CAPACITY APPROXIMATION FOR UNCORRELATED MIMO CHANNELS USING RANDOM MATRIX METHODS

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Abstract—In this paper, we present random matrix methods for determining the capacity of uncorrelated MIMO channels having partial channel state information, i.e., statistical knowledge of the channel parameters. We show that it is possible to approximate the MIMO channel capacities by Taylor series expansions in order to apply large-scale analysis using random matrix methods. For an adequate approximation quality, we introduce a new estimation method of the Taylor expansion points.

I. INTRODUCTION

The Multiple-Input Multiple-Output (MIMO) technology in communications engineering is expected to play a dominant part in future wireless communication systems, which is due to several reasons [1]. One of them is the spatial diversity at receiver (Rx) side, another is the high channel capacity for a limited bandwidth. Hence the available frequency range can be used in a more efficient way.

Random matrix (RM) theory [2], [3] is a tool for analyzing quantities and methods depending on random matrices, i.e., matrices of a certain and known size whose entries are independent, identically distributed (i. i. d.) random variables. At this point, it is important to mention that the respective kind of distribution of the random variables does not play any role at all. For instance, we can consider random matrices whose entries are zero-mean complex normal, zero-mean real normal distributed random variables, or zero-mean uniformly distributed discrete random variables [3], [4]. The only restriction that the random entries have to fulfill is the i. i. d. property and a vanishing mean value [2]. This enables the application of RM methods both to the analytical analysis of the asymptotic properties of large-scale code-multiplexed [4] and antenna array-based communication systems [3], [5]. One possible area of application are reduced-rank receive filters which have both been designed for a multi-user code division multiple access (CDMA) systems [4] and for MIMO systems [3], [5]. In this paper, we will explore the possibility of the additive white Gaussian noise (AWGN) MIMO channel capacity approximation with RM methods.

Compared to [3], we derive in this paper an alternative way of determining MIMO channel capacity approximations by means of an adequate application of Taylor series expansions. Additionally, we introduce a new estimation method for the expansion points for an adequate approximation quality. The random matrices are considered to be infinitely large, which enables conclusions about the asymptotic eigenvalue distribution of the product between the considered RM and its Hermitian.

Section II describes the used MIMO channel model. In the following section, the approximation for the channel capacity is introduced. In Section IV, the quality of the approximation for the MIMO channel capacity is investigated and shown by Monte-Carlo simulation results. Finally, Section V states conclusions.

II. CHANNEL MODEL

This section introduces the channel model that will be the basis for the calculations of the MIMO channel capacity having partial channel state information (CSI) only. A realization of the frequency-selective MIMO channel comprising $R$ Rx antennas and $T$ transmit (Tx) antennas can be described by the matrix-valued channel impulse response

$$H[k] = \sum_{i=0}^{L-1} H_i \delta[k - i],$$  \hspace{1cm} (1)

where $H[k] \in \mathbb{C}^{R \times T}$, $\delta[k]$ denotes the discrete unit impulse response. The channel impulse response in Eqn. (1) consists of matrix-valued taps $H_i \in \mathbb{C}^{R \times T}$, $i = 0, \ldots, L - 1$ which are all realizations of the random matrix $H \in \mathbb{C}^{R \times T}$. The random matrix $H$ consists of entries that are i. i. d. with the complex normal distribution $\mathcal{N} C(0, 1/(T RL))$. Finally, we introduce the noise model which is also essential for the MIMO channel capacity approximations in Section III. The receive signal vector $r[k] \in \mathbb{C}^R$ after transmission over the AWGN MIMO channel can be described as

$$r[k] = H[k] * s[k] + n[k].$$  \hspace{1cm} (2)

$\star$ denotes the convolution operator and the noise vector $n[k] \in \mathbb{C}^R$ is the realization of a complex normal distribution $\mathcal{N}_C(0_{R \times 1}, \sigma^2 I_R)$. As a consequence, we have $R_n = \sigma^2 I_R$ as noise covariance matrix.

III. MIMO CHANNEL CAPACITY APPROXIMATIONS

In this section, we derive a MIMO channel capacity approximation using RM theory. Since the principles and derivations are the same for frequency-flat and frequency-selective channels, we will only perform the derivation for the capacity approximations for $L = 1$ in Eqn. (1) (frequency-flat MIMO channel). Further we assume no channel knowledge at the transmitter and $T < R$. The capacity
of the uncorrelated MIMO channels (cf. Section II) is the relative channel capacity per Rx antenna and is given by [3]

$$C_{\text{MIMO}} = \frac{1}{R} \log_2 \det \left( I_T + H^H R^{-1} H \right). \quad (3)$$

We can simplify Eqn. (3) by using the assumption $R_n = \sigma^2 I_R$ (white Gaussian noise) and the scaled eigenvalue decomposition $H^H H T = Q \Lambda Q^H$:

$$C_{\text{MIMO}} = \frac{1}{R} \log_2 \det \left( I_T + \mathcal{A}/(\sigma^2 T) \right)$$

$$= \beta \frac{1}{T} \sum_{i=1}^{T} \ln \left( 1 + \lambda_i / (\sigma^2 T) \right) / \ln 2 \ \text{[bits/s/Hz]}. \quad (4)$$

The parameter $\beta = T/R$ represents the size ratio between columns and rows of the considered random channel tap matrices $H \in \mathbb{C}^{R \times T}$. Hence, Eqn. (4) gives an analytic expression of the relative MIMO channel capacity $C_{\text{MIMO}}$, which we have obtained from principles of linear algebra. The idea in the following is to apply the Taylor series expansion of the natural logarithm

$\ln(1 + x) \approx \ln(1 + x_0) + \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k(1 + x_0)^k} (x - x_0)^k, \ |x - x_0| < 1$ \quad (5)

to Eqn. (4). For numerical computations, the expansion is truncated after the power $N$. The powers $(x - x_0)^k$, $k = 1, \ldots, N$ can be expanded according to the binomial theorem

$$(x - x_0)^k = \sum_{n=0}^{k} \binom{k}{n} x^n (-1)^{n-k} x_0^{n-k}. \quad (6)$$

Thus, the truncated Taylor series in Eqn. (5) can be written as a polynomial of the form $\sum_{k=0}^{N} p_k x^k$ with $p_k, k = 0, \ldots, N$ dependent on $x_0$.

Since the quality of the approximation using the truncated Taylor series in Eqn. (5) depends very much on a correct choice of the expansion point $x_0$, we present a new method to determine a well-suited $x_0$. The idea is based on the theorem of Gerschgorin circles [6]: Given any Hermitian matrix $A \in \mathbb{C}^{n \times n}$, $A = a_{ij} |_{i,j=1}^n$, we obtain the following boundaries for the eigenvalues $\lambda_i$, $i = 1, \ldots, n$:

$$|\lambda_i - a_{ii}| \leq \sum_{j=1, j \neq i}^{n} |a_{ij}|. \quad (7)$$

Applying the theorem of Gerschgorin circles to Eqn. (4), we first replace $H$ in Eqn. (3) by the RM $H$. Using the expectation operator $E\{\cdot\}$ over the Gramian product of $H$, we have

$$E \left\{ H^H H \right\} = \begin{bmatrix} \frac{1}{T} & \cdots & \frac{1}{T} \\ \vdots & \ddots & \vdots \\ \frac{1}{T} & \cdots & \frac{1}{T} \end{bmatrix} \in \mathbb{C}^{T \times T}. \quad (8)$$

If we apply the statement given by Eqn. (8), we conclude $|\lambda_i| \leq 1$ for the eigenvalues in Eqn. (4). As a consequence, a good choice for the expansion point $x_0$ in Eqn. (5) is given by the reciprocal noise value $1/\sigma^2$.

Due to the linearization of Eqn. (4), the approximation yields

$$C_{\text{MIMO}} = \beta \frac{1}{\ln 2} \sum_{k=0}^{N} \frac{p_k}{(\sigma^2 T)^k} \left( \frac{1}{T} \sum_{i=1}^{T} \lambda_i^k \right). \quad (9)$$

Now we apply the RM theory to the MIMO channel capacity approximation. The Taylor expansion led to expressions of the form $R \sum_{i=1}^{T} \lambda_i^k$, $k = 1, \ldots, N$, which represents the $k$-th moment of the eigenvalue distribution. RM theory enables conclusions about the $k$-th statistical moment of the eigenvalue distribution by assuming $R$ and $T$ becoming infinitely large. As a result, we observe that the explicit realization of the channel matrix $H$ does not play any role any more.

In the following, we will derive the statistical moments of the considered eigenvalue distribution [3] in an alternative way [5], [7]. $R$ represents random matrices emerging from the Gramian products of $H \in \mathbb{C}^{R \times T}$. The key idea for all RM based methods is to assume the existence of an asymptotic eigenvalue distribution $F_R(x)$ [3], i.e.,

$$F_R(x) = \lim_{R \to \infty} \frac{1}{R} \left\{ \lambda_i : \lambda_i < x \right\}, \quad (10)$$

where $|\cdot|$ denotes the cardinality of a set. At this point, it is necessary to introduce the Stieltjes transform [2]

$$G_R(z) = \int \frac{1}{x - z} dF_R(x), \quad (11)$$

whose importance consists in the calculation of the statistical moments according to a series expansion [3]

$$G_R(z^{-1}) = \sum_{k=0}^{\infty} m_k z^k. \quad (12)$$

Having computed the Stieltjes transform, all required moments of the eigenvalue density are known. The entries in the RM $H \in \mathbb{C}^{R \times T}$ are assumed according to Section II to be distributed with expectation value 0 and a constant variance $1/R$ after an adequate rescaling. As a consequence, we can apply the Silverstein-Bai theorem [8] to calculate the needed Stieltjes transform $G_R(z)$. The Silverstein-Bai theorem deals with random matrices $R \in \mathbb{C}^{R \times R}$ composed of $H \in \mathbb{C}^{R \times T}$, $P \in \mathbb{C}^{T \times T}$ and $N \in \mathbb{C}^{R \times R}$ according to $R = N + HPH^H$. Performing the transition from finite size matrices to $T, R \to \infty$ with $T/R \to \beta$, $\beta < \infty$, then the following fix-point equation holds for $G_R(z)$:

$$G_R(z) = \int \frac{1}{x - z} f_R(x) dx$$

$$= G_R \left( z - \beta \int \frac{x f_P(x)}{1 + z G_R(x)} dx \right). \quad (13)$$

The Stieltjes transforms of $P = I_T$ and $N = 0_{R \times R}$ can be calculated as

$$G_N(z) = \int \frac{1}{x - z} f_N(x) dx = \frac{-1}{z}, \quad (14)$$

$$G_P(z) = \int \frac{1}{x - z} f_P(x) dx = \frac{1}{1 - z}. \quad (15)$$
leading to a square equation for $G_R(z)$ having the solution

$$G_R(z) = -\frac{1}{2} + \frac{\beta - 1}{2z} + \sqrt{\left(\frac{1-\beta}{4z^2} + \frac{1 + \beta}{2z}\right)}.$$  \hspace{1cm} (16)

We have to calculate $G_R(z^{-1})/(-z)$ and develop the result into power series. The binomial series expansion

$$\sqrt{1 + x} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{k}\right) x^k, |x| < 1$$

yields capacity values which are at most $0$ capacities. In the SNR range between $0$ dB and $20$ dB, there is no conclusion possible how the capacity difference depends on the number of Tx antennas: the choice of an adequate expansion point $x_0$ for the Taylor series in Eqn. (5) becomes difficult both for increasing SNR and a growing number of Tx antennas.

The values of the exact MIMO channel capacities and the corresponding RM-based capacity approximations for $R = 8$ Rx antennas are plotted versus SNR [dB] in Figure 1. We observe an increasing MIMO channel capacity per Rx antenna for an increasing SNR. By means of the adequate application of RM theory, we obtain MIMO channel capacity approximations which yield virtually the exact MIMO channel capacity in Eqn. (4) for SNR values between $-10$ dB and $0$ dB independent of the number of Tx antennas. Figure 2 shows the difference between the RM-based capacity approximations and the exact MIMO channel capacities. In the SNR range between $0$ dB to $20$ dB, we observe in both figures that the RM-based MIMO channel capacity estimation diverges from the exact MIMO channel capacity. The RM-based MIMO channel capacity estimation yields capacity values which are at most $0.37$ bits/s/Hz larger than the exact MIMO channel capacities. Moreover, in the SNR range between $0$ dB and $20$ dB, there is no conclusion possible how the capacity difference depends on the number of Tx antennas: the choice of an adequate expansion point $x_0$ for the Taylor series in Eqn. (5) becomes difficult both for increasing SNR and a growing number of Tx antennas.

The simulation plots in Figures 3 and 4 assume now a number of $R = 16$ Rx and $T = 2, 4, 6, 8$ Tx antennas in order to investigate the performance of the RM-based MIMO channel capacity approximations. Figures 3 and 4 show the same simulation results as Figures 1 and 2 but for $8$ Rx antennas, respectively. The observations in Figures 3 and 4 are similar to those already described for Figures 1 and 2. The transition from Figures 1 and 2 to Figures 3 and
4 reveals that the RM-based MIMO channel capacities are for some constellations closer to the exact MIMO channel capacities for increasing dimensions of the random MIMO channel matrices. This result confirms the expectations we had from the derivation of Eqn. (20). As for larger number of Rx antennas, the RM approximation of the empirical eigenvalue moments converges to the corresponding statistical ones, differences between $C_{\text{MIMO,RM}}$ and $C_{\text{MIMO}}$ mainly result from the truncated Taylor series and uncertainties in the Taylor expansion point $x_0$. From Figure 4, we see contrary to Figure 2 a monotonously increasing relationship between capacity difference and number of Tx antennas.

The simulation results point out inherent problems of the capacity approximation $C_{\text{MIMO,RM}}$ in Eqn. (20). The first one is the unit of the reciprocal noise value $1/\sigma^2$, which is given in dB and grows therefore by the power of 10. Especially for increasing SNR values $1/\sigma^2 \gg 1$, the estimate for the expansion point $x_0$ becomes worse.

The second problem has its origin in the application of the theorems concerning random matrices [8], which require a rescaling of the channel matrices as described in Section II. This scaling factor $T$ is paid attention to by replacing $\sigma^2$ by $T\sigma^2$. Hence the boundary in Eqn. (7) only permits the conclusion $|\lambda_i| \leq T$, which is not quite a sharp boundary. Nevertheless, we will choose the expansion points $x_0$ as reciprocal noise values $1/\sigma^2$ also for $T > 1$ although we know that this fact could slightly degrade the performance of our approximation.

V. CONCLUSIONS

In this paper, we have presented methods for determining MIMO channel capacities having partial CSI only. Random matrix theory enabled the calculation of capacity approximations by assuming infinitely large channel matrices. Especially in case of low SNR values and a relatively small number of Tx antennas compared to the number of Rx antennas, Monte-Carlo simulations showed that we can approximate the exact MIMO channel capacity very good by the adequate use of RM theory. Thereby, we resorted to the Taylor series expansion and introduced a new estimation method for the expansion point. The cases in which the approximation performance degraded—high SNR values and an increasing number of Tx antennas—are due to inherent construction properties (too rough determination of the expansion point for the truncated Taylor series).

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REFERENCES