Phase Retrieval via Structured Modulations in Paley-Wiener Spaces

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***Abstract*—This paper considers the recovery of continuous time signals from the magnitude of its samples. It uses a combination of structured modulation and oversampling and provides sufficient conditions on the signal and the sampling system such that signal recovery is possible. In particular, it is shown that an average sampling rate of four times the Nyquist rate is sufficient to reconstruct almost every signal from its magnitude measurements.**

***Index Terms*—Bernstein spaces, Paley-Wiener spaces, phase retrieval, sampling**

I. Introduction

In many applications, only intensity measurements are available to reconstruct a desired signal x. This is widely known as the phase retrieval problem which for example occurs in diffraction imaging applications such as X-ray crystallography, astronomical imaging or speech processing.

In the past, several efforts have been made on the recovery of finite n-dimensional signals from the modulus of their Fourier transform. In general however, they require strong limitations on the signal such as constraints on its z-transform [1] or knowledge of its support [2]. Analytic frame-theoretic approaches were considered in [3], [4] and an algorithm was presented which requires that the number of measurements grows proportionally with the square of the space dimension. Ideas of sparse signal representation and convex optimization where applied in [5], [6] to allow for lower computational complexity. Recently in [7], results in the context of entire functions theory have derived a sampling rate of $4n-4$.

Note that all of the above approaches addressed finite dimensional signals and the question is whether similar results can be obtained for continuous signals in infinite dimensional spaces. In [8] it was shown that real valued bandlimited signals are completely determined by their magnitude samples taken at twice the Nyquist rate. In the present work we are looking at complex valued continuous signals in Paley-Wiener spaces. Our approach extends ideas from [3], [4], [6] and involves two steps: first we apply a bank of modulators to the signal and sample the subsequent intensity measurements in the Fourier domain. In this step, finite blocks of intensity samples are obtained and a finite dimensional algorithm from [4] can be used to recover the complex signal samples up to a constant phase. Secondly, by ensuring an overlap between subsequent blocks, the unimodular factor in all blocks is matched and well-known interpolation theorems and the inverse Fourier transform are used to obtain the time signal. Therewith we are able to reconstruct the infinite dimensional signals from samples taken at a rate of four times the Nyquist rate, which asymptotically coincides with the value for the finite dimensional case in [4].

Basic notations for sampling and reconstruction in Paley-Wiener spaces are recaptured in Sec. II, Sec. III describes our sampling setup. In Sec. IV we provide sufficient conditions for perfect signal reconstruction from magnitude measurements of the Fourier transform. The paper closes with a short discussion in Sec.V.

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II. Sampling in Paley-Wiener Spaces

Let $\mathbb{S} \subseteq \mathbb{R}$ be an arbitrary subset of the real axis $\mathbb{R}$. For $1 \leq p \leq \infty$ we write $\mathcal{L}^p(\mathbb{S})$ for the usual Lebesgue space on $\mathbb{S}$. In particular, $\mathcal{L}^2(\mathbb{S})$ is the Hilbert space of square integrable functions on $\mathbb{S}$ with the inner product

$$\langle x, y\rangle_{\mathcal{L}^2(\mathbb{S})} = \int_{\mathbb{S}} x(\theta)\, \overline{y(\theta)}\, \mathrm{d}\theta\ ,$$

where the bar denotes the complex conjugate. In finite dimensional spaces $\langle x, y\rangle = y^* x$ where * denotes the conjugate transpose. Let $T > 0$ be a real number. Throughout this paper $\mathbb{T} = [-T/2, T/2]$ stands for the closed interval of length T, and $\mathcal{PW}_{T/2}$ denotes the *Paley-Wiener space* of entire functions of exponential type $T/2$ whose restriction to $\mathbb{R}$ belongs to $\mathcal{L}^2(\mathbb{R})$. The Paley-Wiener theorem states that to every $\widehat{x} \in \mathcal{PW}_{T/2}$ there is an $x \in \mathcal{L}^2(\mathbb{T})$ such that

$$\widehat{x}(z) = \int_{\mathbb{T}} x(t)\, \mathrm{e}^{\mathrm{i}tz}\, \mathrm{d}t \quad \text{for all } z \in \mathbb{C}\ , \tag{1}$$

and vice versa. If not otherwise noted, our signal space will be $\mathcal{L}^2(\mathbb{T})$, i.e. we consider signals of finite energy which are supported on the finite interval $\mathbb{T}$. These are natural assumptions for signals in reality. In the following we will call x the signal in the *time domain* and $\widehat{x}$ the signal in the *Fourier domain*, since its restriction to the real axis is a Fourier transform.

A sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ of complex numbers is said to be *complete interpolating* for $\mathcal{PW}_{T/2}$ if and only if

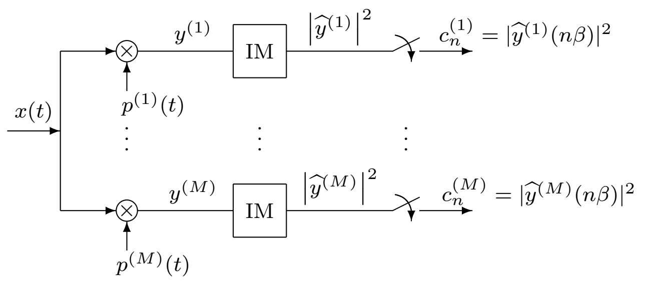

Fig. 1. Measurement setup: In each branch, the unknown signal x is modulated with a different sequence $p^{(m)}$, $m = 1, 2, \dots, M$. Subsequently, the intensities of the resulting signals $y^{(m)}$ are measured (IM) and uniformly sampled in the frequency domain.

the functions $\{\phi_n(t) := \mathrm{e}^{-\mathrm{i}\lambda_n t}\}_{n\in\mathbb{Z}}$ form a Riesz basis for $\mathcal{L}^2(\mathbb{T})$ [9]. Let $x \in \mathcal{L}^2(\mathbb{T})$ be arbitrary. Then (1) shows that

$$\widehat{x}(\lambda_n) = \langle x, \phi_n\rangle_{\mathcal{L}^2(\mathbb{T})} \quad \text{for all } n \in \mathbb{Z}\ .$$

Since $\{\phi_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{L}^2(\mathbb{T})$ the signal x can be reconstructed from the samples $\widehat{x}(\Lambda) = \{\widehat{x}(\lambda_n)\}_{n\in\mathbb{Z}}$ by

$$x(t) = \textstyle\sum_{n\in\mathbb{Z}} \langle x, \phi_n\rangle\ \psi_n(t) = \sum_{n\in\mathbb{Z}} \widehat{x}(\lambda_n)\, \psi_n(t)\ , \quad (2)$$

where $\{\psi_n\}_{n\in\mathbb{Z}}$ is the unique dual Riesz basis of $\{\phi_n\}_{n\in\mathbb{Z}}$ [10]. It is well-known that in the Fourier domain

$$\widehat{\psi}_n(z) = \frac{S(z)}{S'(\lambda_n)(z-\lambda_n)} \text{ with } S(z) = z^{\delta_\Lambda} \lim_{R\to\infty} \prod_{\substack{|\lambda_n|<R \\ \lambda_n\neq 0}} \Big(1 - \frac{z}{\lambda_n}\Big)$$

with $\delta_\Lambda = 1$ if $0 \in \Lambda$ and $\delta_\Lambda = 0$ otherwise. S is an entire function of exponential type $T/2$, and the infinite product converges uniformly on compact subsets of $\mathbb{C}$ if Λ is a complete interpolating sequence (see [11]).

Example 1: The well known Shannon sampling series is obtained for regular sampling with $\lambda_n = n\frac{2\pi}{T}$, $n \in \mathbb{Z}$. Then $S(z) = \sin(\frac{T}{2}z)$ and $\widehat{\psi}_n(z) = \operatorname{sinc}(\frac{T}{2}[z - n\frac{2\pi}{T}])$ where $\operatorname{sinc}(x) := \sin(x)/x$. This corresponds to $x(t) = \sum_{n\in\mathbb{Z}} \widehat{x}(\lambda_n)\, \mathrm{e}^{-\mathrm{i}n\frac{2\pi}{T}t} \mathbb{1}_{\mathbb{T}}(t)$ in the time domain, where $\mathbb{1}_{\mathbb{T}}(t)$ denotes the indicator function on $\mathbb{T}$.

III. Measurement Methodology

We apply a measurement methodology which uses oversampling in connection with structured modulations of the desired signal, inspired by the approach in [6]. Suppose $x \in \mathcal{L}^2(\mathbb{T})$ is the signal of interest. In our sampling scheme in Fig. 1, we assume that x is multiplied with M known modulating functions $p^{(m)}$. In optics, these modulations may be different diffraction gratings between the object (the desired signal) and the measurement device [6]. This way we obtain a collection of M representations (or illuminations) $y^{(m)}$ of x. Afterwards, the modulus of the Fourier spectra $\widehat{y}^{(m)}$ are measured and uniformly sampled with frequency spacing β.

Let $p^{(m)}$ have the following general form

$$p^{(m)}(t) := \textstyle\sum_{k=1}^{K} \overline{\alpha_k^{(m)}}\, \mathrm{e}^{\mathrm{i}\lambda_k t} \quad (3)$$

where λ_k and $\alpha_k^{(m)}$ are complex coefficients. The samples in the mth branch are then given by

$$\begin{aligned} c_n^{(m)} &= |\widehat{y}^{(m)}(n\beta)|^2 = \left|\sum_{k=1}^{K} \overline{\alpha_k^{(m)}}\, \widehat{x}(n\beta + \lambda_k)\right|^2 \\ &= |\langle \widehat{\mathbf{x}}_n, \boldsymbol{\alpha}^{(m)}\rangle|^2 \end{aligned} \quad (4)$$

with the length K vectors

$$\boldsymbol{\alpha}^{(m)} := \begin{pmatrix} \alpha_1^{(m)} \\ \vdots \\ \alpha_K^{(m)} \end{pmatrix} \quad \text{and} \quad \widehat{\mathbf{x}}_n := \begin{pmatrix} \widehat{x}(n\beta + \lambda_1) \\ \vdots \\ \widehat{x}(n\beta + \lambda_K) \end{pmatrix}.$$

We will show that if $\boldsymbol{\alpha}^{(m)}$ and the interpolation points $\{\lambda_{n,k} := n\beta + \lambda_k\}_{n\in\mathbb{Z}}^{k=1,\dots,K}$ are properly chosen, it is possible to reconstruct x from all samples $\mathbf{c} = \{c_n^{(m)}\}_{n\in\mathbb{Z}}^{m=1,\dots,M}$.

A. Choice of the coefficients $\alpha_k^{(m)}$

The first recovery step determines the vector $\widehat{\mathbf{x}}_n \in \mathbb{C}^K$ from the M intensity measurements $c_n^{(m)}$ for every $n \in \mathbb{Z}$, using a result from [4]. It states that if the family of $\mathbb{C}^K$-vectors $\mathcal{A} = \{\boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\alpha}^{(M)}\}$ constitutes a 2-uniform M/K-tight frame which contains $M = K^2$ vectors or $\mathcal{A}$ is a union of $K+1$ mutually unbiased bases in $\mathbb{C}^K$, then every $\widehat{\mathbf{x}}_n \in \mathbb{C}^K$ can be reconstructed up to a constant phase from the magnitude of the inner products (4). For simplicity, we only discuss the first case here and therefore fix $M = K^2$.

Condition A: *A sampling system as in Fig. 1 is said to satisfy Condition A if $\mathcal{A}$ constitutes a 2-uniform M/K-tight frame.*

Then reconstruction will be based on the following formula

$$Q_{\widehat{\mathbf{x}}_n} = \frac{(K+1)}{K} \sum_{m=1}^{M} c_n^{(m)} Q_{\boldsymbol{\alpha}^{(m)}} - \frac{1}{K} \sum_{m=1}^{M} c_n^{(m)} I \quad (5)$$

with rank-1 matrices $Q_{\mathbf{x}} = \mathbf{x}\mathbf{x}^*$. For $K = 2$ a valid choice for $\mathcal{A}$ reads [4]

$$\boldsymbol{\alpha}^{(1)} = \begin{pmatrix} a \\ b \end{pmatrix},\ \boldsymbol{\alpha}^{(2)} = \begin{pmatrix} b \\ a \end{pmatrix},\ \boldsymbol{\alpha}^{(3)} = \begin{pmatrix} a \\ -b \end{pmatrix},\ \boldsymbol{\alpha}^{(4)} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

with $a = \sqrt{\frac{1}{2}(1 - \frac{1}{\sqrt{3}})}$ and $b = \mathrm{e}^{\mathrm{i}5\pi/4}\sqrt{\frac{1}{2}(1 + \frac{1}{\sqrt{3}})}$.

B. Choice of the interpolation points

Now it is necessary to find conditions which allow unique interpolation from the known samples. Let $\{\lambda_k\}_{k=1}^{K}$ be ordered increasingly by their real parts. For each $n \in \mathbb{Z}$, the vector $\widehat{\mathbf{x}}_n$ contains the values of $\widehat{x}$ at K distinct interpolation points in the complex plane

$$\boldsymbol{\lambda}_n^a := \{\lambda_{n,k}^a\}_{k=1}^{K} \quad \text{with} \quad \lambda_{n,k}^a = n\beta + \lambda_k\ , \quad n \in \mathbb{Z}\ . \quad (6)$$

Therein, the parameter $a \in \mathbb{N}$ denotes the number of overlapping points of consecutive sets (6) (cf. also Fig.2). More precisely, we require for every $n \in \mathbb{Z}$ that

$$\lambda_{n,i}^a = \lambda_{n-1,K-i+1}^a \quad \text{for all } i = 1, \dots, a\ . \quad (7)$$

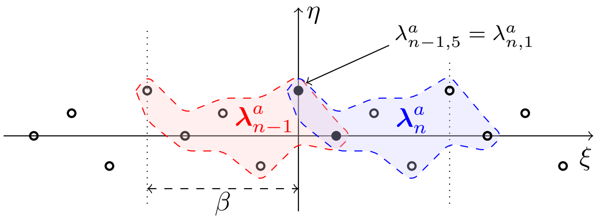

Fig. 2. Illustration for the choice of interpolation points in the complex plane for $K = 6$ in (3) and an overlap $a = 2$.

In the following $\Lambda^a_{O,n} = \boldsymbol{\lambda}^a_n \cap \boldsymbol{\lambda}^a_{n+1}$ is the set of overlapping interpolation points between $\boldsymbol{\lambda}^a_n$ and $\boldsymbol{\lambda}^a_{n+1}$, and we define the overall interpolation sequence

$$\Lambda^a := \bigcup_{n\in\mathbb{Z}} \boldsymbol{\lambda}^a_n \,.$$

In general we allow for $a \geq 1$, but we will see that $a = 1$ is generally sufficient for reconstruction.
As explained in Sec. II, $x \in \mathcal{L}^2(\mathbb{T})$ can be perfectly reconstructed by (2) if Λ^a is complete interpolating for $\mathcal{PW}_{T/2}$. This gives a second condition on our sampling system:

Condition B: *A sampling system as in Fig. 1 is said to satisfy Condition B if the coefficients* $\{\lambda_k\}_{k=1}^K$ *in* (3) *are such that* Λ^a *is complete interpolating for* $\mathcal{PW}_{T/2}$ *and satisfies* (7) *for a certain* $1 \leq a < K$.

In general it is hard to characterize sets which fulfill this condition. One famous example is the set of zeros of a sine-type function of type $\tilde{T}/2 \geq T/2$ which is β-periodic (see, e.g., [9], [11]). Such sine-type functions are entire functions f of exponential type $\tilde{T}/2$ with simple and isolated zeros and for which there exist positive constants A, B, H such that

$$A\,\mathrm{e}^{\frac{\tilde{T}}{2}|\eta|} \leq |f(\xi + \mathrm{i}\eta)| \leq B\,\mathrm{e}^{\frac{\tilde{T}}{2}|\eta|}\,, \quad \text{for } |\eta| \geq H\,.$$

Note that $\sin(\frac{\tilde{T}}{2}z)$ is a trivial example for a sine-type function (cf. Example 1). Moreover, shifting the zeros of one sine-type functions arbitrarily in their imaginary parts yields the zero set of another sine-type function [12]. The complete interpolating property is also preserved under small shifts in the real part (see Katsnelson's theorem, e.g. in [11]).

IV. Phaseless Signal Recovery

We assume a sampling scheme as described in Section III which satisfies Condition A and B. For this setup, we show that almost every $x \in \mathcal{L}^2(\mathbb{T})$ (up to a set of first category) can be reconstructed from the samples (4). The proof provides an explicit algorithm for perfect signal recovery.

Theorem 1: *Let* $x \in \mathcal{L}^2(\mathbb{T})$ *be sampled according to the scheme in Section III which satisfies Condition A and B, and let* $\mathbf{c} = \{c_n^{(m)}\}_{n\in\mathbb{Z}}^{m=1,\dots,M}$ *be the sampling sequence in* (4)*. If the set* $\widehat{x}(\Lambda^a_{O,n})$ *contains at least one non-zero element for each* $n \in \mathbb{Z}$*, then* x *can be perfectly reconstructed from* $\mathbf{c}$ *up to a constant phase.*

Proof: According to Condition B of the sampling system, Λ^a is complete interpolating for $\mathcal{PW}_{T/2}$. Therefore the signal x can be reconstructed from the vectors $\{\widehat{\mathbf{x}}_n\}_{n\in\mathbb{Z}}$ using (2). It remains to show that $\{\widehat{\mathbf{x}}_n\}_{n\in\mathbb{Z}}$ can be determined from $\boldsymbol{c}$.
Let $n \in \mathbb{Z}$ be arbitrary. Since the sampling system satisfies Condition A, we can use (5) to obtain the rank-1 matrix $Q_n := \widehat{\mathbf{x}}_n \widehat{\mathbf{x}}_n^*$ from the measurements $\{c_n^{(m)}\}_{m=1}^M$. Then $\widehat{\mathbf{x}}_n \in \mathbb{C}^K$ is obtained by factorizing Q_n. However, such a factorization is only unique up to a constant phase factor. If the phase $\phi_{n,i}$ of one element $[\widehat{\mathbf{x}}_n]_i$ is known, the vector $\widehat{\mathbf{x}}_n$ can be completely determined from Q_n by

$$\widehat{x}(n\beta + \lambda_k) = \sqrt{[Q_n]_{k,k}}\,\mathrm{e}^{\mathrm{i}(\phi_{n,i} - \arg([Q_n]_{i,k}))}, \;\; \forall k \neq i\,. \quad (8)$$

Assume that we start the recovery of the sequence $\{\widehat{\mathbf{x}}_n\}_{n\in\mathbb{Z}}$ at a certain $n_0 \in \mathbb{Z}$ and set the constant phase of $\widehat{\mathbf{x}}_{n_0}$ arbitrarily to $\theta_0 \in [-\pi, \pi]$. In the next step, we determine $\widehat{\mathbf{x}}_{n_0+1}$. After the factorization of Q_{n_0+1}, we use the nonempty overlap to carry over the phase from n_0 to $n_0 + 1$. Since by assumption the overlapping point, say $\lambda^a_{n_0+1,i}$, can be chosen such that it is non-zero, the propagation of the constant phase can be ensured. Thus, we can completely determine $\widehat{\mathbf{x}}_{n_0+1}$ and successively all $n = n_0 \pm 1, n_0 \pm 2, \dots$ using (8) to obtain $\widehat{x}(\Lambda^a)\,\mathrm{e}^{\mathrm{i}\theta_0}$. The arbitrary setting of the phase of the initial vector $\widehat{\mathbf{x}}_{n_0}$ yields a constant phase shift θ_0 for all $\widehat{\mathbf{x}}_n$ which persists after the reconstruction of the time signal as in (2). ∎

Theorem 1 states that $x \in \mathcal{L}^2(\mathbb{T})$ can only be reconstructed if $\widehat{x} \in \mathcal{PW}_{T/2}$ has at most $a - 1$ zeros on the overlapping interpolation sets $\Lambda^a_{O,n}$. However, this restriction is not too limiting. On the one hand, it is not hard to see that the subset of all $x \in \mathcal{L}^2(\mathbb{T})$ which does not satisfy this condition is of first category [13]. On the other hand, it is known that the zeros of an entire function of exponential type can not be arbitrarily dense. For example, defining $\mathcal{Z}_n := \{z \in \mathbb{C} : n\pi/T < |z| \leq (n+1)\pi/T\}$, the result in [14] states that for every $\widehat{x} \in \mathcal{PW}_{T/2}$ there exist only finitely many sets $\mathcal{Z}_n$ which contain more than one zero of $\widehat{x}$. Consequently, choosing the spacing of the interpolation points in the overlapping sets $\Lambda^a_{O,n}$ less than π/T, it is very unlikely that a randomly chosen function from $\mathcal{PW}_{T/2}$ fails to satisfy the condition of Theorem 1, especially for $a > 1$.

When the overall energy of the signal is known, even such pathological cases can be avoided such that the last condition in Theorem 1 always holds true. To this end, we first state a simple variant of a lemma by Duffin, Schaeffer [15].

Lemma 2: *Let* $\widehat{x}(z) \in \mathcal{PW}_{T/2}$ *be an entire function of* $z = \xi + \mathrm{i}\eta$ *satisfying* $|\widehat{x}(\xi)| \leq M$ *on the real axis. Then for every* $T' > T$ *the function*

$$\widehat{v}(z) = M\cos(\tfrac{T'}{2}z) - \widehat{x}(z) \quad (9)$$

belongs to the Bernstein space $\mathcal{B}^{\infty}_{T'/2}$ and there exists a constant $H = H(T,T')$ such that $|\widehat{v}(z)| > 0\ \forall z : |\eta| > H$.

A proof can be found in [13]. The Bernstein space $\mathcal{B}^{\infty}_{T'/2}$ is the set of all entire functions of exponential type $T'/2$ whose restriction to $\mathbb{R}$ is in $\mathcal{L}^{\infty}(\mathbb{R})$. Upon this we can establish a corollary for signals which have a known maximal energy W_0.

Corollary 3: *Let $x \in \mathcal{L}^2(\mathbb{T}) : \|x\|_{\mathcal{L}^2(\mathbb{T})} \leq W_0$ be sampled according to the scheme in Sec. III. Then there exist interpolation sequences Λ^a with overlap $a \geq 1$ such that every x can be perfectly reconstructed (up to a constant phase) from the measurements* (4).

Sketch of proof: The theorem of Plancherel-Pólya implies that there exists a constant M independent of x such that $|\widehat{x}(\xi)| \leq MW_0$ for all $\xi \in \mathbb{R}$. Using $T' > T$ we can define $\widehat{v}$ by (9) which only has zeros for $|\eta| \leq H$ by Lemma 2. In the measurement scheme this corresponds to adding a cosine to the signal. Subsequently, the function $\widehat{v}$ is modulated and sampled at interpolation points Λ^a, which we choose as the zero set of a sine-type function of type $\tilde{T}/2 > T'/2$. By [12] we can shift the imaginary parts of the interpolation points such that $|\eta_k| > H$ for all k while Λ^a remains to be the zero set of a sine-type function denoted by S. Since $\widehat{v} \in \mathcal{B}^{\infty}_{T'/2}$ and Λ^a is the set of zeros of a sine-type function, the sequence $\{d_n = \widehat{v}(\lambda_n)\,\mathrm{e}^{\mathrm{i}\theta_0}\}_{n\in\mathbb{Z}}$ is in ℓ^{∞}, and we apply a generalization of [11, Lec. 21] (see [13]) to reconstruct $\widehat{v}$ from the sequence $\{d_n\}_{n\in\mathbb{Z}}$ by

$$\widehat{v}(z)\,\mathrm{e}^{\mathrm{i}\theta_0} = \textstyle\sum_{n\in\mathbb{Z}} d_n \frac{S(z)}{S'(\lambda_n)} \left[\frac{1}{z-\lambda_n} + \frac{1}{\lambda_n}\right] ,$$

where the second term in the sum is omitted when $\lambda_n = 0$. Since θ_0 is unknown, we can only obtain

$$\begin{aligned}\tilde{x}(z) &= MW_0 \cos\left(\tfrac{T'}{2}z\right) - \widehat{v}(z)\,\mathrm{e}^{\mathrm{i}\theta_0}\\ &= \widehat{x}(z)\,\mathrm{e}^{\mathrm{i}\theta_0} + MW_0\cos\left(\tfrac{T'}{2}z\right)(1-\mathrm{e}^{\mathrm{i}\theta_0}).\end{aligned}$$

However, applying the inverse Fourier transform yields $x(t)\,\mathrm{e}^{\mathrm{i}\theta_0}$ for $t \in \mathbb{T}$ which is the desired signal up to a constant phase since the distributional Fourier transform of a cosine vanishes within $\mathbb{T}$. ■

V. Discussion and Outlook

To determine the sampling system in Fig.1, one has to fix K, M, a and β. The number $K \geq 2$ can be chosen arbitrarily. Then $M = K^2$ is fixed, and $1 \leq a \leq K-1$. The sampling period β has to be chosen such that the sampling system satisfies Condition B and in particular that Λ^a is complete interpolating for $\mathcal{PW}_{T/2}$. As discussed before, one possible choice could be the zeros of the function $\sin(\frac{\tilde{T}}{2}z)$ with $\tilde{T} > T' > T$. Then $\delta := \lambda_k - \lambda_{k-1} = 2\pi/\tilde{T}$ such that $\beta = (K-a)\,\delta$, and the total sampling rate becomes

$$R(a,K,\tilde{T}) = \frac{M}{\beta} = \frac{K^2}{(K-a)\,\delta} = \frac{K^2}{K-a}\frac{\tilde{T}}{2\pi} = \frac{K^2}{K-a}\frac{\tilde{T}}{T}R_{\mathrm{Ny}}$$

where $R_{\mathrm{Ny}} := T/(2\pi)$ is the Nyquist rate. It is apparent that $R(a,K,\tilde{T})$ grows asymptotically proportional with K and increases with the overlap a. $R(a,K,\tilde{T})$ is bounded below by

$$\inf_{\substack{1\leq a<K,\\ K\geq 1,\tilde{T}>T}} R(a,K,\tilde{T}) = \inf_{\tilde{T}>T} R(1,2,\tilde{T}) = 4R_{\mathrm{Ny}}\,.$$

Since $\tilde{T}/T$ can be made arbitrarily close to 1 using Theorem 1 and Corollary 3, we can sample at a rate which is almost as small as $4R_{\mathrm{Ny}}$ while still ensuring perfect reconstruction. This corresponds to the findings in [3] for finite dimensional spaces, where it was shown that basically any $x \in \mathbb{C}^N$ can be reconstructed from $M \geq 4N-2$ magnitude samples.

We note that the above framework can be applied exactly the same way for bandlimited signals. To this end, one only has to exchange the time and frequency domain. Then the modulators in Fig. 1 have to be replaced by linear filters and the sampling of the magnitudes has to be done in the time domain. In future works, our approach will be extended to larger signal spaces [13] and the influence of sampling errors will be investigated in detail.

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