Optimal Interpolation Laws for Stable AR(1) Processes

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***Abstract*—In this paper, we focus on the problem of interpolating a continuous-time AR(1) process with stable innovations using minimum average error criterion. Stable innovations can be either Gaussian or non-Gaussian. In the former case, the optimality of the exponential splines is well understood. For non-Gaussian innovations, however, the problem has been all too often addressed through Monte Carlo methods. In this paper, based on a recent non-Gaussian stochastic framework, we revisit the AR(1) processes in the context of stable innovations and we derive explicit expressions for the optimal interpolator. We find that the interpolator depends on the stability index of the innovation and is linear for all stable laws, including the Gaussian case. We also show that the solution can be expressed in terms of exponential splines.**

I. Introduction

Autoregressive (AR) processes are popular tools for modeling natural phenomena such as speech signals [1]. The processes are usually characterized by an all-pole filter that acts on the innovation process (white excitation noise). They are indexed by the number n of poles of the filter, as AR(n). The AR family contains both stationary and non-stationary models.

The AR processes were historically founded upon Gaussian statistics. Extensions to non-Gaussian scenarios were introduced later, for instance in financial applications, where the data follow a fat-tailed distribution [3], [4]. Besides, fat-tailed distributions are promising models for representing sparse/compressible data [5]. This fact is recently employed in the framework of *sparse stochastic processes* [6], [7] which proposes a unified approach towards Gaussian and non-Gaussian cases.

The estimation problems arising from AR processes are conventionally studied in a finite-dimensional state-space, resulting in the Kalman filter. Under Gaussian statistics, the Kalman filter coincides with the Bayesian estimator (posterior mean estimator) that minimizes the mean-square error. In non-Gaussian scenarios, however, it is common to either apply the Bayesian estimator on approximated posterior distributions [8], [9], [10] or to realize the Bayesian filter numerically [11], [12], [13].

In this paper, we focus on continuous-time AR(1) processes and investigate the problem of Bayesian interpolation between the samples. Our formulation is based on the characteristic forms introduced in [6]. We show that the Bayesian interpolator is linear with respect to the samples when the process follows a symmetric α-stable distribution. The demonstration of linearity is constructive, in the sense that we derive explicit forms for the Bayesian interpolator.

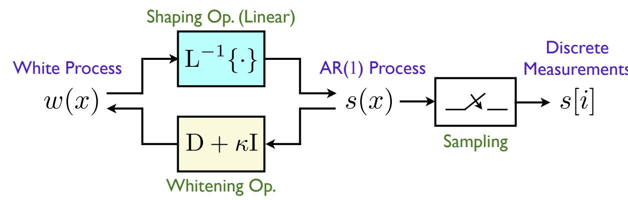

Fig. 1. Generation of the stochastic AR(1) process $s(x)$ based on the excitation white noise $w(x)$. The inverse linear operator L^{-1} includes the possible boundary condition.

II. AR(1) Model

The model in this paper is a special case of [6] adapted for AR(1) processes. The schematic diagram of the continuous-time model is given in Figure 1. The process of interest, s, satisfies the stochastic differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}s(x) + \kappa s(x) = w(x), \tag{1}$$

where w is a stationary white α-stable excitation with $1 \leq \alpha \leq 2$ and $\kappa \in \mathbb{R}$ is a constant. Equation (1) suggests the filter $\mathrm{D} + \kappa\mathrm{I}$ as the whitening operator, where D and I stand for the derivative and identity operators, respectively. This whitening operator has a one-dimensional null space spanned by the function $\mathrm{e}^{-\kappa x}$.

For a proper definition of the process, the shaping operator L^{-1} (inverse of the whitening operator), which transforms the innovations into the main process, needs to be stable. For $\kappa \neq 0$, the system $\mathrm{D}+\kappa\mathrm{I}$ has a unique stable inverse which is shift-invariant and corresponds to the impulse response $\mathrm{e}^{-\kappa x}\chi_{\mathbb{R}_0^+}(x)$ for $\kappa > 0$ and $\mathrm{e}^{-\kappa x}\chi_{\mathbb{R}_0^+}(-x)$ for $\kappa < 0$, where $\chi_{\mathbb{R}_0^+}(\cdot)$ denotes the characteristic function of the nonnegative real numbers (step function).

For $\kappa = 0$, however, there exists no stable inverse. It is shown in [6] that $\mathrm{L}^{-1} = \int_0^x$, which is weakly stable (finite-input finite-output), is a valid choice for $\kappa = 0$. Nevertheless, it imposes $s(0) = 0$ (boundary condition) and makes the process

s non-stationary. A more general way of setting the boundary condition is given by

$$\mathrm{L}^{-1}\{w\}(x) = \int_0^x w(\tau)\mathrm{d}\tau + \langle w, \phi\rangle, \quad (2)$$

where ϕ is an anti-causal function that decreases rapidly and $\langle w, \phi\rangle = \int w(\tau)\phi(\tau)\mathrm{d}\tau$ in the sense of generalized functions. The anti-causal choice of ϕ shows that, for all $x > 0$, the random variable $\mathrm{L}^{-1}\{w\}(x)$ is statistically independent of $w(\tau)$ for $\tau > x$. This will later help us in simplifying the estimation procedure.

Since the innovation process is white and the impulse response of the shaping operator for $(-\kappa)$ is the time-reversal of the one for $\kappa \neq 0$, we expect to obtain the interpolation results of $(-\kappa)$ by time-reversing the results for κ. Therefore, without loss of generality, we shall assume $\kappa \geq 0$.

Finally, the samples of the AR(1) process are taken at the integers $0, 1, \dots, m$. They are then used to interpolate the process values in the interval $[0, m]$. For the sake of simplicity, we use $s[k]$ to denote the sample $s(x)|_{x=k}$ for $k = 0, 1, \dots, m$.

III. Interpolation

Our approach to the interpolation problem is to estimate the process values on a finer grid with spacing T that contains the integers. For this reason, we set $T = \frac{1}{N}$, where N is an arbitrary large positive integer. In this approach, we can get arbitrarily close to any desired point by increasing N. We represent the values $s(x)|_{x=kT}$ for $k = 0, 1, \dots, mN$, which we want to estimate, by $s_T[k]$. Clearly, $s_T[kN]$ (or $s_1[k]$) represent the known samples and we do not need to estimate them. Since the definition of the process s might include a boundary condition, it is not necessarily stationary, which complicates our analysis. Hence, we prefer to work with the generalized-increment process defined as

$$u_T[k] = s_T[k] - \mathrm{e}^{-\kappa T} s_T[k-1]. \quad (3)$$

To relate the generalized increments to the innovation process, recall that

$$s_T[k] = \mathrm{L}^{-1}w(x)\Big|_{x=kT} = \int_{\eta_\kappa}^{kT} w(\tau)\mathrm{e}^{-\kappa(kT-\tau)}\mathrm{d}\tau + c_{w,\kappa}, \quad (4)$$

where $\eta_\kappa = -\infty$ and $c_{w,\kappa} = 0$ for $\kappa > 0$, and $\eta_\kappa = 0$ and $c_{w,\kappa} = \langle w, \phi\rangle$ for $\kappa = 0$. By substituting $s_T[k]$ and $s_T[k-1]$ from (4) into (3), we see that the null-space term vanishes and we obtain

$$u_T[k] = \int_{(k-1)T}^{kT} w(\tau)\mathrm{e}^{-\kappa(kT-\tau)}\mathrm{d}\tau. \quad (5)$$

The outcome can be written in form of an inner product as

$$u_T[k] = \big\langle w \,,\, \beta_{\kappa,T}(kT - \cdot)\big\rangle, \quad (6)$$

where

$$\beta_{\kappa,T}(x) = \mathrm{e}^{-\kappa x}\Big(\chi_{\mathbb{R}_0^+}(x) - \chi_{\mathbb{R}_0^+}(x - T)\Big). \quad (7)$$

The function $\beta_{\kappa,T}$ is usually known as the exponential B-spline for the grid spacing T [14]. It is supported on $[0, T)$.

A. Preliminary Results

To further proceed in solving the interpolation problem, we need to use a few results regarding the increment process which we state below in the form of 3 lemmas.

Lemma 1: Let k, k' be nonnegative integers and T, T' be positive reals. For the generalized increments u_T we have

(i) $u_T[k]$ and $u_{T'}[k']$ are independent if $\frac{T}{T'} \notin (\frac{k'-1}{k}, \frac{k'}{k-1})$;
(ii) $u_T[k]$ and $s_{T'}[k']$ are independent if $\frac{k'}{k-1} \geq \frac{T}{T'}$;
(iii) $u_T[k]$ and $u_T[k']$ are identically distributed.

Proof From (6) and since $\beta_{\kappa,T}$ is of finite support, we know that the statistics of $u_T[k]$ are completely determined by $w\big((k-1)T < x \leq kT\big)$. Condition (i) guarantees that the parts of the innovation contributing to $u_T[k]$ and $u_{T'}[k']$ are disjoint. Since the innovation is white, the two are independent. Similarly, Condition (ii) implies disjointness of the innovation parts involved in forming $u_T[k]$ and $s_{T'}[k']$: the LSI part of $s_{T'}[k']$, due to the use of causal filters, depends only on $w(x \leq k'T')$, while the boundary condition is fully determined by $w(x \leq 0)$. Thus, for nonnegative k', $s_{T'}[k']$ is statistically independent of $w(k'T' < x)$. The validity of (iii) is a direct consequence of the stationarity of the innovation. ■

Lemma 2: For any positive integer n, we have that

$$u_{nT}[k] = \sum_{i=0}^{n-1} \mathrm{e}^{-i\kappa T} u_T[kn - i]. \quad (8)$$

Proof We show this property by pointing out the refinement equation of $\beta_{\kappa,nT}$

$$\begin{aligned}\beta_{\kappa,nT}(x) &= \mathrm{e}^{-\kappa x}\big(\chi_{\mathbb{R}_0^+}(x) - \chi_{\mathbb{R}_0^+}(x - nT)\big)\\ &= \sum_{i=0}^{n-1} \mathrm{e}^{-i\kappa T}\mathrm{e}^{-\kappa(x-iT)}\big(\chi_{\mathbb{R}_0^+}(x-iT) - \chi_{\mathbb{R}_0^+}(x-iT-T)\big)\\ &= \sum_{i=0}^{n-1} \mathrm{e}^{-i\kappa T}\beta_{\kappa,T}(x - iT). \quad (9)\end{aligned}$$

Now, it is easy to conclude the claim by applying (9) to (6). ■

Lemma 3: For any positive integer i, we have that

$$s_T[k+i] - \mathrm{e}^{-i\kappa T}s_T[k] = \sum_{\theta=1}^{i} \mathrm{e}^{-(i-\theta)\kappa T}u_T[k+\theta]. \quad (10)$$

Proof The proof requires only the substitution of u_T by its definition in (3). ■

B. Minimum Conditional Mean-Square Error

The well-known minimum mean-square error (MMSE) estimation of a random variable x based on the multidimensional random variable $\mathbf{y}$ (observations) is the function $\hat{x}(\mathbf{y}) = \mathbb{E}\{x|\mathbf{y}\}$ that minimizes the cost $\mathbb{E}\{(\hat{x}(\mathbf{y}) - x)^2\}$. Note that the averaging applies over both x and $\mathbf{y}$. Consider now that we are estimating x based on a deterministic measurement vector $\mathbf{y}$ that is an observed *realization* of some multivariate random variable. In this case, we should modify the cost to

$\mathbb{E}_x\big\{(\hat{x}(\mathbf{y})-x)^2|\mathbf{y}\big\}$, which again results in $\hat{x}(\mathbf{y})=\mathbb{E}\{x|\mathbf{y}\}$ (*i.e.*, the Bayesian estimator). More precisely, the Bayesian estimator $\hat{x}(\mathbf{y})=\mathbb{E}\{x|\mathbf{y}\}$ not only minimizes the average quadratic cost over all realizations, but also minimizes the cost for every individual realization. The distinction is revealed when $\mathbf{y}$ follows a heavy-tail distribution with infinite variance (*e.g.*, a non-Gaussian α-stable). Here, the cost function for each realization $\mathbf{y}$ might be finite while the average over all $\mathbf{y}$ often does not exist. In other words, the conditional expectation defines an optimal estimator for the modified cost, while the MSE might not be defined. It is obvious that the Bayesian estimator coincides with the MMSE estimator when it exists.

With respect to the conditional MSE criterion, the optimal interpolation for $s_T[k]$, using the given samples $s[l]_{l=0}^m$, is given by $\mathbb{E}\big\{s_T[k]\,\big|\,s[l]_{l=0}^m\big\}$. By using Lemma 3, for $0\leq k<m$ and $0<i<N$ where $T=\frac{1}{N}$, we have that

$$\mathbb{E}\Big\{s_T[kN+i]\,\Big|\,\{s[l]\}_{l=0}^m\Big\}-\mathrm{e}^{-i\kappa T}s[k] = \sum_{\theta=1}^{i}\mathrm{e}^{-(i-\theta)\kappa T}\mathbb{E}\Big\{u_T[kN+\theta]\,\Big|s[l]_{l=0}^m\Big\}. \quad (11)$$

The one-to-one mapping between the sets $s[l]_{l=0}^m$ and $\{u_1[l]\}_{l=1}^m\cup\{s[0]\}$ allows us to rewrite the conditional expectations as

$$\mathbb{E}\Big\{u_T[kN+\theta]\Big|s[l]_{l=0}^m\Big\}=\mathbb{E}\Big\{u_T[kN+\theta]\,\Big|u_1[l]_{l=1}^m,s[0]\Big\}. \quad (12)$$

It follows from (12) and Lemma 1 that $u_T[kN+\theta]$ is independent of $s[0]$ and $u_1[l]_{l=1}^m$ except for $l=k+1$. Thus,

$$\mathbb{E}\Big\{s_T[kN+i]\,\Big|\,s[l]_{l=0}^m\Big\}-\mathrm{e}^{-i\kappa T}s[k] = \textstyle\sum_{\theta=1}^{i}\mathrm{e}^{-(i-\theta)\kappa T}\mathbb{E}\Big\{u_T[kN+\theta]\,\Big|\,u_1[k+1]\Big\}. \quad (13)$$

To simplify the notations, we represent the random variables $u_T[kN+\theta]$ by X_θ and the weights $\mathrm{e}^{-\theta\kappa T}$ by d_θ. Lemma 1 shows that X_θ are i.i.d., and from Lemma 2 we know that

$$u_1[k+1]=\sum_{l=1}^{N}\mathrm{e}^{-(N-l)\kappa T}u_T[kN+l]=\sum_{l=1}^{N}d_{N-l}X_l. \quad (14)$$

Hence,

$$\mathbb{E}\Big\{u_T[kN+i]\,\Big|\,u_1[k+1]\Big\} = \mathbb{E}\Big\{X_i\,\Big|\,\textstyle\sum_{l=1}^N d_{N-l}X_l\Big\}. \quad (15)$$

The summary of the results in (11)–(15) is

$$\hat{s}_T[kN+i]=\frac{s[k]}{\mathrm{e}^{i\kappa T}} + \frac{\sum_{\theta=1}^{i}\mathbb{E}\big\{d_{N-\theta}X_\theta\,\big|\,\sum_{l=1}^N d_{N-l}X_l=u_1[k+1]\big\}}{\mathrm{e}^{(i-N)\kappa T}}. \quad (16)$$

C. Stable Distributions

Up to this point, our results were generic and applicable to all innovation models. We now concentrate on the symmetric α-stable innovations and try to extract the conditional expectations explicitly. For an α-stable innovation w, the inner product $\langle w,\varphi\rangle$ follows an α-stable distribution for any acceptable test function φ[15]. In particular, the distribution of u_T (or X_θ) is α-stable from (6). If we denote the probability density and characteristic functions (Fourier transform of the density function) of u_T by p_X and $\hat{p}_X$, respectively, the α-stable law implies $\hat{p}_X(\omega)=\exp\big(-\sigma|\omega|^\alpha\big)$ for some positive real σ. Unfortunately, there is no closed form for the density function in general. In addition, the characteristic function of the random variable $\sum_i c_iX_i$, which again follows an α-stable distribution, is given by $\exp\big(-\sigma|\omega|^\alpha\sum_i|c_i|^\alpha\big)$ [15]. This shows that, if $Y_1=d_{N-\theta}X_\theta$ and $Y_2=\sum_{l=1,l\neq\theta}^N d_{N-l}X_l$, then we should have

$$\begin{cases}\hat{p}_{Y_1} &= \exp\big(-\sigma|\omega|^\alpha|d_{N-\theta}|^\alpha\big),\\ \hat{p}_{Y_2} &= \exp\big(-\sigma|\omega|^\alpha\sum_{l=1,l\neq\theta}^N|d_{N-l}|^\alpha\big).\end{cases} \quad (17)$$

Note that Y_1 and Y_2 are independent and that the conditional expectations in (16) are equal to

$$\mathbb{E}\Big\{d_{N-\theta}X_\theta\mid\sum_{l=1}^N d_{N-l}X_l=u_1[k+1]\Big\} = \mathbb{E}\Big\{Y_1\mid Y_1+Y_2=u_1[k+1]\Big\} = \frac{\int_{\mathbb{R}}y\,p_{Y_1}(y)\,p_{Y_2}\big(u_1[k+1]-y\big)\,\mathrm{d}y}{p_{Y_1+Y_2}\big(u_1[k+1]\big)}. \quad (18)$$

The latter integral can be converted to the Fourier domain by employing Parseval's theorem, which results in

$$\begin{aligned}&\int_{\mathbb{R}}y\,p_{Y_1}(y)\,p_{Y_2}\big(u_1[k+1]-y\big)\,\mathrm{d}y\\ &=\int_{\mathbb{R}}\mathcal{F}_y\Big\{y\,p_{Y_1}(y)\Big\}(\omega)\overline{\mathcal{F}_y\Big\{p_{Y_2}\big(u_1[k+1]-y\big)\Big\}(\omega)}\,\mathrm{d}\omega\\ &=\int_{\mathbb{R}}\frac{\mathrm{d}}{\mathrm{d}\omega}\hat{p}_{Y_1}(\omega)\overline{\hat{p}_{Y_2}(\omega)}\frac{\mathrm{e}^{-\mathrm{j}\omega u_1[k+1]}}{\mathrm{j}}\,\mathrm{d}\omega\\ &=|d_{N-\theta}|^\alpha\int_{\mathbb{R}}\frac{-\sigma\alpha|\omega|^\alpha\mathrm{e}^{-\mathrm{j}\omega u_1[k+1]-\sigma|\omega|^\alpha\sum_{l=0}^{N-1}|d_l|^\alpha}}{\mathrm{j}\omega}\mathrm{d}\omega.\end{aligned} \quad (19)$$

On one hand, the main message from (18) and (19) is that

$$\frac{\mathbb{E}\Big\{d_{N-\theta}X_\theta\mid\sum_{l=1}^N d_{N-l}X_l=u_1[k+1]\Big\}}{|d_{N-\theta}|^\alpha}=const., \quad (20)$$

where $const.$ does not depend on θ. On the other hand,

$$\sum_{\theta=1}^{N}\mathbb{E}\Big\{d_{N-\theta}X_\theta\mid\sum_{l=1}^N d_{N-l}X_l=u_1[k+1]\Big\}=u_1[k+1]. \quad (21)$$

Now, by combining (20) and (21), we can evaluate the conditional expectations without performing the integration, as

$$\mathbb{E}\Big\{d_{N-\theta}X_\theta\mid\sum_{l=1}^N d_{N-l}X_l=u_1[k+1]\Big\}=\frac{|d_{N-\theta}|^\alpha u_1[k+1]}{\sum_{l=1}^N|d_{N-l}|^\alpha}. \quad (22)$$

The main result of this paper is given in Theorem 1 which is now easy to verify from (16) and (22).

***Theorem** 1:* For the AR(1) process s associated with the whitening operator $\mathrm{D} + \kappa \mathrm{I}$ with α-stable innovations, the optimal Bayesian interpolation at the point $x^* = k + \lambda$, where $0 \leq \lambda \leq 1$ is a rational number and k is a nonnegative integer, depends only on the neighboring samples $s(x = k)$ and $s(x = k+1)$. Moreover, the dependence is linear and can be expressed as

$$\hat{s}(x^*) = \pi_\lambda\, s(k) + \nu_\lambda\, s(k+1), \tag{23}$$

where

$$\begin{cases} \pi_\lambda &= \mathrm{e}^{(\frac{\alpha}{2}-1)\lambda\kappa}\, \frac{\sinh(\frac{\alpha}{2}(1-\lambda)\kappa)}{\sinh(\frac{\alpha}{2}\kappa)}, \\ \nu_\lambda &= \mathrm{e}^{(\frac{\alpha}{2}-1)(\lambda-1)\kappa}\, \frac{\sinh(\frac{\alpha}{2}\lambda\kappa)}{\sinh(\frac{\alpha}{2}\kappa)}, \end{cases} \tag{24}$$

if $\kappa \neq 0$ and, otherwise,

$$\begin{cases} \pi_\lambda &= 1 - \lambda \\ \nu_\lambda &= \lambda. \end{cases} \tag{25}$$

It is interesting that, for $\kappa = 0$ (Lévy process) and independently of the stability index (α), the optimal interpolator is the simple first-degree B-spline. Also, to compare the result with the classical Gaussian theory, we use $\alpha = 2$ and obtain

$$\begin{cases} \pi_\lambda &= \frac{\sinh(1-\lambda)\kappa}{\sinh\kappa}, \\ \nu_\lambda &= \frac{\sinh\lambda\kappa}{\sinh\kappa}. \end{cases} \tag{26}$$

IV. Simulations

To show the impact of our results, we have applied our interpolator to MATLAB simulated data. For this purpose, we have plotted a realization of an α-stable AR(1) process with $\alpha = \frac{3}{2}$ and $\kappa = 5$ in Figure 2. We have used the values at the integers as the samples for interpolating the process. As is evident in Figure 2, the curves connecting the points deviate from straight lines and are not even piecewise monotonic (*e.g.*, the part corresponding to the interval [9,10]). In fact, the statistics of the model show that, for each pair of adjacent samples, the distribution of the values between them is biased in favor of one of the sides of the line connecting the two samples. It is comforting to observe that the curve of the optimal interpolator is bent towards the same direction. From Figure 2, it is evident that the optimal interpolator takes advantage of knowing the system parameters and better follows the process than the outcome of the uninformed first-degree B-spline.

V. Conclusion

In this paper, we studied the interpolation problem for the first-order autoregressive processes generated from stable innovations, including non-Gaussian ones. We applied the Bayesian estimator which minimizes the mean-square error under Gaussian distributions and conditional mean-square error under stable laws that have infinite variance. We derived explicit forms for the optimal interpolator in a general setting and found that it is linear with respect to the samples.

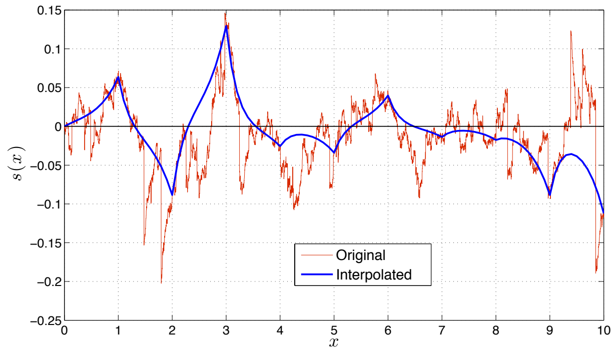

Fig. 2. A realization of the AR(1) process with $\kappa = 5$ and $\alpha = 1.5$, and the interpolated function using the samples at the integers.

Moreover, it depends on the stability index that characterizes stable innovations. Our derivations rely on exponential splines.

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