PROBABILITY OF ERROR FOR BPSK MODULATION IN DISTRIBUTED BEAMFORMING WITH PHASE ERRORS

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ABSTRACT

This paper presents an investigation into the error probability performance for binary phase-shift keying modulation in distributed beamforming with phase errors. The effects of the number of nodes on the beamforming performance are examined as well as the influences of the cumulative phase errors and the total transmit power. Simulation results show a good match with the mathematical analysis of error probability in both static and time-varying channels.

1. INTRODUCTION

Recently, there has been interest in applying beamforming techniques into wireless sensor networks. The motivation is to reduce the energy requirement for each sensor node in signal transmission, and extend the communication range to a far field receiver. The individual sensor nodes share the collected information and transmit it in such a way that the signals add coherently at the destination. Transmit beamforming requires accurate synchronization in frequency and phase among sensors, and accurate channel estimation between each sensor node and the receiver. Although certain techniques have been designed in [1], [2], [3] to minimize the phase errors among sensor nodes, phase errors cannot be eliminated due to hardware constraints. Minimizing total transmit power using quantized channel state information has been studied in [4]. The beam pattern performance of distributed beamforming has been studied in [5] and [6] with synchronous phase errors among sensor nodes. From a more practical view, in this paper, we investigate the probability of error for binary phase-shift keying (BPSK) modulation in distributed beamforming with synchronous phase errors and noise.

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The rest of the paper is organized as follows. Section 2 introduces the system model. In Section 3 we give an equivalent channel concept to simplify the whole beamforming process. In Section 4 the mathematical analysis of the average bit error ratio (BER) for BPSK in both static and time-varying channels are presented. In Section 5 we analyze the beamforming gain with constant total transmit power. Section 6 then presents simulation results to compare with the theoretical analysis and Section 7 draws conclusions for the paper.

2. SYSTEM MODEL

We consider a system of N sensor nodes collaboratively beamforming a narrowband message signal $s(t) = A \cdot m(t)$ to a distant coherent receiver, where A is the amplitude of the message signal. This is performed in a distributed manner by each sensor node modulating $s(t)$ with a RF carrier signal, as illustrated in Fig. 1.

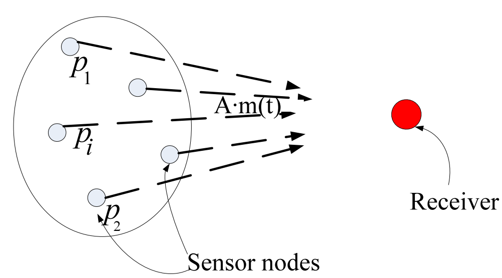


Fig. 1. System model for distributed beamforming

We assume that each sensor node and the receiver are equipped with one single ideal omnidirectional antenna, and there are no mutual coupling effects among the antennas. The receiver has the ability to retrieve the overall channel phase from the received signal. All sensor nodes are synchronized so that they can transmit at the same carrier frequency, and signals transmitted from each sensor node will be added coherently at the receiver. The complex baseband model of the received signal is given by

$$r(t) = \sum_{i=1}^{N} |h_i(t)p_i(t)| e^{j\phi_i(t)} s(t) + n(t) \tag{1}$$

where $p_i(t)$ is the amplification factor and $h_i(t)$ is the channel gain for sensor node i, $\phi_i(t)$ is the cumulative phase error of the carrier signal from the synchronization process among sensor nodes and the estimation of the channel gain for sensor node i, $n(t) \sim CN(0, \sigma_n^2)$ is additive white Gaussian noise. We assume all phase errors $\phi_i(t)$ are independently and uniformly distributed within the range $(-\phi_0, \phi_0)$, which is the assumption adopted in previously reported investigations [1], [2].

A. Static Channel

In a static channel scenario, $h_i(t)$ is set equal to a constant. For simplicity, we set coefficients $h_i(t)$, $p_i(t)$ to be unity. Then the system model is expressed as:

$$r(t) = \sum_{i=1}^{N} e^{j\phi_i(t)} s(t) + n(t) \tag{2}$$

B. Time-Varying Channel

In our time-varying model, the channel coefficients are independent circularly symmetric complex Gaussian distributed, denoted as $h_i(t) \sim CN(0, 1)$, which corresponds to non-line of sight or Rayleigh fading channels. By applying maximal ratio combining, where the pre-amplification gain of each channel is made proportional to the received signal level, we set $|p_i(t)| = |h_i(t)|$ and the system model is then expressed as:

$$r(t) = \sum_{i=1}^{N} |h_i(t)|^2 e^{j\phi_i(t)} s(t) + n(t) \tag{3}$$

3. ANALYSIS OF THE EQUIVALENT CHANNEL

If we view the whole beamforming process as an equivalent channel, denoted as $H(t)$, the system model becomes:

$$r(t) = H(t)s(t) + n(t) \tag{4}$$

where $H(t) = \sum_{i=1}^{N} e^{j\phi_i(t)}$ for the static channel scenario, and $H(t) = \sum_{i=1}^{N} |h_i(t)|^2 e^{j\phi_i(t)}$ for the Rayleigh fading channel scenario. With a coherent receiver, the signal-to-noise ratio (SNR) gain, $\|H(t)\|^2$, is the key element deciding the error probability for distributed beamforming and the communication range for power limited sensor networks.

A. Static Channel

By the central limit theorem, with a large number of sensor nodes N, and the independent identically distributed (i.i.d.) random variables $\phi_i(t)$, we have:

$$\begin{aligned} \|H(t)\|^2 &= \left\| \sum_{i=1}^{N} e^{j\phi_i(t)} \right\|^2 \\ &= \left\| \sum_{i=1}^{N} \cos\phi_i(t) + j \sum_{i=1}^{N} \sin\phi_i(t) \right\|^2 \\ &= \|a_S + jb_S\|^2 \\ &= a_S^2 + b_S^2 \end{aligned} \tag{5}$$

where $a_S = \sum_{i=1}^{N} \cos\phi_i(t) \sim N(\mu_{a_S}, \sigma_{a_S}^2)$, $b_S = \sum_{i=1}^{N} \sin\phi_i(t) \sim N(\mu_{b_S}, \sigma_{b_S}^2)$, using the subscript S for the static channels.

Since the variables $\phi_i(t)$ are independently and uniformly distributed within the range $(-\phi_0, \phi_0)$, the means and variances of a_S and b_S can be obtained as:

$$\begin{aligned} \mu_{a_S} &= N \cdot E[\cos\phi_i(t)] \\ &= N \frac{\sin\phi_0}{\phi_0} \end{aligned} \tag{6}$$

$$\mu_{b_S} = 0 \tag{7}$$

$$\begin{aligned} \sigma_{a_S}^2 &= N\left(\quad E[\cos^2\phi_i(t)] - (E[\cos\phi_i(t)])^2 \quad \right) \\ &= N\left(\frac{1}{2} + \frac{\sin 2\phi_0}{4\phi_0} - \left(\frac{\sin\phi_0}{\phi_0}\right)^2 \right) \end{aligned} \tag{8}$$

$$\begin{aligned} \sigma_{b_S}^2 &= N\left(\quad E[\sin^2\phi_i(t)] - (E[\sin\phi_i(t)])^2 \quad \right) \\ &= N\left(\frac{1}{2} - \frac{\sin 2\phi_0}{4\phi_0} \right) \end{aligned} \tag{9}$$

From (8) and (9), we see, for the equivalent channel $H(t)$, the variance of the real part $\sigma_{a_S}^2$ and the variance of the imaginary part $\sigma_{b_S}^2$ are not equal, which means that the probability density function (PDF) of $\|H(t)\|^2$ is not easily obtained from the joint PDF of $H(t)$, $p(a_S, b_S)$.

B. Rayleigh Fading Channel

For the Rayleigh fading channels, similarly, with a large number of sensor nodes N, and the i.i.d. random variables $h_i(t)$ which are independent from the i.i.d. random variables $\phi_i(t)$, we have:

$$\begin{aligned}
\|H(t)\|^2 &= \left\| \sum_{i=1}^{N} |h_i(t)|^2 e^{j\phi_i(t)} \right\|^2 \\
&= \left\| \sum_{i=1}^{N} |h_i(t)|^2 \cos\phi_i(t) + j \sum_{i=1}^{N} |h_i(t)|^2 \sin\phi_i(t) \right\|^2 \\
&= \|a_R + jb_R\|^2 \\
&= a_R^2 + b_R^2 \qquad (10)
\end{aligned}$$

where $a_R = \sum_{i=1}^{N} |h_i(t)|^2 \cos\phi_i(t) \sim N(\mu_{a_R}, \sigma_{a_R}^2)$, and $b_R = \sum_{i=1}^{N} |h_i(t)|^2 \sin\phi_i(t) \sim N(\mu_{b_R}, \sigma_{b_R}^2)$, using the subscript R for the Rayleigh fading channels.

Based on the previous assumptions that the channel coefficients $h_i(t)$ are independent circularly symmetric complex Gaussian distributed $h_i(t) \sim CN(0, 1)$, and $\phi_i(t) \sim (-\phi_0, \phi_0)$, we derived the means and variances of a_R and b_R as follows:

$$\begin{aligned}
\mu_{a_R} &= N \cdot E[|h_i(t)|^2 \cos\phi_i(t)] \\
&= N \cdot E[|h_i(t)|^2] \cdot E[\cos\phi_i(t)] \\
&= N \frac{\sin\phi_0}{\phi_0} \qquad (11)
\end{aligned}$$

$$\mu_{b_R} = 0 \qquad (12)$$

$$\begin{aligned}
\sigma_{a_R}^2 &= N\Big(E[(|h_i(t)|^2 \cos\phi_i(t))^2] - (E[|h_i(t)|^2 \cos\phi_i(t)])^2 \Big) \\
&= N\Big(E[|h_i(t)|^4] \cdot E[\cos^2\phi_i(t)] - (E[|h_i(t)|^2 \cos\phi_i(t)])^2 \Big) \\
&= N\left(1 + \frac{\sin 2\phi_0}{2\phi_0} - (\frac{\sin\phi_0}{\phi_0})^2 \right) \qquad (13)
\end{aligned}$$

$$\begin{aligned}
\sigma_{b_R}^2 &= N\Big(E[(|h_i(t)|^2 \sin\phi_i(t))^2] - (E[|h_i(t)|^2 \sin\phi_i(t)])^2 \Big) \\
&= N\Big(E[|h_i(t)|^4] \cdot E[\sin^2\phi_i(t)] - (E[|h_i(t)|^2 \sin\phi_i(t)])^2 \Big) \\
&= N\left(1 - \frac{\sin 2\phi_0}{2\phi_0} \right) \qquad (14)
\end{aligned}$$

Similarly, from (13) and (14) we see, for the Rayleigh fading channel scenario, $\sigma_{a_R}^2$ and $\sigma_{b_R}^2$ are not equal, thus the expression of the PDF of $\|H(t)\|^2$ is difficult to compute.

4. MATHEMATICAL ANALYSIS OF ERROR PROBABILITY

The BER of BPSK over a fixed channel in the presence of AWGN is given by [7] in Chapter 5:

$$P_e(\gamma) = \frac{1}{2}\text{erfc}(\sqrt{\gamma}) \qquad (15)$$

where γ is the received signal-to-noise ratio per bit, and erfc(.) is the complementary error function.

When the channel gain is random, the average BER for BPSK over all values of γ is given by [7] in Chapter 14:

$$P_e = \int_0^\infty P_e(\gamma) p(\gamma) d\gamma \qquad (16)$$

where $\gamma = \|H(t)\|^2 \frac{A^2}{\sigma_n^2}$ in our system model described in Section 2.

In Section 3, we have analyzed the SNR gain $\|H(t)\|^2$ of the distributed beamforming system, and the expression of the PDF of $\|H(t)\|^2$ was not obtained due to the variances of the real part and the imaginary part of the equivalent channel being unequal. Consequently, $p(\gamma)$ is not available in either the static channel scenario or the Rayleigh fading channel scenario. Formula (16) cannot be solved directly to get a closed-form expression of the integration for our model, and can only be evaluated by numerical techniques. Instead, we provide another method to approximate the BER results as follows.

Method 1:

For both the static channel scenario and the Rayleigh fading channel scenario, we set the variances of the real part and the imaginary part of $H(t)$ to be equal and use the maximum value between them:

$$\sigma_S^2 = \max(\sigma_{a_S}^2, \sigma_{b_S}^2) \qquad (17)$$

for the static channels, and

$$\sigma_R^2 = \max(\sigma_{a_R}^2, \sigma_{b_R}^2) \qquad (18)$$

for the Rayleigh fading channels.

Because the real part and the imaginary part of the equivalent channel $H(t)$ now have different means but same variances, the magnitude gain of $H(t)$ is approximated as a Rician distribution.

The closed-form of BER for BPSK through Rician fading channel with a coherent receiver is given by [8]:

$$P_E = Q_1(u, w) - \frac{1}{2}\left(1 + \sqrt{\frac{d}{1+d}}\right) \exp\left(-\frac{u^2 + w^2}{2}\right) I_0(uw) \qquad (19)$$

where

$$d = 2\sigma^2 \frac{A^2}{\sigma_n^2} \qquad (20)$$

$$u = \sqrt{\frac{\mu_a^2 + \mu_b^2}{2\sigma^2} \cdot \frac{1 + 2d - 2\sqrt{d(1+d)}}{2(1+d)}} \qquad (21)$$

$$w = \sqrt{\frac{\mu_a^2 + \mu_b^2}{2\sigma^2} \cdot \frac{1 + 2d + 2\sqrt{d(1+d)}}{2(1+d)}} \tag{22}$$

and $I_0(x)$ is the zeroth-order-modified Bessel function of the first kind, defined as:

$$I_0(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!\Gamma(k+1)}, \qquad x \geq 0 \tag{23}$$

$Q_1(x, y)$ is the Marcum Q-function, defined as:

$$Q_1(x, y) = \int_y^{\infty} z \cdot \exp\left(-\frac{z^2 + x^2}{2}\right) I_0(xz)dz \tag{24}$$

Using (19) to (24), we can get the BER for our static channel scenario by substituting (6), (7), (17) for μ_a, μ_b, σ^2 in (20), (21), (22), and get the BER for our Rayleigh fading channel scenario by substituting (11), (12), (18) for μ_a, μ_b, σ^2 in (20), (21), (22).

An approximation of $I_0(x)$ is given by [9] in Chapter 6:

$$I_0(x) \approx \frac{1}{\sqrt{2\pi x}} \exp(x), \qquad x \gg 0 \tag{25}$$

and after manipulation, (19) can be simplified as:

$$P_E = Q_1(u, w) - \frac{1}{2\sqrt{2\pi uw}} \left(1 + \sqrt{\frac{d}{1+d}}\right) \exp\left(-\frac{(u-w)^2}{2}\right) \tag{26}$$

Method 2:

We are currently investigating the approximation of the BER performance by an additive white Gaussian noise formula. This is the subject of ongoing work.

5. DISTRIBUTED BEAMFORMING GAIN WITH CONSTANT TOTAL TRANSMIT POWER

As the received signal-to-noise ratio cannot show the advantages of beamforming gain, and is uncertain due to independent and random phase errors ϕ_i, our simulation results are plotted as BER vs total transmit power. Before we present our simulation results, we first analyze the beamforming gain with constant total transmit power. We use P to represent the total transmit power of all the sensor nodes. In the static channel scenario,

$$\begin{aligned} P &= \sum_{i=1}^{N} A^2 \\ &= A^2 \cdot N \end{aligned}$$

In the Rayleigh fading channel scenario,

$$\begin{aligned} P &= \sum_{i=1}^{N} (A|p_i(t)|)^2 \\ &= A^2 \sum_{i=1}^{N} |p_i(t)|^2 \end{aligned}$$

With large N, by the law of large numbers, it becomes:

$$P \approx A^2 \cdot N$$

Generally, with a constant P, we can represent A as:

$$A = \sqrt{\frac{P}{N}} \tag{27}$$

Putting (27) and $s(t) = A \cdot m(t)$ into (4), we obtain:

$$r(t) = H(t)\sqrt{\frac{P}{N}} m(t) + n(t) \tag{28}$$

and

$$\gamma = \frac{1}{N} \|H(t)\|^2 \frac{P}{\sigma_n^2} \tag{29}$$

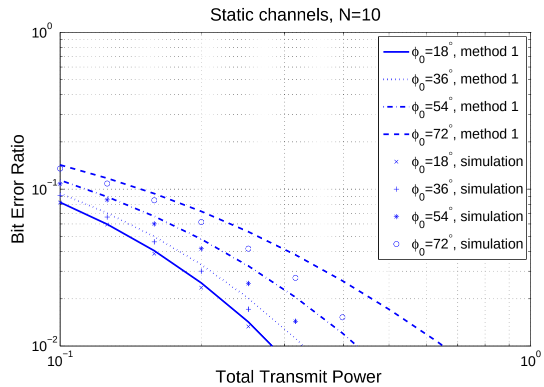
Since the mean of $\|H(t)\|^2$ grows linearly with N^2, the mean of the received signal-to-noise ratio per bit $\gamma_{mean} \propto P{\cdot}N$, and with a constant P, γ_{mean} is proportional to N.

6. SIMULATION RESULTS

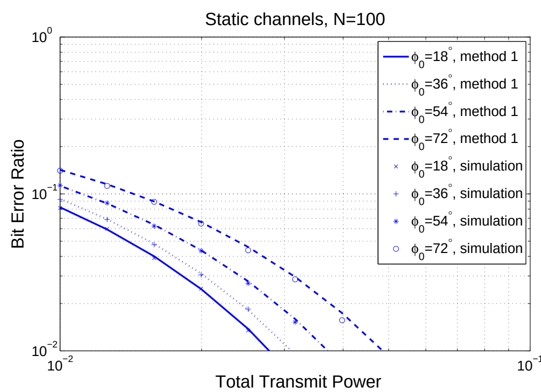
In this section, we present some simulation results in accordance with our previous assumptions, and compare them with our mathematical analysis given in Section 4.

Fig. 2 shows the comparison of the simulation results with the mathematical analysis based on method 1 for BPSK modulation over *static channels* with phase errors. The simulation results are conducted over 10^5 symbols with different number of nodes $N = 10, 100, 1000$, and different phase error ranges $\phi_0 = 18°, 36°, 54°, 72°$. We set $n(t) \sim CN(0, 1)$. All curves in Fig. 2 are drawn by (19) except the curves for $\phi_0 = 18°$ with $N = 100$ in part (b) and $\phi_0 = 18°, 36°, 54°$ with $N = 1000$ in part (c). These four curves cannot be drawn out by (19) because of the overflow caused by the function $I_0(x)$ in (23) used in MATLAB. Instead of (19), We use (26) to draw these four curves in Fig. 2.

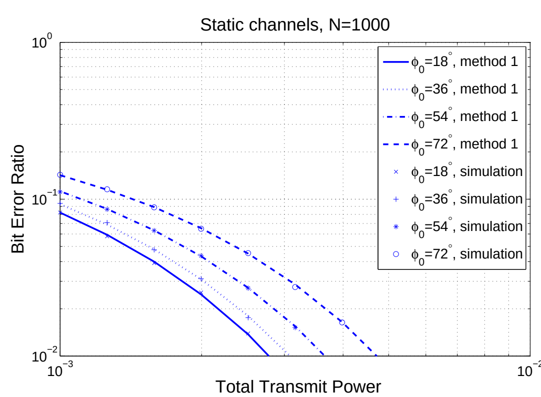
By comparing the simulation results plotted in parts (a), (b), (c) in Fig. 2 and noting the order of magnitude difference of total transmit power in (a), (b), (c), we find that, with similar BER performance in each part, when increasing the number of nodes N by a factor of 10, the total transmit power is reduced by a factor of 10, which means the energy transmitted by each node is reduced by a factor of 10^2. Thus, we have the conclusion that increasing the number of nodes N



(a)

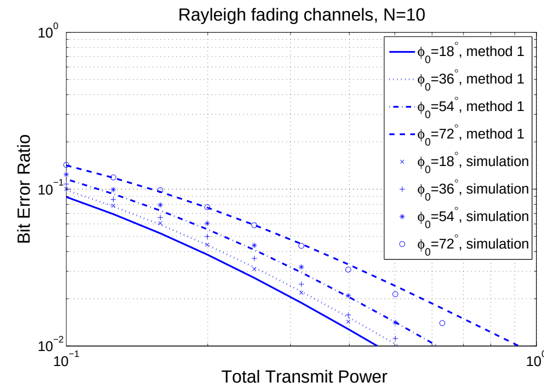


(b)

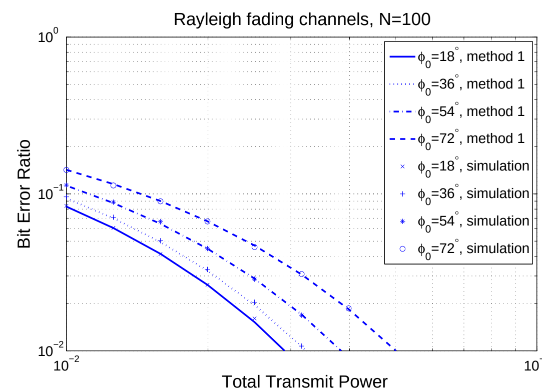


(c)

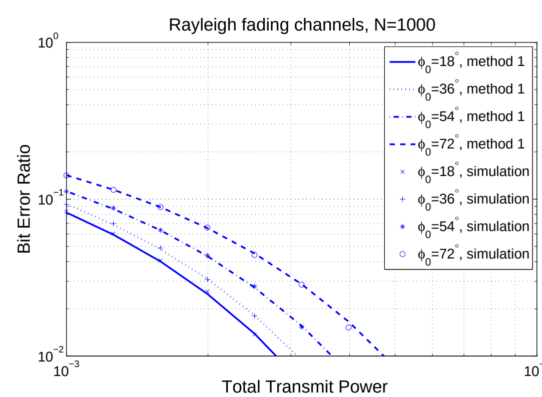
Fig. 2. Comparison of mathematical analysis based on method 1 with simulation results of BER versus total transmit power in the static channel scenario with different numbers of nodes N =(a)10, (b)100, and (c)1000, and different phase error ranges $\phi_0 = 18°, 36°, 54°, 72°$.



(a)



(b)



(c)

Fig. 3. Comparison of mathematical analysis based on method 1 with simulation results of BER versus total transmit power in the Rayleigh fading channel scenario with different numbers of nodes N =(a)10, (b)100, and (c)1000, and different phase error ranges $\phi_0 = 18°, 36°, 54°, 72°$.

can dramatically reduce the energy requirement for each sensor node subject to the same BER performance, and the number of nodes N has a much larger effect on BER performance than the phase error range ϕ_0.

From Fig. 2, we see, on the one hand, with a large number of nodes $N = 1000$, the BER analysis based on method 1 matches the simulation results accurately. On the other hand, with a small number of nodes $N = 10$, the BER analysis based on method 1 has a slight difference with the simulation results. This is due to the limitation that central limit theorem does not apply for a small number of nodes.

Fig. 3 shows the comparison of the simulation results with the mathematical analysis based on method 1 for BPSK modulation over *Rayleigh fading channels* with phase errors. The simulation results are also conducted over 10^5 symbols with different number of nodes $N = 10, 100, 1000$, and different phase error ranges $\phi_0 = 18°, 36°, 54°, 72°$. We also set $n(t) \sim CN(0, 1)$.

Similarly, from Fig. 3 we see, with large N, method 1 gives an accurate prediction of the BER, but with small N, method 1 gives a better prediction in the Rayleigh fading channel scenario than that in the static channel scenario.

By comparing the simulation results plotted in parts (a), (b), (c) in Fig. 3, we can also have the conclusion that increasing the number of nodes N can dramatically reduce the energy requirement for each sensor node subject to the same BER performance, and the number of nodes N has a much larger effect on BER performance than the phase error range ϕ_0.

By comparing the simulation results plotted in Fig. 2 with those in Fig. 3, we see when increasing the number of nodes N, the BER performance in the Rayleigh fading channel scenario comes close to that in the static channel scenario, which highlights the ability to mitigate fading through path diversity.

7. CONCLUSION

We have simulated the BER performance for BPSK modulation in distributed beamforming with phase errors in the static channel scenario and the Rayleigh fading channel scenario, where the results show a good match with our mathematical analysis. The whole beamforming process has been viewed as an equivalent channel and the system performance has been analyzed for different numbers of nodes and different phase error ranges. As the closed-form expression of BER is not easily obtained, we provide a method to approximate the BER results. Generally, method 1 gives a better prediction in the Rayleigh fading channel scenario than the static channel scenario. We are currently working on other approximations of the BER performance, such as method 2 outlined above. The effect of the energy limitation of each sensor node on the BER performance, and BER analysis for other modulation schemes in distributed beamforming with phase errors are also of particular interest for future work.

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