology based on flatness and collocation. We prove here that we can cut down time of orbital maneuvers to increase duration of data acquisition. Simulation results demonstrate the potential and advantages of the proposed approach with respect to the commonly used Jerk profiles for slew maneuvers.

I. INTRODUCTION

The role of a path-planning unit within the control scheme is to determine the reference signals using a nominal model of the plant. In this paper, we will consider a satellite actuated by means of reaction wheels for its maneuver. The contribution of this paper is twofold. First, it is shown that the non-linear model of reaction wheels actuated satellite satisfies the flatness property. Second, a path planner methodology based on flatness (see for instance [1], [2]) and collocation ([3]) allow to reduce time of maneuver especially for large motion. The proposed method leads to a computationally efficient trajectory optimization technique. The developed approach can also be used for versatile real-time trajectory generation see for instance [4], [5].

This paper is organized as follows: in the following section, a quick overview of flatness definitions used in this work is presented. In section III, we show that with the right choice of orientation parameters, the satellite’s dynamic model is flat. In section IV, trajectory generation methodology and results are presented; comparisons with classical methods in satellite large rotation maneuvers are made.

II. SHORT REVIEW OF DIFFERENTIAL FLATNESS

Differential flatness, or shortly flatness, has been introduced by Fliess et al. [6] in 1992. It was first defined using the formalism of differential algebra where a system is viewed as differential field generated by state and input variables. Consider a non linear system:

\[ \ddot{x} = f(x, u) \]  

where \( x \) is the \( n \)-component state vector and \( u \) the \( m \)-component control vector. We assume that \( m \leq n \). The system is said to be flat if one can find a set of \( m \) variables such that the system is algebraic over the differential field generated by this set of variables. These variables are so-called flat outputs. Roughly speaking a system (1) is flat if we can find \( z \in \mathbb{R}^m \) such that all states and inputs can be determined from \( z \) components and a finite number their time derivatives without integration [6]:

\[ x = \phi(\dot{z}, z, \ldots, z^{(s)}) \]  
\[ u = \psi(\dot{z}, z, \ldots, z^{(s+1)}) \]

for a certain integer vector \( s = (s_1, \ldots, s_m) \). Moreover \( z \) is a function of \( x, u \) and a finite number of derivatives of \( u \):

\[ z = h(x, u, \dot{u}, \ddot{u}, \ldots, u^{(r)}) \]

for a certain integer \( r \).

More recently, flatness has been described using the formalism of infinite dimensional differential geometry of jets and prolongations [1]. In this geometric context, flatness is described in terms of notion of absolute equivalence. One approach is to use Lie-Bäcklund framework to define equivalence between systems. Flatness is defined as Lie-Bäcklund equivalence between a non linear system and a trivial system [1]. Besides the flat outputs are states of this trivial system. Therefore the components of \( z \) and their successive time derivatives \( \dot{z}, \ddot{z}, \ldots \) are independent. In this context, flatness can be described as property of a non linear system to have a linear geometric structure in spite of its natural non linear representation.

III. FLATNESS OF SATELLITE’S MODEL

A. Satellite model & State representation

1) Orientation parameters and kinematic equations: It is well known that three parameters are sufficient to describe a general rotation but they may bring some singularities for specified orientations (see for instance [7]). In [8] it is shown that three orientation parameters deduced from the stereographic projection of the four normalized orientation parameter quaternions’ unit sphere have only one singularity point. Interested reader can refer to [8] for more details about this last point. This kind of parameters is called Rodrigues parameters We are interested in this work in the modified Rodrigues parameters (MRP) presented in [9]. They go singular only for the principal rotation of \( \pm 300^\circ \). The kinematic differential equations of modified Rodrigues parameters \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T \) are defined by:
Note that equation (5) can also be written as $\dot{\sigma} = \Theta(\sigma)\omega$ where $\Theta(\sigma)$ is square matrix and $\omega = (\omega_1, \omega_2, \omega_3)^T$ is the velocity vector. We can verify that for all $\sigma$ equation (5) is always inversible.

Note that, as every stereographic projection, the shadow point $\sigma^S$ has the opposite qualitative behavior of $\sigma$ : they go singular for a zero rotation and go to zero at $\pm 360$ rotation. Their kinematic equations are:

$$\dot{\sigma}^S = \frac{1}{\sigma^T \sigma} \left( \bar{\sigma} - \frac{1}{2}(1 + \sigma^2)\sigma \times \omega - (1 - \sigma^2)\frac{\omega}{4} \right)$$

where $\sigma^2 = \sigma^T \sigma$. Consequently it’s possible to describe every rotation avoiding singularities and with only three parameters, using both $\sigma$ and $\sigma^S$ but with a discontinuity at switching point. A convenient way to switch is switching on the sphere $\sigma^T \sigma = 1$ where $\sigma = -\sigma^S$.

2) Dynamical equations: The satellite motion equations are deduced from the equation of kinetic momentum variation:

$$\left( \frac{dH_{tot}}{dt} \right)_{/Ri} = \sum \text{external torques}$$

where $H_{tot}$ is the total kinetic momentum of the satellite and $Ri$ the inertial frame. $(dH_{tot}/dt)_{/Ri}$ can also be written in the body frame $Rv$ as follow:

$$\left( \frac{dH_{tot}}{dt} \right)_{/Ri} = \left( \frac{dH_{tot}}{dt} \right)_{/Rv} + \Omega_{Re/Ri} \cdot H_{tot}$$

where $\Omega_{Re/Ri}$ is the rotation velocity of $Rv$ in respect of $Ri$. Actually $\Omega_{Re/Ri} = \hat{\omega}$. Due to the fact that the system considered consists of the body of satellite and the reaction wheels we can set that $H_{tot} = H_{body} + H_{int}$ where $H_{body} = I_{body} \cdot \dot{\omega}$ and $H_{int} = I_{wheel} \cdot \Omega_{wheel}$ are respectively kinetic momentum of the body and the reaction wheels. Then the only considered torques are internal to the system because they are produced by the reaction wheels so $H_{tot}/Ri = \text{constant}$. We can now write the dynamical equations of a satellite moved by reaction wheels:

$$I_{body} \ddot{\omega} + \dot{H}_{int} + \omega \cdot H_{tot} = 0$$

3) State representation: Let now establish the state representation of satellite with above parameters. So we set states and input vector as follow:

$$X = [\sigma_1, \sigma_2, \sigma_3, \omega_1, \omega_2, \omega_3]^T$$

$$U = [H_{int,1}, H_{int,2}, H_{int,3}]^T$$

Now, we can write explicitly the system $\dot{X} = f(X, U)$ from equations (5),(10):

$$\begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \\ \dot{\sigma}_3 \end{bmatrix} = \begin{bmatrix} (1 + \sigma_1^2 - \sigma_2^2 - \sigma_3^2)\omega_1 + 2(\sigma_1\sigma_2 - \sigma_3)\omega_2 \\ 2(\sigma_1\sigma_2 + \sigma_3)\omega_1 + (1 - \sigma_1^2 + \sigma_2^2 - \sigma_3^2)\omega_2 \\ 2(\sigma_1\sigma_3 - \sigma_2)\omega_1 + 2(\sigma_2\sigma_3 + \sigma_1)\omega_2 \\ \cdots + 2(\sigma_2\sigma_3 - \sigma_1)\omega_3 \\ \cdots \\ \cdots \end{bmatrix}$$

$$+ \begin{bmatrix} 2(\sigma_1\sigma_2 + \sigma_3)\omega_1 + (1 - \sigma_1^2 + \sigma_2^2 - \sigma_3^2)\omega_2 \\ 2(\sigma_1\sigma_3 - \sigma_2)\omega_1 + 2(\sigma_2\sigma_3 + \sigma_1)\omega_2 \\ \cdots + 2(\sigma_2\sigma_3 - \sigma_1)\omega_3 \\ \cdots \end{bmatrix}$$

Notice that by switching from $MRP$ to shadow $MRP$, we also replace (5) by (7).

B. Flatness parametrization

The problem of finding appropriate flat outputs and flatness parametrization is still an open problem. So the parametrization developed in this section is supposed to be valid for the model (13). Let $\tilde{z}$ be the vector of flat output chosen as:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$$

The mapping $h$ from equation (4) is set. We will note the vector $\bar{Z} = [\tilde{z}, \tilde{z}, \ldots, \tilde{z}^{(k)}, \ldots]$. By inverting equation (5), we obtain $\omega$ in respect of $\bar{Z}$:

$$\omega(\bar{Z}) = \Theta(\bar{Z})^{-1} \tilde{z}$$

The kinetic momentum conservation equation gives us the input variable in respect of $\bar{Z}$. Next, we can write the internal kinetic momentum as a function of the rotation velocity and total kinetic momentum:

$$\begin{bmatrix} H_{int,1} \\ H_{int,2} \\ H_{int,3} \end{bmatrix} = \begin{bmatrix} H_{tot,1} \\ H_{tot,2} \\ H_{tot,3} \end{bmatrix} - \Omega \begin{bmatrix} \omega_1(\bar{Z}) \\ \omega_2(\bar{Z}) \\ \omega_3(\bar{Z}) \end{bmatrix}$$

Finally, using the conservation of total kinetic momentum, we obtain the control torque. In this way:

$$\frac{\dot{H}_{tot}}{H_{tot}} = 0$$

$$\hat{U} = \frac{\dot{H}_{int}}{H_{int}} = -\Omega \ddot{\omega} - \omega \cdot H_{tot}$$

The flatness of model (13) is now straightforward since the existence of $\phi$ and $\psi$ (see equation (2) and (3)) is established. In the following section, this flatness property is used to generate trajectories. Note that the similar demonstration can be made with shadow $MRP$.

IV. TRAJECTORY GENERATION

In orbital control domain, the path planning design is usually based on linear approaches, for instance Bang-Bang or Jerk profiles.
A. Optimal Control Problem

The problem of trajectory generation can be formulated as Optimal Control Problem (OCP). The OCP in general terms consists of minimization of $J$ under consideration of trajectory constraints:

$$\min_u J(x, u, t) \quad \text{subject to:} \begin{cases} \dot{x} = f(x, u) \\ x(t_i) = x_i, \quad u(t_i) = u_i \\ x(t_f) = x_f, \quad u(t_f) = u_f \\ lb(t) \leq \Gamma(z(t), u(t)) \leq ub(t) \end{cases} \quad (19)$$

where $\dot{x} = f(x, u)$ represents the considered flat non linear system, the others constraints represent initial and final conditions and the constraints on the trajectory.

In the case that a model is proved to be flat, we can write all trajectories $(x(t), u(t))$ satisfying the dynamics of the non linear systems in terms of the flat outputs and its derivatives (cf. equations (2), (3)). The optimal control problem is then transformed into the problem of finding the mapping $t \mapsto Z(t)$ such that

$$\min_{\mathbb{Z}} J(Z) \quad \text{subject to:} \begin{cases} Z(t_i) = z_i \\ \dot{Z}(t) = \dot{z}_f \\ lb \leq \Gamma(Z(t)) \leq ub \end{cases} \quad (20)$$

The main problem when handling these constraints in optimization process, is that $S$ defined as $S = \{Z| lb \leq \Gamma(Z(t)) \leq ub\}$ is usually non convex.

Remark 1 A key feature is that integration is no more needed to find $(z(t), u(t))$. Moreover, since flatness properties set that flat output are state variables of a trivial system (chain of integrators), one can select any $C^\infty$ vector fields is convenient for mapping $t \mapsto z(t)$.

B. From OCP to NLP

In this section, a B-splines collocation based method is used to transform the original infinite dimensional OCP (22) into a computationally tractable Non Linear Programming (NLP). Collocation was first used for OCP in [3]. This method consists of a parametrization of the decision variables.

So we parameterize the components of the flat outputs $z$ as follow

$$z_i(t) = \sum_{k} C_{i,k} B_{i,k}(t), \quad i = 1, 2, 3 \quad (21)$$

where the $B_{i,k}$ are $k^{th}$ order B-spline, and $C_{i}$ coefficients are called control points. A $k^{th}$ order B-spline are $k$ piece-wise polynomials of $k - 1$ degree. All materials about B-splines in particular the Cox-deBoor iterative algorithm that define $B_{i,k}$ can be found in [10]. Indeed the control points $C$ become the decision variables.

Then we discretize the constraints problem. In fact the constraints will be enforced only at a finite number of point $\{\tau_j\}$ called collocation points. The choice of $\{\tau_j\}$ is then a degree of freedom in path planning process and involves some technical expertise in the technological constraint field.

Now, we can state the NLP to be solve

$$\min_{\mathbb{C}} J(C) \quad \text{subject to:} \begin{cases} C_1 = z_i \\ C_n = z_f \\ lb(\tau_j) \leq \Gamma(C) \leq ub(\tau_j) \end{cases} \quad (22)$$

V. SIMULATION RESULTS

As example, we will consider the problem of steering the satellite from an equilibrium point to another in a given duration $[t_i; t_f]$. Because of equilibrium, we have $\dot{x}(t_i) = 0$, $u(t_i) = 0$ and $\dot{x}(t_f) = 0$, $u(t_f) = 0$. Recall that one of the flatness properties is that non linear system in variable $z$ and the trivial system in variable $\dot{z}$ are Lie-Bäcklund equivalents [1]. So equilibrium points are preserved: all $\dot{z}$ derivatives are null. Initial guess of OCP can be every polynomial trajectories that satisfy equilibrium at the beginning and the end of the motion.

Simulations were run on Pentium 4 at 2.8GHz computer using Optimization and Spline toolbox of Matlab. The satellite model correponds to earth observer satellite like Jason family from Alcatel Alenia Space. The considered mission consists of maneuver which will pass the cardan angles from $(\theta_x = \theta_y = \theta_z = 0)$ to $(\theta_x = \theta_y = \theta_z = 30)$. The cost function to be minimized is chosen to be $J = \int_{t_i}^{t_f} \| \dot{H}_{int} \|_2$. This choice is motivated by energy consumption. We also consider saturation constraints on input variables $|H_{int,i}(t)| \leq H_{int,\max}$, $|\dot{H}_{int,i}(t)| \leq \dot{H}_{int,\max}$, $i = 1, 2, 3$. These constraints represent respectively the maximum rotation velocity of reaction wheels and the maximum torques can be applied. We set $H_{int,\max} = 10N.m.s$ et $\dot{H}_{int,\max} = 5N.m$.

For considered satellite, the results of the jerk path planning are given in figure 1. To giving comparisons we calculate $J = \int_{t_i}^{t_f} \| \dot{C}_{thrust} \|_2 = 123.2N.m.s$. One can notice that the overall duration of the mission is about 24 seconds. In figure 2 and 3, we present results for flat path planner for $6^{th}$ and $12^{th}$ order Bsplines. The overall duration of the mission is 20 seconds and cost function is $J = 50.47N.m.s$ and $J = 41.42N.m.s$. Note that although time of maneuver was not optimized, we get much better results in these cases.
VI. CONCLUDING REMARKS

After proving the flatness of reaction-wheel-actuated satellites with the right choice of orientation parameters, we proposed a tractable OCP technique based on collocation and non linear programming. We show that computation time is related with collocation parameters but optimality of solutions is very tight with the choice of these parameters too. Next step of our research are to detect global optimum using more efficient non linear programming algorithm and a convex approximation of the set of constraints.

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