Approximation by Shannon sampling operators in terms of an averaged modulus of smoothness

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***Abstract*—The aim of this paper is to study the approximation properties of generalized sampling operators in $L^p(\mathbb{R})$-space in terms of an averaged modulus of smoothness.**

I. Introduction

For the uniformly continuous and bounded functions $f \in C(\mathbb{R})$ the generalized sampling series are given by ($t \in \mathbb{R}$; $w > 0$)

$$(S_w f)(t) := \sum_{k=-\infty}^{\infty} f(\frac{k}{w}) s(wt-k), \qquad (1)$$

where the condition for the operator $S_w : C(\mathbb{R}) \to C(\mathbb{R})$ to be well-defined is

$$\sum_{k=-\infty}^{\infty} |s(u-k)| < \infty \quad (u \in \mathbb{R}), \qquad (2)$$

the absolute convergence being uniform on compact intervals of $\mathbb{R}$.

If the kernel function is

$$s(t) = \operatorname{sinc}(t) := \frac{\sin \pi t}{\pi t},$$

we get the classical (Whittaker-Kotel'nikov-)Shannon operator,

$$(S_w^{\mathrm{sinc}} f)(t) := \sum_{k=-\infty}^{\infty} f(\frac{k}{w}) \operatorname{sinc}(wt-k).$$

A systematic study of sampling operators (1) for arbitrary kernel functions s with (2) was initiated at RWTH Aachen by P. L. Butzer and his students since 1977 (see [1], [2], [3] and references cited there).

Since in practice signals are however often discontinuous, this paper is concerned with the convergence of $S_w f$ to f in the $L^p(\mathbb{R})$-norm for $1 \leqslant p < \infty$, the classical modulus of continuity being replaced by the averaged modulus of smoothness $\tau_k(f; 1/w)p$. For the classical (Whittaker-Kotel'nikov-Shannon) operator this approach was introduced by P. L. Butzer, C. Bardaro, R. Stens and G. Vinti (2006) in [4] (see also [5]) for $1 < p < \infty$. For time-limited kernels s this approach was applied for $1 \leqslant p < \infty$ in [6] and [7]. In this paper we use this approach for band-limited kernels for $1 \leqslant p < \infty$.

In this paper we study an even band-limited kernel s, defined by an even window function $\lambda \in C_{[-1,1]}$, $\lambda(0) = 1$, $\lambda(u) = 0$ ($|u| \geqslant 1$) by the equality

$$s(t) := s(\lambda; t) := \int_0^1 \lambda(u) \cos(\pi t u)\, du. \qquad (3)$$

We first used the band-limited kernel in general form (3) in [8], see also [9], [10]. We studied the generalized sampling operators $S_W : C(\mathbb{R}) \to C(\mathbb{R})$ with the kernels in form (3) in [11]-[12]. We computed exact values of operator norms

$$\|S_w\| := \sup_{\|f\|_C \leqslant 1} \|S_w f\|_C = \sup_{u \in \mathbb{R}} \sum_{k=-\infty}^{\infty} |s(u-k)| \qquad (4)$$

and estimated the order of approximation in terms of the classical modulus of smoothness. In this paper we give similar results for $L^p(\mathbb{R})$ norm in terms of the averaged modulus of smoothness. The main result of this paper, Theorem 2, was proved for $f \in C(\mathbb{R})$ in [11].

II. Preliminary results

A. Averaged modulus of smoothness

In this section we follow the approach of Butzer et al [4] of convergence problems of Shannon sampling series in a suitable subspace of $L^p(\mathbb{R})$.

Let $f \in M(\mathbb{R})$ be measurable and bounded on $\mathbb{R}$, and $\delta \geqslant 0$. The k-th averaged τ-modulus of smoothness for $1 \leqslant p \leqslant \infty$ is defined as ([4], Def. 1)

$$\tau_k(f; \delta)p := \|\omega_k(f; \cdot; \delta)\|_p, \qquad (5)$$

where $\omega_k(f; t; \delta)$ is a local modulus of smoothness of order $k \in \mathbb{N}$ at $t \in \mathbb{R}$,

$$\omega_k(f; t; \delta) :=$$
$$:= \sup\{|\Delta_h^k f(x)|; x, x+kh \in [t - \frac{k\delta}{2}, t + \frac{k\delta}{2}]\},$$

where the classical finite forward difference is given by

$$\Delta_h^k f(x) = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} f(x+\ell)h). \qquad (6)$$

The classical modulus of smoothness can be estimated via the τ-modulus (see [4], Proposition 4)

$$\omega_k(f;\delta)_p \leqslant \tau_k(f;\delta)_p \quad (1 \leqslant p < \infty).$$

B. The space Λ^p

Since the sampling series $S_w f$ of (1) of an arbitrary L^p-function f may be divergent, we have to restrict the matter to a suitable subspace. Further, since we want to use the τ-modulus as a measure for the approximation error, we have to ensure that it is finite for all functions under consideration. In [4] it was proved that we can define a suitable subspace as follows

Definition 1 ([4], Def. 10, [6], Def. 2.1):
(a) A sequence $\Sigma := (x_j)_{j\in\mathbb{Z}} \subset \mathbb{R}$ is called an admissible partition of $\mathbb{R}$ or an admissible sequence, if it satisfies

$$0 < \inf_{j\in\mathbb{Z}} \Delta_j \leqslant \sup_{j\in\mathbb{Z}} \Delta_j < \infty, \quad \Delta_j := x_j - x_{j-1}.$$

(b) Let $\Sigma := (x_j)_{j\in\mathbb{Z}} \subset \mathbb{R}$ be an admissible partition of $\mathbb{R}$. The discrete $\ell^p(\Sigma)$-seminorm of a sequence of function values f_Σ on Σ of a function $f : \mathbb{R} \to \mathbb{C}$ is defined for $1 \leqslant p < \infty$ by

$$\|f\|_{\ell^p(\Sigma)} := \left\{ \sum_{j\in\mathbb{Z}} |f(x_j)|^p \Delta_j \right\}^{1/p}.$$

(c) The space Λ^p for $1 \leqslant p < \infty$ is defined by

$$\Lambda^p := \{f \in M(\mathbb{R}); \|f\|_{\ell^p(\Sigma)} < \infty \text{ for each admissible sequence } \Sigma\}.$$

It can be shown (see [4], Proposition 18) that if $f \in \Lambda^p \cap R_{loc}(\mathbb{R})$ for $1 \leqslant p < \infty$ we have

$$\lim_{\delta\to 0} \tau_k(f;\delta)_p = 0, \tag{7}$$

where

$$R_{loc}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C}, \text{ is locally Riemann integrable on } \mathbb{R}\}.$$

The assumption $f \in R_{loc}(\mathbb{R})$ is related to the fact that the τ-modulus on $[a,b]$ tends to zero (with $\delta \to 0+$) if and only if when f is Riemann integrable on $[a,b]$ (see [13], Th. 1.2 and [4], Proposition 6.).

We have for $1 \leqslant p < \infty$ that $B^p_w \subsetneq W^r_p \subsetneq \Lambda^p \subsetneq L^p$, where B^p_w is the Bernstein class (e.g. [14], Def. 6.5) and

$$W^r_p := \{f \in L^p; f \in AC^r_{loc}, f^{(r)} \in L^p\}$$

is the classical Sobolev space.

In the following we consider the uniform partitions $\Sigma_w := (j/w)_{j\in\mathbb{Z}} \subset \mathbb{R}$ for $w > 0$ only. For these partitions we have ([6], Proposition 2.2)

$$\|f\|_{\ell^p(w)} \leqslant \|f\|_p + \frac{1}{w}\|f'\|_p, \quad f \in W^r_p. \tag{8}$$

Proposition 1 ([6], Th. 2.8): Let $(L_w)_{w>0}$ be a family of linear operators mapping Λ^p into L^p, $1 \leqslant p < \infty$, satisfying the properties

$$(i) \quad \|L_w f\|_p \leqslant K\|f\|_{\ell^p(w)}, \quad f \in \Lambda^p, \tag{9}$$

$$(ii) \quad \|L_w g - g\|_p \leqslant K_r \frac{1}{w^s}\|g^{(r)}\|_p, \quad g \in W^r_p, \tag{10}$$

for some fixed $r, s \in \mathbb{N}$, $(s \leqslant r)$ and a constant K_r depending only on r. Then for each $f \in \Lambda^p$ there holds the estimate

$$\|L_w f - f\|_p \leqslant c\,\tau_r(f; \frac{1}{W^{s/r}})_p, \quad W > 0, \tag{11}$$

the constant c depending only on r, K and K_r.

To use Proposition 1 for Shannon sampling operators we need the following proposition.

Proposition 2 (cf. [4], Proposition 25): For $1 \leqslant p \leqslant \infty$, for some $r \in \mathbb{N}$ and $s = 0, 1, \ldots, r$ there exists a constant $c_r > 0$ such that for each $f \in W^r_p$ and $w > 0$ one can find a function $g_w \in B^p_{\pi w}$ satisfying

$$\|f^{(s)} - g^{(s)}_w\|_p \leqslant c_r \frac{1}{w^{r-s}}\|f^{(r)}\|_p.$$

C. Sampling operators

The kernel for the sampling operators S_w in (1) is defined in the following way.

Definition 2 ([3], Def. 6.3): If $s : \mathbb{R} \to \mathbb{C}$ is a bounded function such that

$$\sum_{k=-\infty}^{\infty} |s(u-k)| < \infty \quad (u \in \mathbb{R}), \tag{12}$$

the absolute convergence being uniform on compact subsets of $\mathbb{R}$, and

$$\sum_{k=-\infty}^{\infty} s(u-k) = 1 \quad (u \in \mathbb{R}), \tag{13}$$

then s is said to be a kernel for sampling operators (1).

For $f \in \Lambda^p$ we have:

Proposition 3 ([6], Proposition 3.2): Let $s \in M(\mathbb{R}) \cap L^1(\mathbb{R})$ be a kernel. Then $\{S_w\}_{w>0}$ defines a family of bounded linear operators from Λ^p into L^p, $1 \leqslant p < \infty$ (and also from $C(\mathbb{R})$ into $C\mathbb{R}$ with the norm (4)), satisfying $(1/p + 1/q = 1)$

$$\|S_w f\|_p \leqslant \|S_w\|^{1/q}\|s\|_1^{1/p}\|f\|_{\ell^p(w)} \quad (w > 0). \tag{14}$$

If the kernel s is time-limited, i.e. there exists $T_0, T_1 \in \mathbb{R}$, $T_0 < T_1$ such that $s(t) = 0$ for $t \notin [T_0, T_1]$, then in case $f \in \Lambda^p \cap R_{loc}(\mathbb{R})$ for $1 \leqslant p < \infty$, we have (see [6], Th. 4.4)

$$\lim_{w\to\infty} \|S_w f - f\|_p = 0. \tag{15}$$

I this paper we prove analogous result for band-limited kernels.

D. Band-limited kernels

In the following we assume that our kernel (3) belongs to B^1_π. For the band-limited functions $s \in B^p_\pi \subset L^p(\mathbb{R})$ the operator norm $\|S_w\|$ is related to the norm $\|s\|_p$ by Nikolskii's inequality.

Proposition 4 (Nikolskii inequality; [14], Th. 6.8): Let $1 \leqslant p \leqslant \infty$. Then, for every $s \in B^p_\sigma$,

$$\|s\|_p \leqslant \sup_{u\in\mathbb{R}} \left\{ \sum_{k=-\infty}^{\infty} |s(u-k)|^p \right\}^{1/p} \leqslant (1+\sigma)\|s\|_p.$$

From the Nikolskii's inequality we see that our assumption $s \in L^1(\mathbb{R})$ is sufficient for (12) and thus s in (3) is indeed a kernel in the sense of Definition 2.

These types of kernels arise in conjunction with window functions widely used in applications (e.g. [15], [16], [17], [18]), in Signal Analysis in particular. Unfortunately bandlimited kernels do not have compact support. Many kernels can be defined by (3), e.g.

1) $\lambda(u) = 1$ defines the sinc function;

2) $\lambda_j(u) := \cos\pi(j+1/2)u,\ j = 0,1,2,\dots$ defines the Rogosinski-type kernel (see [9]) in the form

$$r_j(t) := \frac{1}{2}\left(\operatorname{sinc}(t+j+\frac{1}{2}) + \operatorname{sinc}(t-j-\frac{1}{2})\right) \qquad (16)$$

3) $\lambda_H(u) := \cos^2\frac{\pi u}{2} = \frac{1}{2}(1+\cos\pi u)$ defines the Hann kernel (see [12])

$$s_H(t) := \frac{1}{2}\frac{\operatorname{sinc} t}{1-t^2}; \qquad (17)$$

III. Subordination by typical (Zygmund) sampling operators

In [11] we introduced typical (Zygmund) sampling series $Z^r_w f$ for $f \in C(\mathbb{R})$ with kernels $z_r \in B^1_\pi$ defined via (3) using the window function

$$\lambda_{Z,r}(u) := 1 - u^{2r}, \quad r > 0.$$

We proved an estimate ([11], Th. 1)

$$\|Z^r_w\| \leqslant \frac{2}{\pi}\log r + C \qquad (18)$$

Consider now an even bandlimited kernel $s_r \in B^1_\pi$ defined via (3) using the window function λ_r, which has a representation

$$\lambda_r(u) := 1 - \sum_{j=r}^{\infty} c_j\, u^{2j}, \quad r \geqslant 1. \qquad (19)$$

The condition (19) is satisfied for many kernels $s \in B^1_\pi$.

If $\sum_{j=r}^{\infty} |c_j|\log j < \infty$ then substituting (19) in (3) and the last one into (1) gives a double series, where interchanging of the order of summation is justified. Therefore, for generalized sampling series in (1) defined by the kernel s_r one has the subordination equalities

$$S^r_w f = \sum_{j=r}^{\infty} c_j Z^j_w f \qquad (20)$$

$$S^r_w f - f = \sum_{j=r}^{\infty} c_j (Z^j_w f - f). \qquad (21)$$

Theorem 1: Let $f \in \Lambda^p$ for $1 \leqslant p < \infty$, $r \in \mathbb{N}$. Then

$$\|Z^r_w f - f\|_p \leqslant M_r \tau_{2r}(f; \frac{1}{w})_p. \qquad (22)$$

The constants M_r are independent of f and w. Moreover, if $f \in \Lambda^p \cap R_{loc}(\mathbb{R})$ for $1 \leqslant p < \infty$, we have

$$\lim_{w\to\infty} \|Z^r_w f - f\|_p = 0. \qquad (23)$$

PROOF: We apply Proposition 1. For (9) in Proposition 1 we have for $f \in \Lambda^p$ by Proposition 3, (18) and Nikolski inequality

$$\|Z^r_w f\|_p \leqslant \|Z^r_w\|^{1/q}\|z_r\|_1^{1/p}\|f\|_{\ell^p(w)} \leqslant \|Z^r_w\|\|f\|_{\ell^p(w)}.$$

Now we show that (10) in Proposition 1 holds. Let $g \in B^p_{\pi w}$. For $f \in W^{2r}_p$ we have

$$\|Z^r_w f - f\|_p \leqslant \|Z^r_w(f-g)\|_p + \|Z^r_w g - g\|_p + \|f-g\|_p \quad (24)$$

By Proposition 3 and (8) we have

$$\begin{aligned}\|Z^r_w(f-g)\|_p &\leqslant \|Z^r_w\|^{1/q}\|z_r\|_1^{1/p}\|f-g\|_{\ell^p(w)} \\ &\leqslant \|Z^r_w\|^{1/q}\|z_r\|_1^{1/p}(\|f-g\|_p + \frac{1}{w}\|f'-g'\|_p). \quad (25)\end{aligned}$$

If $g \in B^p_{\pi w}$, then $S^{sinc}_w g = g$ i.e.

$$g(t) = \sum_{k\in\mathbb{Z}} g\Big(\frac{k}{w}\Big)\int_0^1 \cos(\pi(wt-k)u)\,du.$$

Hence on the right hand side the series is uniformly convergent and after term-by-term differentiation we get also a uniformly convergent series (cf. [2], Th. 3.3). Therefore for $r \in \mathbb{N}$

$$\frac{(-1)^r}{(\pi w)^{2r}} g^{(2r)}(t) = \sum_{k\in\mathbb{Z}} g\Big(\frac{t}{w}\Big)\int_0^1 u^{2r}\cos(\pi(wt-k)u)\,du \qquad (26)$$

Now by the definition of Z^r_w it follows

$$\begin{aligned}\|Z^r_w g - g\|_p &= \frac{1}{(\pi w)^{2r}}\|g^{(2r)}\|_p \\ &\leqslant \frac{1}{(\pi w)^{2r}}(\|f^{(2r)} - g^{(2r)}\|_p + \|f^{(2r)}\|_p). \quad (27)\end{aligned}$$

Substituting (25) and (27) in (24) and choosing finally the function g as $g_w \in B^p_{\pi w}$ from Proposition 2 it follows

$$\|Z^r_w f - f\|_p \leqslant K_r \frac{1}{w^{2r}}\|f^{(2r)}\|_p$$

and (10) is fullfilled. Proposition 1 yields (22). The last assertion (23) follows from (22) and (7). ■

Theorem 2: Let sampling operator S^r_w $(w > 0)$ be defined by the kernel (3) with $\lambda = \lambda_r$ and for some $r \in \mathbb{N}$ let

$$\lambda_r(u) := 1 - \sum_{j=r}^{\infty} c_j u^{2j}, \quad \sum_{j=r}^{\infty} |c_j|\log j \leqslant \infty. \qquad (28)$$

Then for $f \in \Lambda^p$ $(1 \leqslant p < \infty)$

$$\|S^r_w f - f\|_p \leqslant M_r \tau_{2r}(f; \frac{1}{w})_p. \qquad (29)$$

The constants M_r are independent of f and w. Moreover, if $f \in \Lambda^p \cap R_{loc}(\mathbb{R})$ for $1 \leqslant p < \infty$, we have

$$\lim_{w\to\infty} \|S_w^r f - f\|_p = 0. \tag{30}$$

PROOF: We apply Proposition 1. For (9) in Proposition 1 we have for $f \in \Lambda^p$ by (20), (18), Proposition 3 and Nikolski inequality

$$\|S_w^r f\|_p \leqslant \|f\|_{\ell^p(w)} \sum_{j=r}^{\infty} |c_j| \log j$$

Now we show that (10) in Proposition 1 holds. Let $g \in B_{\pi w}^p$. For $f \in W_p^{2r}$ we have

$$\|S_w^r f - f\|_p \leqslant \|S_w^r (f-g)\|_p + \|S_w^r g - g\|_p + \|f-g\|_p \tag{31}$$

The subordination equality (21) gives the estimate

$$\|S_w^r g - g\|_p \leqslant \sum_{j=r}^{\infty} |c_j| \|Z_w^j g - g\|_p$$

Now we show that for $g \in B_{\pi w}^p$ and $s \leqslant r$ there holds the estimate $\|Z_w^r g - g\|_p \leqslant \|Z_w^s g - g\|_p$. Using (26) and the definition of Z_w^r we have

$$Z_w^j g(t) - g(t) = -(\pi w)^{-2} \Big((Z_w^{j-1} g)''(t) - g''(t) \Big) \tag{32}$$

Applying ([14], Th. 6.11 and Lemma 6.6) we have $Z_w^j g \in B_{\pi w}^1 \subset B_{\pi w}^p$, hence $(Z_w^j g - g) \in B_{\pi w}^p$ and we can use the Bernstein inequality for $1 \leqslant p \leqslant \infty$

$$\|(Z_w^{j-1} g)'' - g''\|_p \leqslant (\pi w)^2 \|Z_w^{j-1} g - g\|_p,$$

hence

$$\|Z_w^j g - g\|_p \leqslant \|Z_w^{j-1} g - g\|_p,$$

and we have

$$\|S_w^r g - g\|_p \leqslant \|Z_w^r g - g\|_p \sum_{j=r}^{\infty} |c_j|.$$

Finally we use (27) and substitute the resulting estimate in (31). The rest of the proof is the same as for Theorem 1. ∎

IV. EXAMPLES

Now we apply Theorem 2 for some sampling operators.

Theorem 3: Let the Rogosinski-type sampling operator $R_{w,j}$ $(j = 0, 1, 2, \ldots)$ be defined by the kernel (16). Then for $f \in \Lambda^p$ $(1 \leqslant p < \infty)$

$$\|R_{w,j} f - f\|_p \leqslant M_j \tau_2(f; \frac{1}{w})_p.$$

The constants M_j are independent of f and w.

PROOF: We have for the Rogosinski-type window function

$$\lambda_j(u) = \cos \pi (j + \frac{1}{2}) u = 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\pi^{2k} (j + 1/2)^{2k}}{(2k)!} u^{2k}$$

and obviously

$$\sum_{k=1}^{\infty} \frac{\pi^{2k} (j+1/2)^{2k}}{(2k)!} \log k < \infty. \quad ∎$$

Theorem 4: Let the Hann sampling operator H_w be defined by the kernel (17). Then for $f \in \Lambda^p$ $(1 \leqslant p < \infty)$

$$\|H_w f - f\|_p \leqslant M \tau_2(f; \frac{1}{w})_p.$$

The constant M is independent of f and w.

PROOF: We have for the Hann window function

$$\lambda_H(u) = \frac{1}{2}(1 + \cos \pi u) = 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\pi^{2k}}{2(2k)!} u^{2k}. \quad ∎$$

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